CONTROL ENGINEERING
Theory and Practice

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Preface

This book is designed to present the major topics in control engineering and is intended for use as a text at the undergraduate engineering level as there is very little demarcation between electrical, electronics, mechanical, industrial, aerospace, and chemical engineering in control system practices. It offers a basic yet comprehensive treatment of both continuous-time and discrete-time control systems with an emphasis on continuous-time systems. A chapter each is devoted to in-depth analysis and design of nonlinear control systems, control system devices which form an important segment of control technology, and optimal control theory. The book also introduces students to the modern concepts of neural fuzzy systems and adaptive learning systems. Written in a readable manner, it also contains many solved examples to reinforce understanding of the theory.

The text is organized into 16 chapters and an appendix. Chapter 1 elucidates the concept of a control system with examples drawn from several disciplines. Models of control systems with the help of differential equations, block diagrams and signal flow graphs are explained. Chapter 2 reviews some mathematical tools, with an application-oriented approach, used in control systems theory for system modelling, such as Laplace transforms, $\mathcal{Z}$-transforms, eigenvalues and eigenvectors, and calculus of variations. Chapter 3 gives transient response analysis of first- and second-order systems and discusses 'types' of feedback control systems with examples. Chapter 4 is devoted to analysis of control systems in state space. The concepts of controllability and observability are defined and illustrated with examples. Chapter 5 presents the basic concepts of stability of linear control systems. This chapter also discusses Hurwitz stability criterion and Routh's stability criterion in detail. Chapter 6 discusses the root-locus analysis. Chapters 7 and 8 deal with the frequency-response analysis of control systems. Bode plots, polar plots, and Nyquist stability criterion are discussed in these chapters. Chapter 9 explains the need of compensation in control systems. The design and characteristics of lead, lag, and lag-lead compensators are discussed.

Chapter 10 provides in-depth coverage of nonlinear systems. It begins with Liapunov stability analysis and also treats at length the phase plane and describing function methods of nonlinear analysis. Different examples of nonlinear systems involving saturation, friction, backlash, deal zone, and relay are described. Chapter 11 introduces students to the constituents of sampled-data control systems. The sampling procedure is described in detail. The Jury's stability and Schücohn-stability tests are explained with examples. Analysis of stability with the help of bilinear transformation is also discussed.

The principles of operation of control system devices and components are described in Chapter 12. The devices discussed include potentiometers, synchros, differential transformers, servomotors, tachogenerators, magnetic amplifiers, stepper motors, and gyroscopes. In Chapter 13, mathematical procedures for optimal control design are fully explained. Chapter 14 provides a brief description of recent advances in control systems, namely adaptive control, neural network control, and fuzzy logic control.

The entire text is presented in a simple, lucid manner so that the students acquire a thorough understanding of the subject matter discussed. The book contains a lot of solved examples within the chapters. Typical solved examples of greater analytical nature (including some requiring use of MATLAB) objective type questions and exercises and included in concluding Chapters 15 and 16 constitute a few of the outstanding features that distinguish this book from other undergraduate control system books.

I am deeply indebted to my Professor Late Dr. S.K. Nag of Jadavpur University. His valuable support and constant encouragement was the major driving force for the birth of this book.

I would also like to express my deep respect to my beloved mother Late Smt. Susoma Bandyopadhyay for inspiration to write this book.

I am also thankful to my sisters Smt. Ramala Chatterjee, Smt. Krishna Roy, Smt. Swapna Banerjee, brother-in-law Mr. Asim Roy and sister-in-law Miss Meera Chatterjee for their kind words of support and encouragement over the years.

Last and the most, I am deeply grateful to my wife Ila and son Soumendra for their loving understanding throughout the countless hours spent working on the manuscript.

Manabendra Nath Bandyopadhyay

Chapter 1 Introduction

1.1 CONCEPT OF CONTROL SYSTEMS

A control system is a system by virtue of which any quantity or condition (called *controlled variable*) of interest of a machine, mechanism or equipment can be controlled as per desire. Usually, the system has a command signal applied at the input and the controlled variable exhibited at the output. For example, the force applied on to an accelerator pedal causes the speed of the engine vehicle to increase. Here, force is the command signal and the speed of the engine is the controlled variable.

1.1.1 Closed-loop and Open-loop Control

Figure 1.1 shows the schematic diagram of the automatic closed-loop control system. The controlled output is compared with the reference input in the error detector. The controlled output is passed through the feedback elements before it is compared with the reference or desired input. The difference of the two (i.e. the error signal) is used to control the output. As a closed loop (or feedback loop) is formed in the controlled system, it is termed *closed-loop control system* or *feedback control system*.

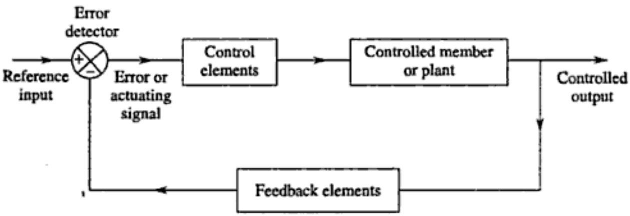


Fig. 1.1 Closed-loop control system.

The *open-loop control system* is that system where the output remains constant for a constant input signal provided the conditions external to the system remain unchanged. Though output in this system may be changed to any required value by changing the input signal accordingly, the main problem in this system is that the variations in the external conditions or internal parameters of the system may cause, in an uncontrolled manner, unwanted variations of the output from the required value.

1.1.2 Examples of Closed-loop Control Systems

Figure 1.2 is an example of a closed-loop (feedback) control system. In this system, the different devices shown operate in the following manner:

Float is the feedback path element.
Potentiometer is the error detector.
Height h_1 of the liquid is the reference input.
Slider P of the potentiometer is positioned as per the liquid level h_1.
Motor drive with the mechanical link and valve T_2 is the controlled member.

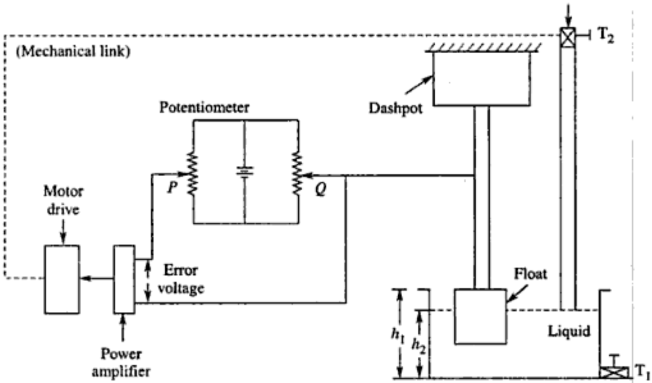


Fig. 1.2 Control system to maintain tank level.

Any variation in the desired liquid level h_1 (say, from h_1 to h_2) appears as an error voltage to the input of the power amplifier. The output of the amplifier drives the motor which is connected mechanically to valve T_2. The movement of the valve automatically adjusts the height of level of liquid such that the height h_2 comes back to the desired height h_1.

Servomechanism is one of the examples of the feedback control system where the controlled variable is mechanical position or time derivative of position, for example, velocity or acceleration. Figure 1.3 is a schematic diagram of a position control system. This is nothing but a servo system. In this system, θ_o and θ_{ref} are the output and reference positions respectively. When there is a difference between θ_{ref} and θ_o, it acts as an error in the angular position and the same is converted to an equivalent error voltage by the potentiometer. The error voltage is amplified and changes the field current of the dc generator. It ultimately changes the speed of the motor. The motor moves the load through the gear mechanism such that the output position θ_o becomes equal to the reference position θ_{ref}.

A missile launching and guidance system is a very important application of the closed-loop control system.

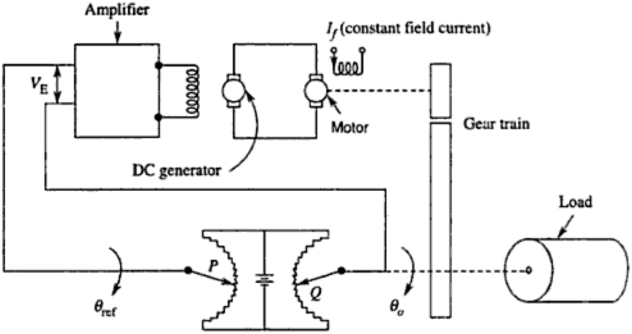


Fig. 1.3 Example of a position control system.

Figure 1.4 is a schematic diagram of a missile launching and guidance system. The radar detects the presence of the target aircraft through its rotating antenna and passes the detection signal to the launch computer indicating the velocity and position of the target. The computer calculates the firing angle which is the launch command signal. This command signal is passed to the launcher, i.e. the drive motor, through the power amplifier. The launcher angular position is fed back to the launch computer and the missile is fired at the moment when the difference

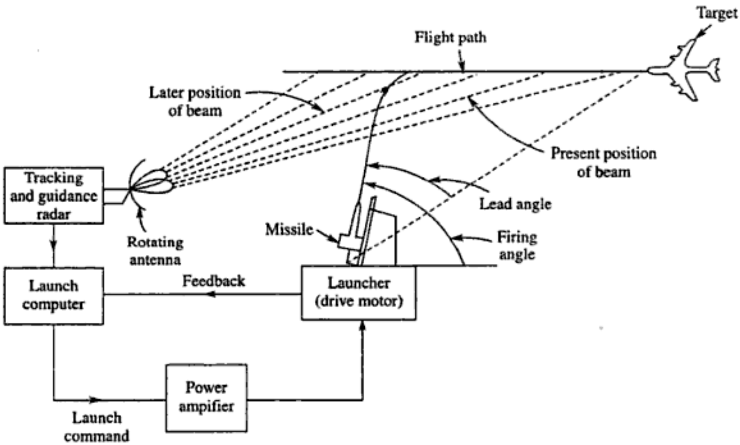


Fig. 1.4 Use of control system in missile launching.

between the launch command signal and the missile firing angle becomes zero. After the missile is launched, the guiding signal is from the radar beam itself as the radar antenna is locked on to the target and it continuously tracks the target. The missile after being launched comes under the guidance of the radar beam. The control system available within the missile now obtains the guidance signal from the beam which automatically adjusts the control surfaces of the missile in such a manner that the missile moves along the beam. Finally, the target will be hit.

Automatic frequency control in the radio communication system is also another example of a closed-loop control system. Figure 1.5 shows a schematic diagram of a superheterodyne receiver with the automatic frequency control system. Here, the discriminator is a frequency modulating detector. The output of the intermediate frequency amplifier (IF amplifier) drives the discriminator. The automatic frequency control (AFC) voltage from the discriminator is applied across the varactor diode (reactance tube) as reverse bias. The varactor diode is connected across the tank circuit of the local oscillator of the AFC voltage. The change of polarity of capacitor will, of course, depend upon the AFC control voltage. Thus the frequency of local oscillator will be changed in magnitude and direction in such a way that the exact intermediate frequency is obtained.

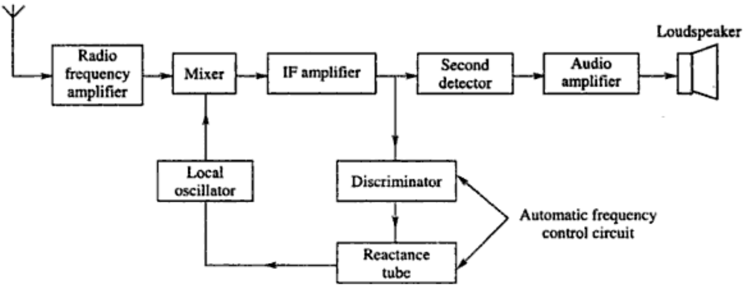


Fig. 1.5 Use of control system in automatic frequency control.

The automatic frequency control in the line-of-sight microwave links is another example of the closed-loop control system. Figure 1.6 shows the block diagram of a microwave line-of-sight transmitter. Different voice signals and other signals are formed into a baseband signal. This baseband signal is then used to modulate a microwave transmitter signal. Here the control part of the transmitting system is the automatic frequency control circuit which compares the intermediate frequency (IF) signal produced by the transmitting circuit with the reference oscillator output in a limiter discriminator circuit followed by the synchronous detector. The resulting dc output, proportional to the frequency difference between the two signals, is passed on to the reflector of the beat reflex klystron for effecting frequency correction in order to make the IF signal exactly equal to 70 MHz and thus eliminate any frequency error.

The self-balancing servo-operated potentiometer is another example of the closed-loop feedback control system. Figure 1.7 shows a schematic diagram of a self-balancing servo-operated

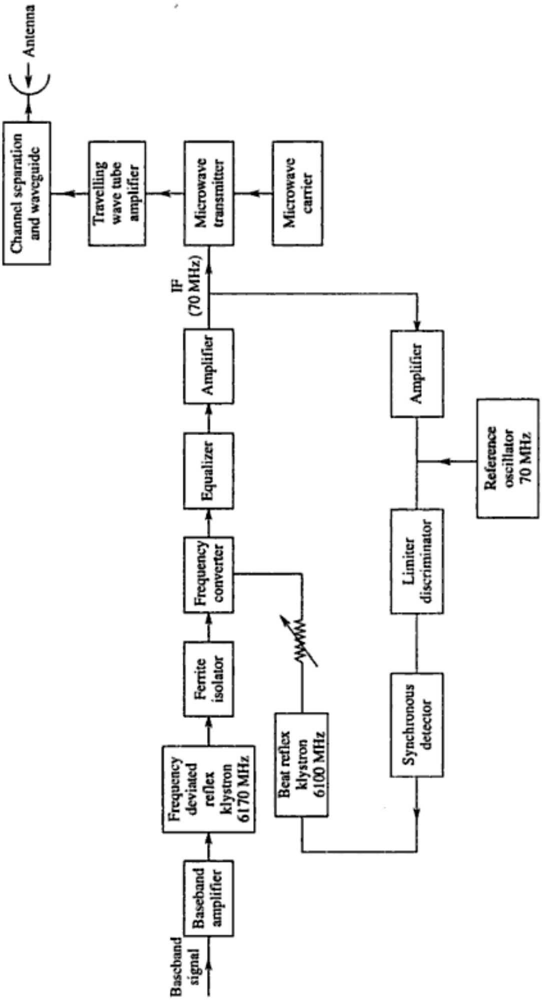


Fig. 1.6 *Control system in a microwave communication transmitter.*

potentiometer. Figure 1.8 is the block diagram of the same. Here, the automatic balancing is provided by an ac servomotor as shown in Fig. 1.7. The servomotor drives the tapping point of the potentiometer, the writing mechanism, and the pointer for indication. The motor comes to rest when the unbalance voltage becomes zero. The output displacement and voltage feedback are related by the calibration constant of the potentiometer wire. The amplifier is tuned to 50 Hz, and has high gain.

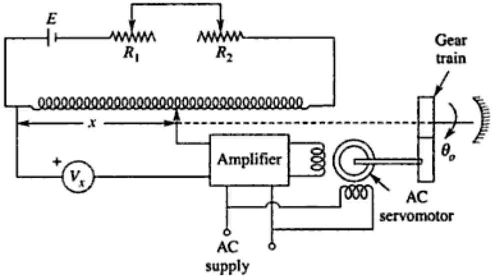


***Fig. 1.7** Self-balancing servo-operated potentiometer.*

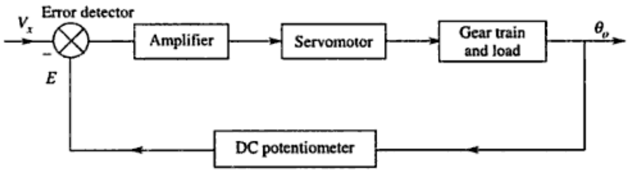


***Fig. 1.8** Block diagram of Fig. 1.7.*

Figures 1.9 and 1.10 show an electromagnetic balance system and a beam balance system respectively. The beam balance system is a feedback measuring system used for comparison of the moments of two forces from which one physical quantity is converted into another physical quantity. Two physical quantities are needed to set up forces proportional to them. Figure 1.9 shows the scheme of an electromagnetic balance in which the beam takes the horizontal position when the moments on account of the unknown and standard masses are equal. The horizontality of the beam is found out by the photo-electric cell and restored by the permanent magnet moving coil system. The output of the photo cell is processed by the amplifier which in turn drives the current I_o through the moving coil. By adjusting the initial coil position and the gain of the amplifier, the system is made to take the null position when the two masses are equal. Whenever there is any difference in the two masses, then the horizontality of the beam is disturbed. It can only be restored by the torque produced by the moving coil on account of change in I_o. The

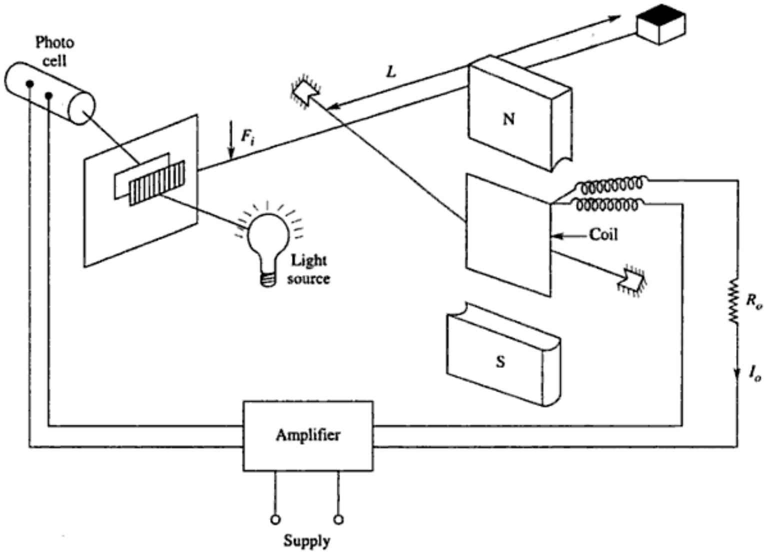


Fig. 1.9 Electromagnetic balance system.

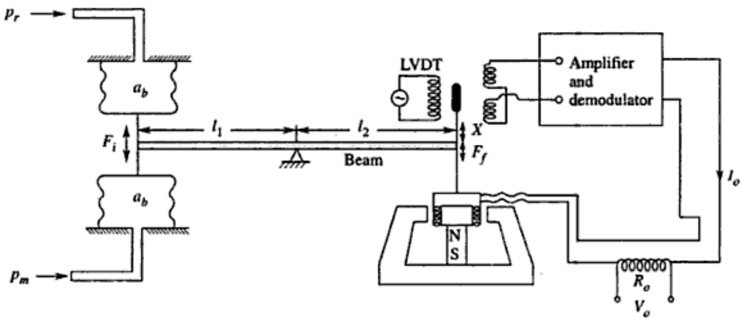


Fig. 1.10 Beam balance system.

change in I_o will be proportional to the difference in the two masses. In Fig. 1.10 the comparison of moments on account of two forces is shown. These two forces are as follows:

(i) One force F_i is on account of the error between the reference and the measured pressures, i.e. $(p_r - p_m)$.
(ii) The other force F_f is on account of the electromagnetic force servo when carrying an output current I_o.

In the feedback system of Fig. 1.10 the gain of the amplifier is kept high such that the moments are balanced when the beam is horizontal. The block diagram is illustrated in Fig. 1.11.

The relationship at null position will be

$$(p_r - p_m)a_b l_1 = K_f l_2 I_o$$

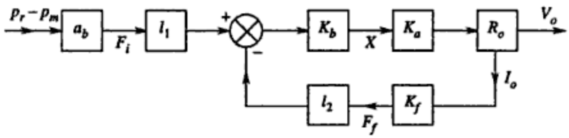


Fig. 1.11 Block diagram of Fig. 1.10.

Merits and demerits of closed- and open-loop control systems

Although the closed-loop control system provides effective control of the system, it develops the possibility of undesirable system oscillations, i.e. hunting. The open-loop control is found to be satisfactory only when the fluctuations in external conditions or internal parameters of the system can be tolerated, or when the design of system components is made in such a way that it limits the variations in parameters and environmental conditions. No doubt that the open-loop systems are by far the simplest and most economic type of control systems, but these are generally inaccurate and unreliable and usually not preferred.

1.1.3 Stages of Development of Control Systems

The stages of development of control systems can be classified into the following four categories:

(i) Open-loop system
(ii) Closed-loop system
(iii) Utilization of adaptive controller in the closed-loop system
(iv) Learning system

The adaptive controller in the closed-loop system helps the control system to adapt itself to the changing external conditions. The learning system is capable of recognizing new situations on the basis of its past experience and finally provides decisions and then acts accordingly. Figures 1.12, 1.13, 1.14, and 1.15 describe the four stages of development of control systems in block diagram forms.

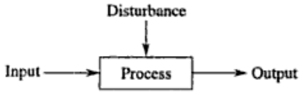


Fig. 1.12 *Open-loop system.*

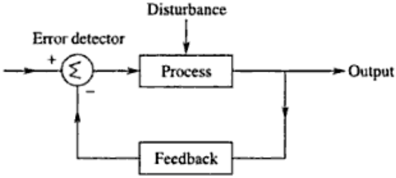


Fig. 1.13 *Closed-loop system.*

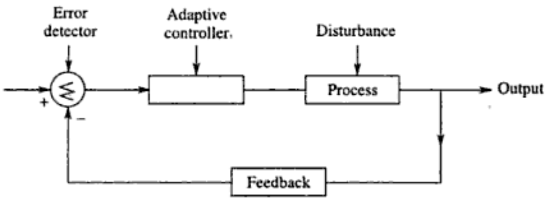


Fig. 1.14 *Adaptive controller.*

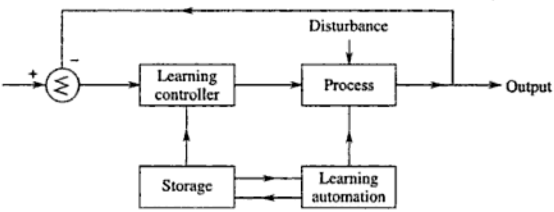


Fig. 1.15 *Learning system.*

Multivariable control system

When a number of variables are controlled in a system, then that system is called the multivariable control system. Figure 1.16 shows a block diagram of a multivariable control system.

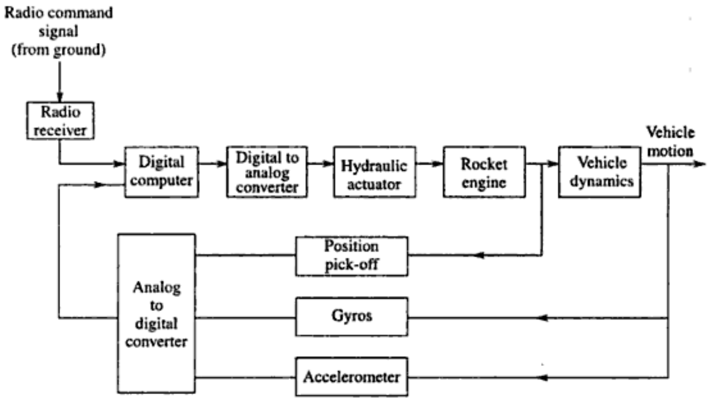


Fig. 1.16 Example of a multivariable control system.

It is nothing but an autopilot system. This system steers a rocket vehicle with the help of a radio command signal. The velocity and acceleration of motion of the vehicle are the feedback signals controlled by motion sensors, i.e. gyros and accelerometers respectively. The position pick-off feeds the information about the angular displacement of the engine. Thus, the direction in which the vehicle is moving is also controlled. On receiving commands from the ground, the onboard computer develops a signal that controls the hydraulic actuator, and the rocket moves giving the proper closed-loop feedback signals about the vehicle's motion.

1.2 MODELS OF THE CONTROL SYSTEM

The following three basic representations of physical components and systems are widely used in the study of control systems:

(i) Differential equations and other mathematical relations
(ii) Block diagrams
(iii) Signal flow graphs

1.3 BLOCK DIAGRAM

Figure 1.17 shows the armature controlled dc motor. The air-gap flux

$$\phi = K_f I_f$$

where I_f is the field current. The torque developed by the motor

$$T_M = K K_f I_f I_a$$

where I_a is the armature current. In the case of an armature controlled dc motor, the field current is constant. Therefore,

$$T_M = K_T I_a$$

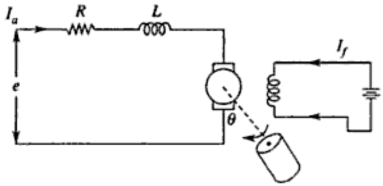


Fig. 1.17 DC motor with armature control.

where K_T is the torque constant. The motor back emf is proportional to the speed and is denoted by

$$e_b = K_b \frac{d\theta}{dt} \tag{1.1}$$

where K_b is the back emf constant. The differential equation of the armature circuit is

$$L\frac{dI_a}{dt} + RI_a + e_b = e \tag{1.2}$$

The torque equation is

$$J\frac{d^2\theta}{dt^2} + F\frac{d\theta}{dt} = T_M = K_T I_a \tag{1.3}$$

where

J = equivalent moment of inertia of motor and load referred to the motor shaft
F = equivalent viscous friction coefficient of motor and load referred to the motor shaft.

Taking Laplace transforms of (1.1), (1.2), and (1.3), we get

$$E_b(s) = K_b s\theta(s) \tag{1.4}$$

$$(Ls + R)I_a(s) = E(s) - E_b(s) \tag{1.5}$$

$$(Js^2 + Fs)\theta(s) = T_M(s) = K_T I_a(s) \tag{1.6}$$

The transfer function of the system $\dfrac{\theta(s)}{E(s)}$ can thus be determined as follows:

$$\theta(s) = \frac{E_b(s)}{K_b s}$$

$$= \frac{E(s) - (Ls + R)I_a(s)}{K_b s}$$

$$= \frac{E(s) - (Ls + R)\dfrac{(Js^2 + Fs)}{K_T}\theta(s)}{K_b s}$$

or

$$\theta(s) \cdot K_b s = E(s) - \frac{(Ls+R)(Js^2+Fs)}{K_T}\theta(s)$$

or

$$\theta(s)\left[K_b s + \frac{(Ls+R)(Js^2+Fs)}{K_T}\right] = E(s)$$

or

$$\frac{\theta(s)}{E(s)} = \frac{K_T}{s[(R+Ls)(Js+F)+K_T K_b]}$$

Figure 1.18 has been developed as the block diagram of the armature controlled dc motor. As $E_b = K_b s\theta(s)$ is nothing but $E_b = K_b\dot{\theta}(s)$, this block diagram can be modified as shown in Fig. 1.19.

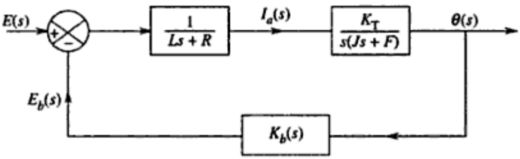


Fig. 1.18 Block diagram of Fig. 1.17.

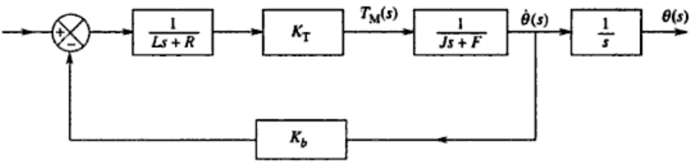


Fig. 1.19 Modified block diagram.

The block diagram of a closed-loop system can also be developed by the following generalized approach. Suppose a negative feedback system is being developed in a closed-loop system. Let

$R(s)$ be the reference input.

$O(s)$ be the output signal.

$f(s)$ be the feedback signal.

$E(s)$ be the actuating signal.

Then,

$G(s) = \dfrac{O(s)}{E(s)}$ = forward path transfer function.

$$H(s) = \text{transfer function of the feedback elements} = \frac{f(s)}{O(s)}$$

and

$$G(s)H(s) = \frac{f(s)}{E(s)} = \text{loop transfer function}$$

$$T(s) = \frac{O(s)}{R(s)} = \text{closed-loop transfer function}$$

Now,

$$O(s) = G(s)E(s)$$

$$\begin{aligned} E(s) &= R(s) - f(s) \\ &= R(s) - G(s)H(s)E(s) \\ &= R(s) - \frac{O(s)}{E(s)}H(s)E(s) \\ &= R(s) - O(s)H(s) \end{aligned}$$

Therefore,

$$\frac{O(s)}{G(s)} = E(s) = R(s) - O(s)H(s)$$

or

$$O(s) = G(s)R(s) - G(s)O(s)H(s)$$

or

$$O(s)[1 + G(s)H(s)] = G(s)R(s)$$

or

$$\frac{O(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

or

$$T(s) = \frac{G(s)}{1 + G(s)H(s)}$$

Hence the generalized block diagrams will be as shown in Figs. 1.20 and 1.21.

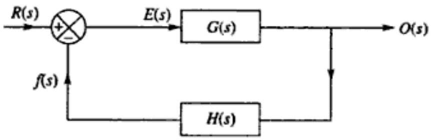


Fig. 1.20 Generalized form of the block diagram of a closed-loop system.

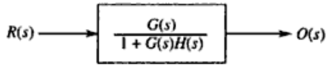


Fig. 1.21 Equivalent block diagram of the closed-loop.

1.3.1 Block Diagram of Multiple Input and Multiple Output Systems

Multiple input and multiple output systems mean that the system consists of a number of inputs and outputs. When multiple inputs are present in a linear system, then each input can be treated independently of the others. Finally, the complete output of the system is obtained by superposition. In other words, the outputs corresponding to each input are added together to obtain the net output. For example,

$$O_i(s) = \sum_{j=i}^{r} G_{ij}(s)\, R_j(s), \qquad i = 1, 2, \cdots, n$$

The above relation indicates that $R_j(s)$ is the jth input and $G_{ij}(s)$ is the transfer function between the ith output and the jth input considering all other inputs having been reduced to zero. Let us take an example of a two-input system as shown in Fig. 1.22.

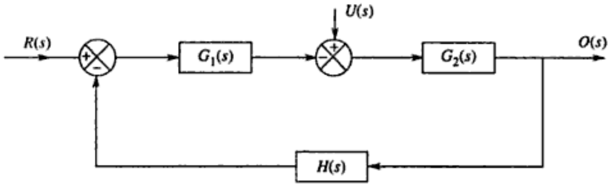


Fig. 1.22 Two-input system.

Here the two inputs are $R(s)$ and $U(s)$. If $R(s)$ and $U(s)$ are taken individually, then the two block diagrams will be as shown in Figs. 1.23 and 1.24, respectively.

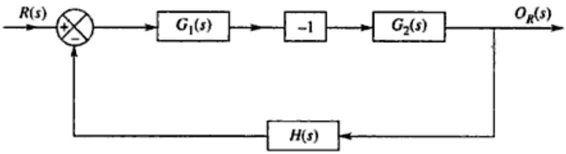


Fig. 1.23 System of Fig. 1.22 with R(s) input.

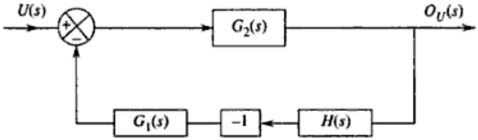


Fig. 1.24 *System of Fig. 1.22 with U(s) input.*

For Fig. 1.23, the output will be

$$O_R(s) = \frac{-G_1(s)\,G_2(s)}{1 - G_1(s)\,G_2(s)\,H(s)}\,R(s)$$

For Fig. 1.24, the output will be

$$O_U(s) = \frac{G_2(s)\,U(s)}{1 - G_1(s)\,G_2(s)\,H(s)}$$

Hence, the total output will be

$$O(s) = O_R(s) + O_U(s)$$

$$= \frac{G_2(s)}{1 - G_1(s)\,G_2(s)\,H(s)}\,[-G_1(s)\,R(s) + U(s)]$$

1.3.2 Sensitivity of the Control System

The main objective of the feedback system is to reduce the sensitivity of the system to parameter variations. The parameter variations may occur, for example, due to aging, environmental change, etc. Therefore, the sensitivity measures how effective the feedback system is for reducing the effect of parameter variations on system performance. It may be noted that for improving the sensitivity, the gain of the system has to be reduced.

In the case of the open-loop system, as shown in Fig. 1.25

$$O(s) = G(s)R(s)$$

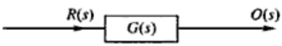


Fig. 1.25 *Open-loop system.*

Suppose on account of parameter variation $G(s)$ is changed to $[G(s) + \Delta G(s)]$, where $|G(s)| >> |\Delta G(s)|$. The output of the open-loop system then changes to

$$O(s) + \Delta O(s) = [G(s) + \Delta G(s)]R(s)$$

Hence, $\Delta O(s) = \Delta G(s)R(s)$.

In the case of the closed-loop system, as shown in Fig. 1.26, the output

$$O(s) = \frac{G(s)}{1 + G(s)H(s)} R(s)$$

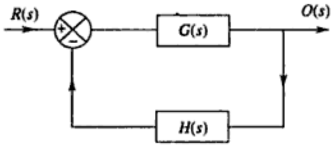


Fig. 1.26 Closed-loop system.

On account of the similar variation as mentioned above

$$O(s) + \Delta O(s) = \frac{[G(s) + \Delta G(s)]R(s)}{1 + G(s)H(s) + \Delta G(s)H(s)}$$

Therefore,

$$\Delta O(s) = \frac{[G(s) + \Delta G(s)]R(s)}{1 + G(s)H(s) + \Delta G(s)H(s)} - \frac{G(s)R(s)}{1 + G(s)H(s)}$$

$$\approx \frac{[G(s) + \Delta G(s)]R(s)}{1 + G(s)H(s)} - \frac{G(s)R(s)}{1 + G(s)H(s)}$$

$$= \frac{\Delta G(s)R(s)}{1 + G(s)H(s)}$$

The term sensitivity is generally defined quantitatively as

$$\frac{\text{Percentage change in the transfer function of the closed-loop system}}{\text{Percentage change in the forward path transfer function}}$$

Here the closed loop transfer function is $T(s) = \dfrac{O(s)}{R(s)}$ and the forward path transfer function is $G(s)$. Hence the sensitivity is

$$\frac{\delta T(s)/T(s)}{\delta G(s)/G(s)} = \frac{\delta T(s)}{\delta G(s)} \cdot \frac{G(s)}{T(s)}$$

$$= \frac{\delta}{\delta G(s)}\left(\frac{G(s)}{1+G(s)H(s)}\right)\cdot\frac{G(s)}{\dfrac{G(s)}{1+G(s)H(s)}}$$

$$= \frac{1+G(s)H(s)-G(s)H(s)}{[1+G(s)H(s)]^2}\cdot[1+G(s)\,H(s)]$$

$$= \frac{1}{1+G(s)H(s)}$$

Therefore, the sensitivity of the closed-loop system is $\dfrac{1}{1+G(s)H(s)}$.

In the case of the open-loop system, the sensitivity is one, since $T(s)$ is equal to $G(s)$.

Thus, it is observed that in the case of the closed-loop system, the sensitivity is reduced by a factor $\dfrac{1}{1+GH}$ compared to the open-loop system. On the other hand, the gain of the closed-loop system is reduced by the same factor. The sensitivity with respect to feedback is expressed as

$$\frac{\text{Percentage change in the transfer function of the closed-loop system}}{\text{Percentage change in the transfer function of the feedback elements}}$$

$$= \frac{\dfrac{\delta T(s)}{T(s)}}{\dfrac{\delta H(s)}{H(s)}} = \frac{\delta T(s)}{\delta H(s)}\cdot\frac{H(s)}{T(s)}$$

$$= \frac{\delta}{\delta H(s)}\left(\frac{G(s)}{1+G(s)H(s)}\right)\cdot\frac{H(s)}{\dfrac{G(s)}{1+G(s)H(s)}}$$

$$= -\frac{G(s)}{[1+G(s)H(s)]^2}\,G(s)\,\frac{1+G(s)H(s)}{G(s)}\,H(s)$$

$$= \frac{-G(s)H(s)}{1+G(s)H(s)}$$

From the above it is clear that if $G(s)H(s) >> 1$, then the sensitivity of the feedback system with respect to H tends to one. Therefore, the changes in $H(s)$ affect the system output. That is why, it is essential to utilize the feedback elements which will not vary with the change of external conditions. $G(s)$ is developed mainly from power elements, whereas $H(s)$ is developed from measuring elements. The measuring elements operate at low power level. Therefore, the cost of making use of accurate measuring elements is much less than that of the $G(s)$ elements.

1.3.3 Improvement of System Dynamics by Feedback

Figure 1.27 shows the unity feedback system, the closed-loop transfer function of which is

$$\frac{O(s)}{R(s)} = \frac{\dfrac{K}{s+a}}{1+\dfrac{K}{s+a}}$$

$$= \frac{\dfrac{K}{s+a}}{\dfrac{s+a+K}{s+a}} = \frac{K}{s+K+a}$$

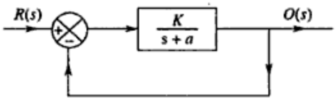


Fig. 1.27 Unity feedback system.

If there is no feedback, then $\dfrac{O(s)}{R(s)} = \dfrac{K}{s+a}$.

If the input $R(s) = 1$ (in the case of the impulse input, the value of $R(s) = 1$), then the output of the non-feedback system will be

$$O(s) = \frac{K}{s+a}$$

or

$$o(t) = Ke^{-at}$$

For unity feedback, the output is

$$O(s) = \frac{K}{s+a+K}$$

or

$$o(t) = Ke^{-(a+K)t}$$

It is quite clear from above that the time constant for the non-feedback system is $1/a$ and that for the unity feedback system is $1/(a + K)$.

Hence, for a positive value of K, the time constant for the feedback system is less than that of the non-feedback system. This indicates that with the increase in value of K, the system dynamics becomes faster. In other words, the transient response decays more quickly. Thus, it is proved that the feedback is a powerful technique for the control of system dynamics. The disturbance signal is also reduced with the help of the closed-loop feedback system. In Fig. 1.28,

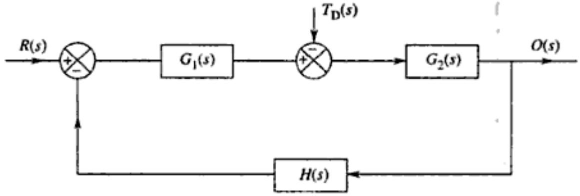


Fig. 1.28 Closed-loop system with disturbance signal.

$T_D(s)$ is the disturbance signal. The output due to the disturbance signal can be calculated as follows (see Fig. 1.29).

$$\frac{O_1(s)}{T_D(s)} = \frac{-G_2(s)}{1 + G_1(s)H(s)G_2(s)}$$

or

$$O_1(s) = \frac{-G_2(s)}{1 + G_1(s)G_2(s)H(s)} T_D(s)$$

If $G_1(s)G_2(s)H(s) >> 1$, then

$$O_1(s) = -\frac{1}{G_1(s)H(s)} T_D(s)$$

Hence, the output due to a disturbance signal in a closed-loop system is reduced to a great extent.

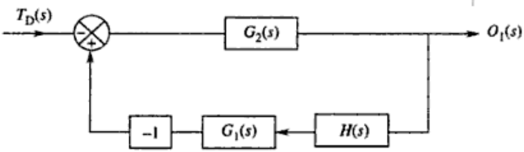


Fig. 1.29 Block diagram showing $O_1(s)$ as the output due to disturbance signal.

1.4 SIGNAL FLOW GRAPH

The signal flow graph is the graphical representation of the relationships between the variables of a group of linear algebraic equations. The signal flow graph can be derived from the block diagram of the closed-loop system in the following manner.

Figure 1.31 is the signal flow graph of the block diagram shown in Fig. 1.30. Here *R* is termed the input node and *O* the output node. The input node of the signal flow graph will have only outgoing branches whereas the output node of the signal flow graph will have only incoming branches. Forward path is the path from the input node to the output node.

For example in Fig. 1.31, *RECO* is the forward path. Loop is the path which originates and terminates at the same node. The forward path gain is the product of the branch gains encountered in traversing the forward path. In Fig. 1.31, it is, $1.G.1 = G$.

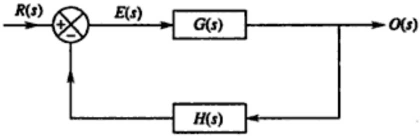


Fig. 1.30 Closed-loop system.

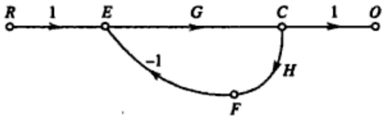


Fig. 1.31 Signal flow graph of Fig. 1.30.

The loop gain is the product of the branch gains encountered in traversing the loop. For example, in Fig. 1.31, it is $G.H.(-1) = -GH$.

1.4.1 Signal Flow Graph Developed From Equations

Let x_1 and x_5 be the input and the output variables and the system be described by the following equations.

$$x_2 = A_{12}x_1 + A_{32}x_3 + A_{42}x_4 + A_{52}x_5 \quad \text{(i)}$$

$$x_3 = A_{23}x_2 \quad \text{(ii)}$$

$$x_4 = A_{34}x_3 + A_{44}x_4 \quad \text{(iii)}$$

$$x_5 = A_{35}x_3 + A_{45}x_4 \quad \text{(iv)}$$

The first equation describes the signal flow graph as shown in Fig. 1.32.

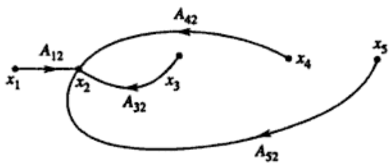


Fig. 1.32 Signal flow graph of Eq. (i).

The second equation describes the signal flow graph as shown in Fig. 1.33.

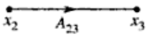


Fig. 1.33 Signal flow graph of Eq. (ii).

The third equation describes the signal flow graph as shown in Fig. 1.34.

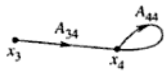


Fig. 1.34 Signal flow graph of Eq. (iii).

The fourth equation describes the signal flow graph as shown in Fig. 1.35.

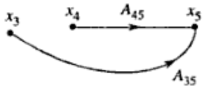


Fig. 1.35 Signal flow graph of Eq. (iv).

If all of the above signal flow graphs are combined, then the signal flow graph of the whole system will be developed as shown in Fig. 1.36.

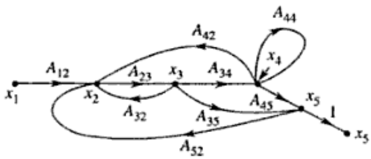


Fig. 1.36 Signal flow graph of the system.

1.4.2 Mason's Gain Formula

Mathematical modelling of a signal flow graph can be easily dealt with by the Mason's gain formula. The application part of the Mason's gain formula is described in this book. The reader interested to know the mathematical proof of this formula may consult books on advanced mathematics.

Mason's gain formula states that

$$T = \frac{1}{\Delta}\sum_K P_K \Delta_K$$

where

T = overall gain of the system

P_K = path gain of the Kth forward path.

Here forward is nothing but the path from the input node to the output node.

Δ is termed the determinant of the graph = 1 − (sum of the loop gains of all individual loops) + (sum of the gain products of all the possible combinations of two non-touching loops) − (sum of the gain products of all the possible combinations of three non-touching loops) + ⋯

Δ_K is the value of Δ for that part of the graph not touching the Kth forward path.

In this example, there are two forward paths.

$$P_1 = A_{12}A_{23}A_{34}A_{45}$$

$$P_2 = A_{12}A_{23}A_{35}$$

Individual loop gain products are:

$$P_{11} = A_{23}A_{32}$$

$$P_{21} = A_{23}A_{34}A_{42}$$

$$P_{31} = A_{44}$$

$$P_{41} = A_{23}A_{34}A_{45}A_{52}$$

$$P_{51} = A_{23}A_{35}A_{52}$$

Gain products of all the possible combinations of two non-touching loops are:
(Loops are termed non-touching if they do not possess any common node)

$$P_{12} = A_{23}A_{32}A_{44}$$

$$P_{22} = A_{23}A_{35}A_{52}A_{44}$$

Therefore,

$$\Delta = 1 - (A_{23}A_{32} + A_{23}A_{34}A_{42} + A_{44}A_{23}A_{34}A_{45}A_{52} + A_{23}A_{35}A_{52}) + (A_{23}A_{32}A_{44} + A_{23}A_{35}A_{52}A_{44})$$

Again, Δ_1 – the value of Δ for that part of the graph not touching the first forward path = 1
Similarly,

$$\Delta_2 = 1 - A_{44}$$

Therefore,

$$P_1\Delta_1 + P_2\Delta_2 = A_{12}A_{23}A_{34}A_{45} + A_{12}A_{23}A_{35}(1 - A_{44})$$

Hence, the overall gain

$$\frac{x_5}{x_1} = \frac{A_{12}A_{23}A_{34}A_{45} + A_{12}A_{23}A_{35}(1 - A_{44})}{1 - A_{23}A_{32} - A_{23}A_{34}A_{42} - A_{44} - A_{23}A_{34}A_{45}A_{52} - A_{23}A_{35}A_{52} + A_{23}A_{32}A_{44} + A_{23}A_{35}A_{52}A_{44}} \tag{1.7}$$

If the above gain is calculated algebraically, then the result will be determined as follows.

$$x_5 = A_{35}x_3 + A_{45}x_4$$

$$= A_{35}A_{23}x_2 + A_{45}\frac{A_{34}}{1 - A_{44}}x_3$$

$$= A_{35}A_{23}x_2 + A_{45}\frac{A_{34}}{1 - A_{44}}A_{23}x_2$$

($\because$ $x_3 = A_{23}x_2$ and $x_4 = A_{34}x_3 + A_{44}x_4$)

Therefore,

$$x_5 = x_2\left[A_{35}A_{23} + A_{45}\frac{A_{34}}{1-A_{44}}A_{23}\right]$$

Again,

$$A_{12}x_1 = x_2 - A_{32}x_3 - A_{42}x_4 - A_{52}x_5$$

or

$$A_{12}x_1 = x_2 - A_{32}A_{23}x_2 - A_{42}\frac{A_{34}}{1-A_{44}}A_{23}x_2 - A_{52}\left[A_{35}A_{23} + A_{45}\frac{A_{34}}{1-A_{44}}A_{23}\right]x_2$$

or

$$x_1 = x_2\left[\frac{1}{A_{12}} - \frac{A_{32}A_{23}}{A_{12}} - \frac{A_{42}A_{34}A_{23}}{(1-A_{44})A_{12}} - \frac{A_{52}A_{35}A_{23}}{A_{12}} - \frac{A_{45}A_{52}A_{34}A_{23}}{(1-A_{44})A_{12}}\right]$$

$$\frac{x_5}{x_1} = \frac{A_{45}A_{34}A_{23}A_{12} + A_{12}A_{35}A_{23}(1-A_{44})}{1-A_{44}-A_{32}A_{23}(1-A_{44})-A_{42}A_{34}A_{23}-A_{52}A_{35}A_{23}(1-A_{44})-A_{45}A_{52}A_{34}A_{23}}$$

$$= \frac{A_{12}A_{23}A_{34}A_{45} + A_{12}A_{23}A_{35}(1-A_{44})}{1-A_{23}A_{32}-A_{23}A_{34}A_{42}-A_{44}-A_{23}A_{34}A_{45}A_{52}-A_{23}A_{35}A_{52}+A_{23}A_{32}A_{44}+A_{23}A_{35}A_{52}A_{44}} \quad (1.8)$$

Equations (1.7) and (1.8) give the same result. But the algebraic solution becomes more and more tiresome as we proceed to solve more complicated networks. That is why the Mason's gain formula is being utilized for the solution of the signal flow graph.

1.4.3 Development of Signal Flow Graph From the Practical Example

Figure 1.37 shows the example of the closed-loop controlled drive of a dc motor.

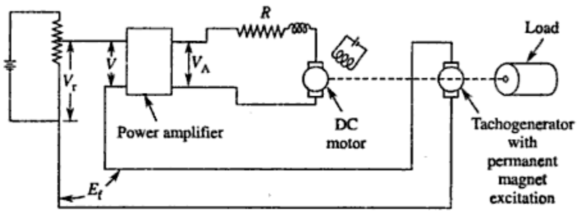


Fig. 1.37 Closed-loop control drive.

Here

V_r is the reference voltage

E_f is the feedback voltage

V is the actuating voltage applied to the control element, i.e. the power amplifier.

The controlled member is the dc motor connected to the load.

The feedback element is the tachogenerator having permanent magnet excitation. Here

$$E_f = K_1\phi\omega = K_2\omega$$

since ϕ is constant on account of permanent magnet excitation. The voltage applied at the armature terminals

$$V_A = K_A(V_r - E_f)$$

Also the armature circuit voltage

$$V_A = I_aR + L\frac{dI_a}{dt} + K_3\omega$$

where K_3 is the back emf constant, R and L are the resistance and inductance of the armature winding. For constant field current, the torque developed by the motor is

$$T_M = K_4I_a$$

where

$$T_M = T_L + J\frac{d\omega}{dt} + F\omega$$

with

T_L = load torque
J = moment of inertia
F = viscous friction coefficient

Taking the Laplace transforms of the above equations developed, we get

$$E_f(s) = K_2\omega(s)$$

$$V_A(s) = K_A[V_r(s) - E_f(s)]$$

$$V_A(s) = I_a(s)R + LsI_a(s) + K_3\omega(s)$$

$$K_4I_a(s) = T_M(s) = Js\omega(s) + F\omega(s) + T_L(s)$$

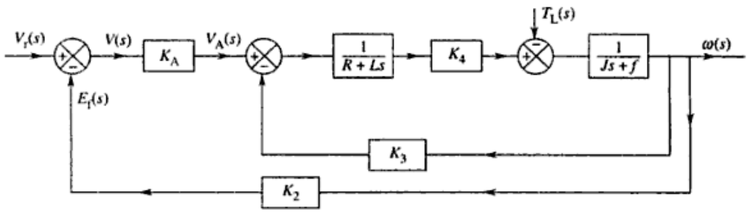


Fig. 1.38 Block diagram of Fig. 1.37.

or

$$T_M(s) - T_L(s) = (Js + F)\omega(s)$$

The signal flow graph of the block diagram (shown in Fig. 1.38) is described in Fig. 1.39.

The Mason's gain formula can easily be applied on the signal flow graph shown in Fig. 1.39 to determine the overall gain.

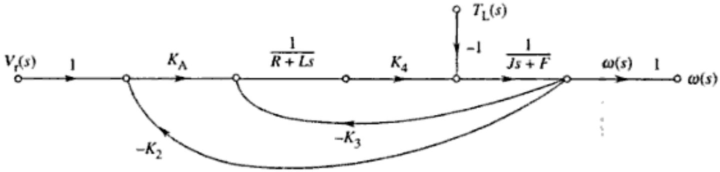


Fig. 1.39 Signal flow graph of Fig. 1.37.

The forward path gain

$$P_1 = \frac{K_A K_4}{(Js + F)(R + Ls)}$$

The individual loops with loop gains

$$P_{11} = \frac{-K_3 K_4}{(Js + F)(R + Ls)}$$

$$P_{21} = \frac{-K_A K_4 K_2}{(R + Ls)(Js + F)}$$

There are no combinations of non-touching loops. Hence,

$$\Delta = 1 - \left[\frac{-K_3 K_4}{(R + sL)(Js + F)} - \frac{K_A K_4 K_2}{(R + Ls)(Js + F)}\right]$$

The forward path is in touch with both the loops, hence $\Delta_1 = 1$.

If $T_L(s)$ is not considered, then

$$\frac{\omega(s)}{V_r(s)} = \frac{\dfrac{K_A K_4}{(R + Ls)(Js + F)}}{1 + \dfrac{K_3 K_4}{(R + Ls)(Js + F)} + \dfrac{K_A K_4 K_2}{(R + Ls)(Js + F)}}$$

$$= \frac{K_A K_4}{(R + Ls)(Js + F) + K_3 K_4 + K_A K_4 K_2} \tag{1.9}$$

If $T_L(s)$ is considered as input and $\omega(s)$ as output, and $V_r(s)$ is not considered, then

$$\frac{\omega(s)}{T_L(s)} = \frac{-\dfrac{1}{Js+F}}{1-\left(\dfrac{1}{(Js+F)}\dfrac{-K_3K_4}{(R+Ls)}+\dfrac{1}{(Js+F)}\dfrac{-K_2K_AK_4}{(R+Ls)}\right)}$$

$$= \frac{-\dfrac{1}{Js+F}}{\dfrac{(R+Ls)(Js+F)+K_3K_4+K_2K_AK_4}{(R+Ls)(Js+F)}}$$

$$= \frac{-1}{Js+F+\dfrac{K_3K_4}{R+Ls}+\dfrac{K_2K_AK_4}{R+Ls}}$$

Therefore, considering both the inputs, the actual $\omega(s)$ will be

$$\omega(s) = \frac{K_AK_4V_r(s)}{(R+Ls)(Js+F)+K_3K_4+K_AK_4K_2} - \frac{T_L(s)}{(Js+F)+\dfrac{K_3K_4}{R+Ls}+\dfrac{K_2K_AK_4}{R+Ls}}$$

The open-loop transfer function of the above system can also be determined from Eq. (1.9) by putting K_2 equal to zero. Moreover, if the load torque is zero, the open-loop transfer function will be

$$\frac{\omega(s)}{V_r(s)} = \frac{K_AK_4}{(R+Ls)(Js+F)+K_3K_4}$$

$$= \frac{K_AK_4}{R\left(1+\dfrac{L}{R}s\right)(Js+F)+K_3K_4}$$

Since the electrical time constant is much small compared to the mechanical one, we get

$$\frac{\omega(s)}{V_r(s)} = \frac{K_AK_4}{R(Js+F)+K_3K_4}$$

$$= \frac{K_AK_4}{RJs+RF+K_3K_4}$$

$$= \frac{\dfrac{K_AK_4}{RF+K_3K_4}}{\dfrac{RJs}{RF+K_3K_4}+1}$$

$$= \frac{K}{\tau s+1}$$

where

$$K = \frac{K_A K_4}{RF + K_3 K_4}$$

$$\tau = \frac{RJ}{RF + K_3 K_4}$$

1.4.4 Methods of Solving Problems of Block Diagram and Signal Flow Graph

1. How do you reduce the block diagram (shown in Fig. 1.40) to canonical form?

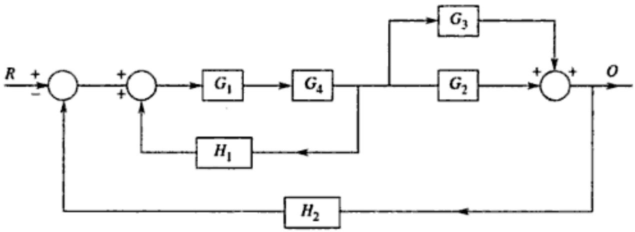


Fig. 1.40

First step

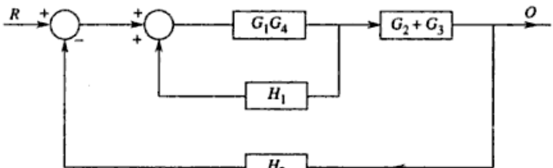


Fig. 1.41

Second step

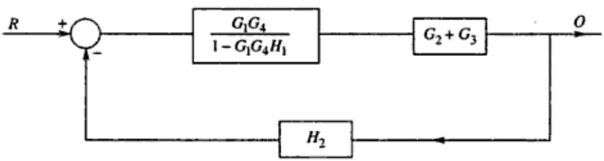


Fig. 1.42

Third step

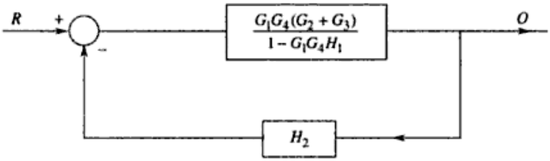


Fig. 1.43

The block developed in the third step (i.e. Fig. 1.43) is the canonical form. From this block, the transfer function can be easily determined.

2. How do you reduce the block diagram (shown in Fig. 1.44) to open-loop form.

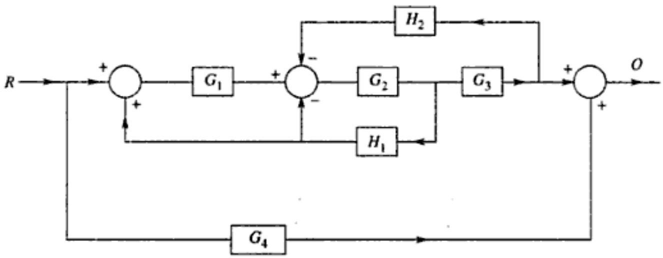


Fig. 1.44

First step

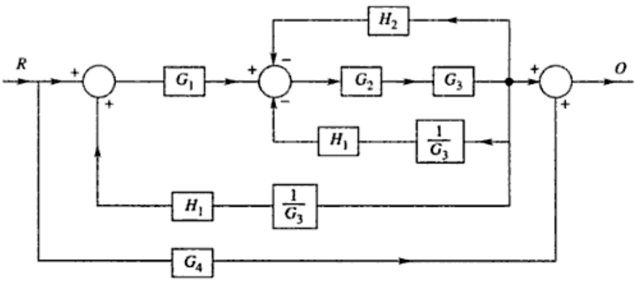


Fig. 1.45

Second step

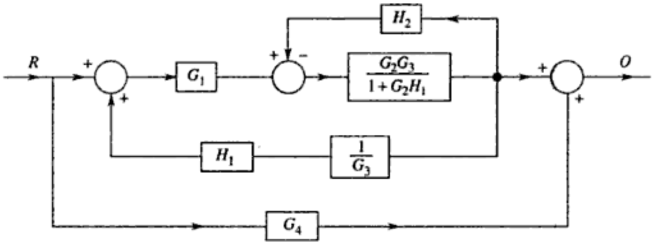


Fig. 1.46

Third step

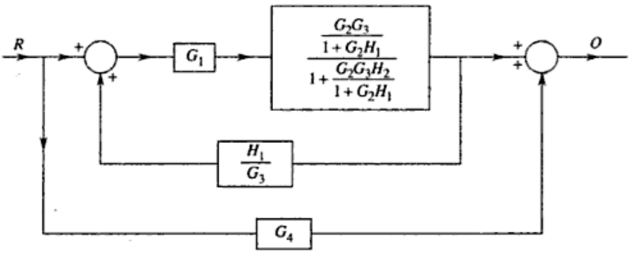


Fig. 1.47

Fourth step

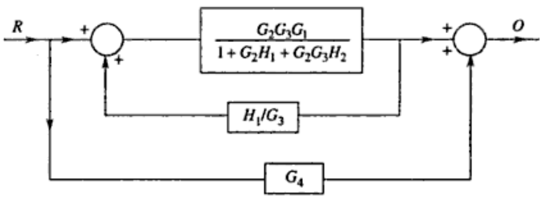


Fig. 1.48

Fifth step

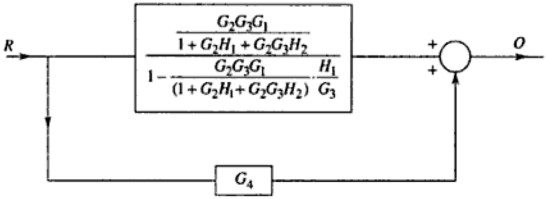


Fig. 1.49

Sixth step

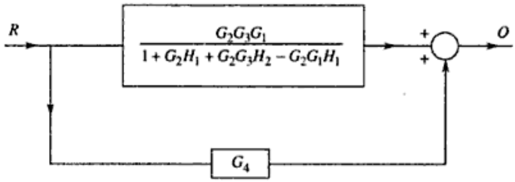


Fig. 1.50

Figure 1.51 describes the block diagram reduced to open-loop form.

Seventh step

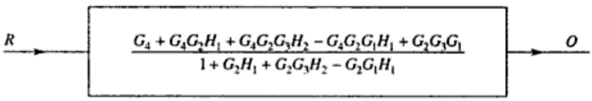


Fig. 1.51

3. Develop the signal flow graph of the circuit shown in Fig. 1.52. Also calculate v_5/v_1.

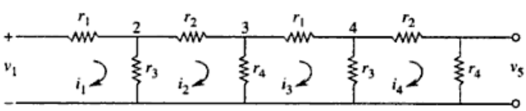


Fig. 1.52 *Circuit diagram.*

From the circuit shown in Fig. 1.52, the following equations are developed.

$$i_1 = \frac{1}{r_1}v_1 - \frac{1}{r_1}v_2 \qquad\qquad v_3 = r_4 i_2 - r_4 i_3$$

$$v_2 = r_3 i_1 - r_3 i_2 \qquad i_3 = \frac{v_3}{r_1} - \frac{v_4}{r_1}$$

$$i_2 = \frac{v_2}{r_2} - \frac{v_3}{r_2} \qquad v_4 = r_3 i_3 - r_3 i_4$$

$$i_4 = \frac{1}{r_2} v_4 - \frac{1}{r_2} v_5$$

$$v_5 = r_4 i_4$$

With the above equations the signal flow graph is drawn as shown in Fig. 1.53.

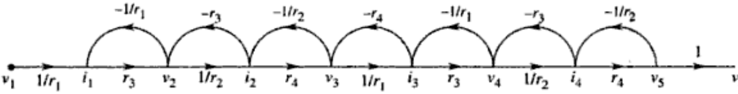


Fig. 1.53 Signal flow graph of Fig. 1.52.

The transfer function can be obtained by applying the Mason's gain formula.
The gains of the seven individual loops are:

$$P_{11} = -\frac{r_3}{r_1} \qquad P_{21} = -\frac{r_3}{r_2}$$

$$P_{31} = -\frac{r_4}{r_2} \qquad P_{41} = -\frac{r_4}{r_1}$$

$$P_{51} = -\frac{r_3}{r_1} \qquad P_{61} = -\frac{r_3}{r_2}$$

$$P_{71} = -\frac{r_4}{r_2}$$

For two non-touching loops, the number of combinations will be ${}^6C_2$ since, two non-touching loops, for a particular loop, will be all except its adjacent one. That is why the combinations will be ${}^6C_2$ instead of ${}^7C_2$. Now,

$${}^6C_2 = \frac{6!}{4!\,2!} = \frac{6 \times 5}{2} = 15$$

$$P_{12} = \frac{r_3 r_4}{r_1 r_2}, \qquad P_{22} = \frac{r_3 r_4}{r_1^2}, \qquad P_{32} = \left(\frac{r_3}{r_1}\right)^2$$

$$P_{42} = \frac{r_3^2}{r_1 r_2}, \qquad P_{52} = \frac{r_3 r_4}{r_1 r_2}, \qquad P_{62} = \frac{r_3 r_4}{r_1 r_2}$$

$$P_{72} = \frac{r_3^2}{r_1 r_2}, \qquad P_{82} = \left(\frac{r_3}{r_2}\right)^2, \qquad P_{92} = \frac{r_3 r_4}{r_2^2}$$

$$P_{10,2} = \frac{r_3 r_4}{r_1 r_2}, \qquad P_{11,2} = \frac{r_3 r_4}{r_2^2}$$

$$P_{12,2} = \left(\frac{r_4}{r_2}\right)^2, \qquad P_{13,2} = \frac{r_3 r_4}{r_1 r_2},$$

$$P_{14,2} = \frac{r_4^2}{r_1 r_2}, \qquad P_{15,2} = \frac{r_3 r_4}{r_1 r_2}$$

Similarly for three non-touching loops, the number of combinations will be ${}^5C_3$. Now

$${}^5C_3 = \frac{5!}{3!\,2!} = 10$$

$$P_{13} = -\frac{r_3^2 r_4}{r_1^2 r_2}, \qquad P_{23} = -\frac{r_3^2 r_4}{r_1 r_2^2}$$

$$P_{33} = -\frac{r_3 r_4^2}{r_1 r_2^2}, \qquad P_{43} = -\frac{r_3^2 r_4}{r_1^2 r_2}$$

$$P_{53} = -\frac{r_3 r_4^2}{r_1^2 r_2}, \qquad P_{63} = -\frac{r_3^2 r_4}{r_1^2 r_2}$$

$$P_{73} = -\frac{r_3^2 r_4}{r_1 r_2^2}, \qquad P_{83} = -\frac{r_3 r_4^2}{r_1 r_2^2}$$

$$P_{93} = -\frac{r_3^2 r_4}{r_1 r_2^2}, \qquad P_{10,3} = -\frac{r_3 r_4^2}{r_1 r_2^2}$$

For four non-touching loops, the number of combinations will be ${}^4C_4 = 1$. That is,

$$P_{14} = P_{11} P_{31} P_{51} P_{71} = \left(\frac{r_3 r_4}{r_1 r_2}\right)^2$$

Therefore,

$$\Delta = 1 - \sum_{i=1}^{7} P_{i1} + \sum_{i=1}^{15} P_{i2} - \sum_{i=1}^{10} P_{i3} + P_{14}$$

As all the loops touch the forward path, $\Delta_1 = 1$
Therefore,

$$\frac{v_5}{v_1} = \frac{P_1\Delta_1}{\Delta} = \frac{\dfrac{(r_3r_4)^2}{(r_1r_2)^2}}{1+\dfrac{r_1r_3+r_1r_4+r_2r_3+r_2r_4+6r_3r_4+2r_3^2+r_4^2}{r_1r_2}+\dfrac{r_3r_4+r_3^2}{r_1^2}+\dfrac{r_3^2+r_4^2+r_3r_4}{r_2^2}}$$

$$= \frac{(r_3r_4)^2}{(r_1r_2)^2+r_1^2(r_3^2+r_4^2+r_3r_4+r_2r_3+r_2r_4)+r_2^2(r_3r_4+r_3^2+r_1r_3+r_1r_4)+6r_1r_2r_3r_4+2r_3^2r_1r_2+r_1r_2r_4^2}$$

4. Draw the signal flow graph of the electrical network shown in Fig. 1.54. Also calculate v_3/v_1.

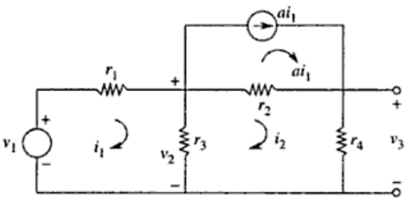


***Fig. 1.54** Electrical network.*

The equations of the circuit shown in Fig. 1.54 are:

$$\frac{v_1 - v_2}{r_1} = i_1$$

$$v_2 = (i_1 - i_2)r_3$$

$$\frac{v_2 - v_3}{r_2} + ai_1 = i_2 - i_1 + i_1$$

$$v_3 = i_2r_4$$

Utilizing the above equations, the signal flow graph is developed as in Fig. 1.55.

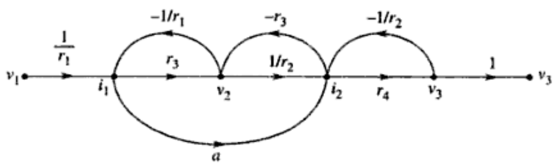


***Fig. 1.55** Signal flow graph of Fig. 1.54.*

Forward path

$$P_1 = \frac{1}{r_1} r_3 \frac{1}{r_2} r_4$$

$$\Delta_1 = 1$$

$$P_2 = \frac{1}{r_1} a \cdot r_4$$

$$\Delta_2 = 1$$

$$\Delta = 1 - \left(-\frac{r_3}{r_1} - \frac{r_3}{r_2} - \frac{r_4}{r_2} + a\frac{r_3}{r_1} \right) + \left(-\frac{r_3}{r_1} \cdot \frac{-r_4}{r_2} \right)$$

$$= 1 + \frac{r_3}{r_1} + \frac{r_3}{r_2} + \frac{r_4}{r_2} - a\frac{r_3}{r_1} + \frac{r_3 r_4}{r_1 r_2}$$

Therefore,

$$\frac{v_3}{v_1} = \frac{\dfrac{r_3 r_4}{r_1 r_2} + \dfrac{a r_4}{r_1}}{\dfrac{r_1 r_2 + r_3 r_2 + r_1 r_3 + r_1 r_4 - a r_3 r_2 + r_3 r_4}{r_1 r_2}}$$

$$= \frac{r_3 r_4 + a r_2 r_4}{r_1 r_2 + r_1 r_3 + r_1 r_4 + r_3 r_2 - a r_3 r_2 + r_3 r_4}$$

5. Figure 1.56 shows a schematic diagram of a liquid-level control system. The flow of liquid in the tank is controlled by the pressure P_o and valve opening V_x. Linearized liquid level is described by the following equation.

$$\Delta Q_i = K_1 \Delta P + K_2 \Delta V_x$$

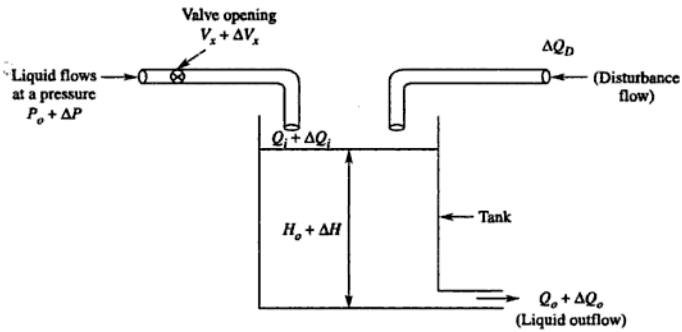


Fig. 1.56 Liquid-level control system.

Develop the block diagram and determine the transfer function $\left|\frac{\Delta Q_o(s)}{\Delta V_x(s)}\right|_{\Delta Q_D = 0}$ with pressure remaining constant. Consider that the tank and output pipe have liquid capacitance C and flow resistance R respectively.

The liquid-level control system can be described by the block diagram shown in Fig. 1.57.

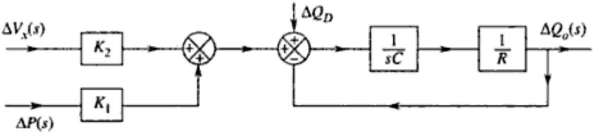


Fig. 1.57 Block diagram of Fig. 1.56.

We know that

$$\Delta Q_i - \Delta Q_o = C \cdot \frac{d}{dt}\Delta H$$

where C is the liquid capacitance. Now,

$$K_1\Delta P + K_2\Delta V_x = \Delta Q_i$$

$$\Delta Q_o = \frac{\Delta H}{R}$$

where R is the flow resistance. Therefore, taking the Laplace transforms

$$\Delta Q_i(s) - \Delta Q_o(s) = sC\Delta H(s)$$

$$\Delta Q_o(s) = \frac{\Delta H(s)}{R}$$

$$\Delta Q_i(s) = K_2\Delta V_x(s)$$

when the pressure remains constant. Therefore,

$$\frac{\Delta Q_i(s)}{\Delta V_x(s)} = K_2$$

or

$$\frac{\Delta Q_o(s) + sC\,\Delta H(s)}{\Delta V_x(s)} = K_2$$

or

$$\frac{\Delta Q_o(s) + sC\,\Delta Q_o(s)}{\Delta V_x(s)} = K_2$$

or

$$\frac{\Delta Q_o(s)}{\Delta V_x(s)}(1 + sCR) = K_2$$

or

$$\frac{\Delta Q_o(s)}{\Delta V_x(s)} = \frac{K_2}{1 + sCR}$$

Obviously the above relation is possible only when $\Delta Q_D = 0$. Therefore,

$$\left|\frac{\Delta Q_o(s)}{\Delta V_x(s)}\right|_{\Delta Q_D = 0} = \frac{K_2}{1 + sCR}$$

6. Find the outputs O_1 and O_2 of the signal flow graph shown in Fig. 1.58.

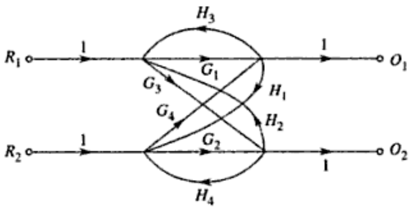


Fig. 1.58 Signal flow graph.

First of all, input R_1 is considered and outputs O_1 and O_2 are calculated separately. Then, input R_2 is considered and outputs O_1 and O_2 are calculated separately.

Then the results are superimposed to get the actual values of O_1 and O_2.

$$\Delta = 1 - (H_3G_1 + H_4G_2 + G_3H_2 + G_4H_1 + G_1H_1G_2H_2 + H_3G_3H_4G_4) + H_3G_1H_4G_2 + G_4H_1G_3H_2$$

when R_1 is the input and O_1 is the output. Then

$$P_1\Delta_1 = G_1(1 - G_2H_4)$$

$$P_2\Delta_2 = G_3H_4G_4 \cdot 1$$

when R_2 is the input and O_1 is the output. Then,

$$P_1\Delta_1 = G_4(1 - G_3H_2)$$

$$P_2\Delta_2 = G_2H_2G_1 \cdot 1.$$

Therefore, the output O_1 will be

$$= \frac{1}{\Delta}[\{G_1(1 - G_2H_4) + G_3H_4G_4\}R_1 + R_2\{G_4(1 - G_3H_2) + G_2H_2G_1\}]$$

where Δ has already been calculated.

Similarly, the output O_2 can be calculated.

7. Calculate the transfer function $X(s)/U(s)$ from the following equations and develop the block diagram.

$$x = x_1 + \alpha_3 u$$

$$\dot{x}_1 = -a_1 x_1 + x_2 + \alpha_2 u$$

$$\dot{x}_2 = -a_2 x_1 + \alpha_1 u$$

Taking the Laplace transforms of the given equations, we get (considering the initial value to be zero)

$$X(s) = X_1(s) + \alpha_3 U(s) \tag{1.10}$$

$$sX_1(s) = -a_1 X_1(s) + X_2(s) + \alpha_2 U(s) \tag{1.11}$$

$$sX_2(s) = -a_2 X_1(s) + \alpha_1 U(s) \tag{1.12}$$

Multiplying Eq. (1.11) by s

$$s^2 X_1(s) + a_1 s X_1(s) = sX_2(s) + s\alpha_2 U(s)$$

or

$$s^2 X_1(s) + a_1 s X_1(s) = -a_2 X_1(s) + \alpha_1 u(s) + s\alpha_2 U(s)$$

or

$$X_1(s)[s^2 + a_1 s + a_2] = \alpha_1 U(s) + \alpha_2 s U(s)$$

or

$$X_1(s) = \frac{\alpha_1 U(s) + \alpha_2 s U(s)}{s^2 + a_1 s + a_2} \tag{1.13}$$

Now,

$$X(s) = X_1(s) + \alpha_3 U(s)$$

Hence,

$$X(s) = \frac{\alpha_1 U(s) + \alpha_2 s U(s)}{s^2 + a_1 s + a_2} + \alpha_3 U(s)$$

or

$$\frac{X(s)}{U(s)} = \frac{\alpha_1 + \alpha_2 s}{s^2 + a_1 s + a_2} + \alpha_3$$

$$= \frac{\alpha_1 + \alpha_2 s + \alpha_3 (s^2 + a_1 s + a_2)}{s^2 + a_1 s + a_2}$$

Figure 1.59 shows the block diagram of the above example.

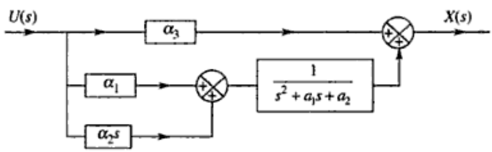


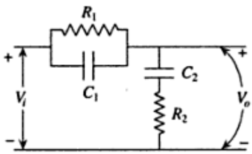
Fig. 1.59 Block diagram.

SUMMARY

The concept of control systems is introduced. The closed-loop and open-loop control systems are described with examples of position control, missile launching, automatic frequency control, microwave communication, and electromagnetic balance. The merits and demerits of open-loop and closed-loop control are discussed. Ideas of learning system and multivariable control are given. Models of control systems with the help of differential equations, block diagrams and signal flow graphs are explained. Methods of simplying block diagrams in the case of multiple input and multiple output systems are also explained. Idea is given of the sensitivity of control systems. Idea is also given of the improvement of system dynamics by feedback. On signal flow graphs, the Mason's gain formula is described. Methods of solving problems of block diagram and signal flow graph are explained with examples.

QUESTIONS

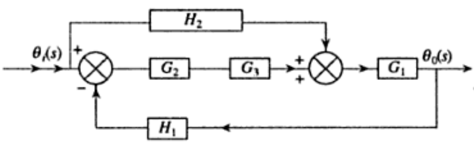
1. How do you differentiate between open-loop control and closed-loop control?
2. Describe an example of a closed-loop control system. What is a servomechanism?
3. What are the merits and demerits of the closed-loop and open-loop control systems?
4. What do you mean by a multivariable control system? Explain with an example.
5. How do you represent a control system by the block diagram? Explain with an example.
6. What do you mean by the sensitivity of the control system? How do you calculate the sensitivity of the closed-loop and open-loop control systems?
7. How do you improve the system dynamics by feedback? Explain with an example.
8. What is signal flow graph? What is its necessity in control systems?
9. Describe the Mason's gain formula with an example.
10. What do you understand by the transfer function of a system? State its properties. Find the transfer function of the lag-lead compensator network shown in the figure below.



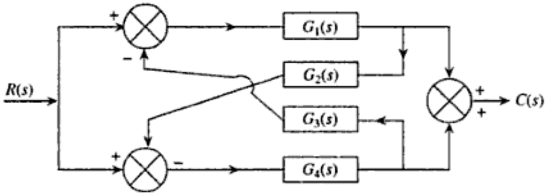
11. The transfer function of the block diagram shown below is

$$\frac{\theta_0(s)}{\theta_i(s)} = \frac{G_2 G_3 G_4 + G_4 H_2}{1 + H_1 G_2 G_3 G_4}$$

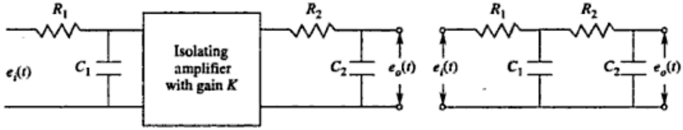
State whether it is 'correct' or 'incorrect'. Give a brief justification for your answer.



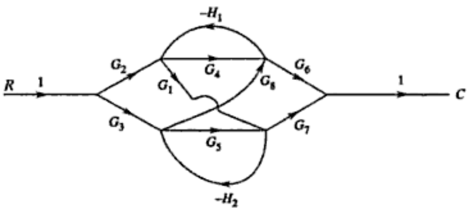
12. Determine the transfer function $C(s)/R(s)$ of the system shown in the following figure.



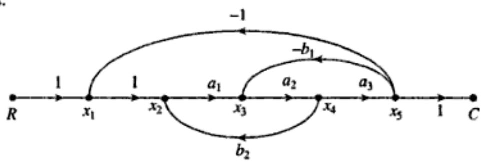
13. Determine the transfer function of the following two systems and explain why these transfer functions are different (not merely by the factor K but even otherwise).



14. Use the following signal flow graph to obtain the transfer function $C(s)/R(s)$.



15. The signal flow graph of a system is shown in the figure below. The internal nodes x_1, x_2, x_3, x_4, and x_5 represent variables of the system and branch gains are indicated next to the branches.



(a) By appropriate elimination of the nodes, determine the overall system gain C/R.

(b) State the Mason's gain formula and explain the terms used. Use this formula to calculate the overall gain C/R of the system shown in the above figure.

Chapter 2

Review of Some Mathematical Tools

2.1 THE LAPLACE TRANSFORM

Before going to detailed analysis of the control system, we will review here the Laplace transformation that we studied in mathematics. Laplace transformation is utilized to solve linear differential equations. In the Laplace transform method, the differential equation in the time domain is transformed into s-plane, where the solution needs some simple algebraic operations. The Laplace transform $F(s)$ of a function $f(t)$ is expressed as

$$F(s) = \mathcal{L}[f(t)] = \int_0^{\infty} f(t)\, e^{-st}\, dt$$

where s is a complex variable $\sigma + j\omega$ which makes $F(s)$ convergent. The inverse Laplace transform operation is expressed as

$$f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi} \int_{c-j\infty}^{c+j\infty} F(s) e^{st}\, ds$$

where c is a real constant and the same is greater than the real part of any of the singularities of $F(s)$.

2.1.1 Laplace Transforms of Some Important Functions

Unit-step function. Consider the unit-step function

$$f(t) = u(t)$$

with

$$u(t) = 0 \qquad \text{for } t < 0$$
$$= 1 \qquad \text{for } t \geq 0$$

The Laplace transform of the above function

$$F(s) = \text{Laplace transform of } f(t)$$

$$= \int_0^{\infty} u(t) e^{-st}\, dt = \int_0^{\infty} 1 e^{-st}\, dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_0^{\infty}$$

$$= -\frac{1}{s}(e^{-\infty \cdot s} - e^0) = -\frac{1}{s}(0-1) = \frac{1}{s}$$

Exponential function. Consider the exponential function

$$f(t) = e^{at}$$

The Laplace transform

$$F(s) = \int_0^{\infty} e^{at} e^{-st}\, dt = \int_0^{\infty} e^{(a-s)t}\, dt$$

$$= \left[\frac{e^{(a-s)t}}{a-s}\right]_0^{\infty} = -\frac{1}{a-s} = \frac{1}{s-a}$$

Sinusoidal functions. Consider the sinusoidal function

$$f(t) = \sin \alpha t$$

Now,

$$\sin \alpha t = \frac{1}{2j}(e^{j\alpha t} - e^{-j\alpha t})$$

Therefore,

$$F(s) = \int_0^{\infty} \sin \alpha t\, e^{-st}\, dt = \int_0^{\infty} \frac{1}{2j}(e^{j\alpha t} - e^{-j\alpha t})\, dt$$

$$= \frac{1}{2j}\left[\frac{1}{s-j\alpha} - \frac{1}{s+j\alpha}\right] = \frac{\alpha}{s^2+\alpha^2}$$

Ramp function. Consider the ramp function

$$f(t) = t \qquad \text{for } t > 0$$
$$= 0 \qquad \text{for } t < 0$$

The Laplace transform

$$F(s) = \int_0^{\infty} t\, e^{-st}\, dt$$

$$= \left[t\int e^{-st}\, dt - \int\left(\int e^{-st}\, dt\right) dt\right]_0^{\infty} \qquad \text{(integration by parts)}$$

$$= \left[\frac{te^{-st}}{-s}\right]_0^{\infty} - \int_0^{\infty}\frac{e^{-st}}{-s}dt$$

$$= (-0+0) - \left[\frac{e^{-st}}{s^2}\right]_0^{\infty} = -\frac{1}{s^2}(0-1) = \frac{1}{s^2}$$

Translated function. If $f_1(t) = f(t - t_0)$ where $t > t_0$

$= 0$ where $t < t_0$

the Laplace transform will be

$$= \int_0^{\infty} f_1(t)\, e^{-st}\, dt$$

$$= \int_{t_0}^{\infty} f(t - t_0)\, e^{-s(t - t_0)}\, e^{-st_0}\, dt$$

Putting $t - t_0 = \tau$, we have

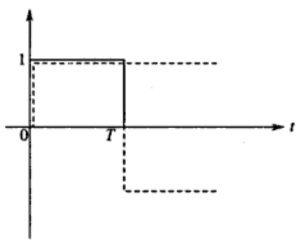
$$\frac{dt}{d\tau} = 1 \qquad \text{or} \qquad dt = d\tau$$

Therefore,

$$F_1(s) = e^{-st_0}\int_{t_0}^{\infty} f(\tau)e^{-s\tau}d\tau = e^{-st_0}F(s)$$

where $F(s)$ is the Laplace transform of $f(t)$.

Unit pulse function. Consider the pulse function of unity height and width T, denoted by $f(t, T)$ and as shown in the figure below:



The Laplace transform of $f(t, T)$ is found by superposition of two step functions as shown in the preceding figure. Thus,

$$\mathcal{L}f(t, T) = \mathcal{L}[u(t)] - \mathcal{L}[u(t-T)]$$

$$= \frac{1}{s} - \frac{e^{-sT}}{s} = \frac{1}{s}(1 - e^{-sT})$$

Unit impulse function. The unit impulse function is the pulse of height $\frac{1}{T}$ and width T starting at $t = 0$ and considering T approaching zero. That is,

$$\delta(t) = \lim_{T \to 0} \frac{1}{T}[u(t) - u(t-T)]$$

Therefore,

$$\mathcal{L}[\delta(t)] = \lim_{T \to 0} \frac{1}{T} \frac{(1 - e^{-sT})}{s}$$

$$= \lim_{T \to 0} \frac{se^{-sT}}{s} \qquad \text{(By L'Hospital's rule)}$$

$$= 1$$

Some Useful Laplace Transforms

$f(t)$	$F(s)$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
e^{-at}	$\frac{1}{s+a}$
e^{at}	$\frac{1}{s-a}$
$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$
$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$
$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2 + \omega^2}$
$e^{-at} \cos \omega t$	$\frac{s+a}{(s+a)^2 + \omega^2}$

Contd.

Some Useful Laplace Transforms *(Contd.)*

$f(t)$	$F(s)$
$\dfrac{t^n}{n!}$	$\dfrac{1}{s^{n+1}}$
$\dfrac{t^{n-1}}{(n-1)!}$	$\dfrac{1}{s^n}$
$\cosh at$	$\dfrac{s}{s^2 - a^2}$
$\sinh at$	$\dfrac{a}{s^2 - a^2}$
$\dfrac{e^{at} t^{n-1}}{(n-1)!}$	$\dfrac{1}{(s-a)^n}$
$e^{at} \sin bt$	$\dfrac{b}{(s-a)^2 + b^2}$
$e^{at} \cos bt$	$\dfrac{s-a}{(s-a)^2 + b^2}$
$t \sin at$	$\dfrac{2as}{(s^2 + a^2)^2}$
$t \cos at$	$\dfrac{s^2 - a^2}{(s^2 + a^2)^2}$
$\dfrac{1}{2a^3}(\sin at - at \cos at)$	$\dfrac{1}{(s^2 + a^2)^2}$

EXAMPLE 2.1 Solve the following differential equation

$$4\frac{dy}{dt} + 3y = 10, \quad y(0) = 1$$

Solution Taking the Laplace transform of the differential equation, we have

$$\mathcal{L}\left[4\frac{dy}{dt}\right] + \mathcal{L}\,[3y] = \mathcal{L}\,[10]$$

or

$$4[sY(s) - y(0)] + 3Y(s) = \frac{10}{s}$$

or

$$(4s + 3)Y(s) = \frac{10}{s} + 4 \qquad [\because y(0) = 1]$$

or

$$Y(s) = \frac{10}{s(4s+3)} + \frac{4}{4s+3}$$

$$= \frac{10}{4s\left(s+\frac{3}{4}\right)} + \frac{4}{4\left(s+\frac{3}{4}\right)}$$

Taking the Laplace inverse of both the sides, we get

$$\mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1}\left[\frac{10}{4\left(s+\frac{3}{4}\right)s}\right] + \mathcal{L}^{-1}\left[\frac{1}{\left(s+\frac{3}{4}\right)}\right]$$

Now $\mathcal{L}^{-1}\left[\frac{10}{4}\frac{1}{\left(s+\frac{3}{4}\right)s}\right]$ can be evaluated as follows:

$$\frac{10}{4}\frac{1}{\left(s+\frac{3}{4}\right)s} = \frac{10}{4}\left[\frac{A}{s+\frac{3}{4}} + \frac{B}{s}\right]$$

Thus,

$$As + B\left(s+\frac{3}{4}\right) = 1$$

Then,

$$A + B = 0 \qquad \text{and} \qquad B\frac{3}{4} = 1$$

or

$$B = \frac{4}{3},\ A = -\frac{4}{3}$$

Hence,

$$\mathcal{L}^{-1}\left[\frac{10}{4}\frac{1}{s\left(s+\frac{3}{4}\right)}\right] = \mathcal{L}^{-1}\frac{10}{4}\left(\frac{-\frac{4}{3}}{s+\frac{3}{4}} + \frac{\frac{4}{3}}{s}\right)$$

$$= -\frac{10}{4}\cdot\frac{4}{3}\cdot e^{-(3/4)t} + \frac{10}{4}\cdot\frac{4}{3} = -\frac{10}{3}e^{-(3/4)t} + \frac{10}{3}$$

Therefore,

$$\mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1}\left[\frac{10}{4\left(s+\frac{3}{4}\right)s}\right] + \mathcal{L}^{-1}\left[\frac{1}{s+\frac{3}{4}}\right]$$

$$= -\frac{10}{3}e^{-(3/4)t} + \frac{10}{3} + e^{-(3/4)t}$$

Hence,

$$y(t) = -\frac{7}{3}e^{-(3/4)t} + \frac{10}{3}$$

EXAMPLE 2.2 Find the Laplace inverse of

$$F(s) = \frac{s+6}{s^2+2s+10}$$

Solution

$$f(t) = \mathcal{L}^{-1}\left[\frac{s+1+5}{(s+1)^2+3^2}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{s+1}{(s+1)^2+3^2}\right] + \mathcal{L}^{-1}\left[\frac{5}{(s+1)^2+3^2}\right]$$

$$= e^{-t}\cos 3t + \frac{5}{3}\mathcal{L}^{-1}\left[\frac{3}{(s+1)^2+3^2}\right]$$

$$= e^{-t}\cos 3t + \frac{5}{3}e^{-t}\sin 3t$$

2.1.2 Properties of Laplace Transform

(1) **Prove that the Laplace transform of** $tf(t)$, that is,

$$\mathcal{L}[tf(t)] = -\frac{d}{ds}F(s)$$

where $F(s) = \mathcal{L}[f(t)]$.

Proof

$$\frac{d}{ds}F(s) = \frac{d}{ds}\int_0^\infty f(t)e^{-st}\,dt = \int_0^\infty f(t)\frac{d}{ds}(e^{-st})\,dt$$

$$= -\int_0^\infty tf(t)e^{-st}\,dt$$

or

$$-\frac{d}{ds}F(s) = \int_0^\infty tf(t)e^{-st}\,dt = \mathcal{L}\,[tf(t)]$$

Hence it is proved that the Laplace transform of $tf(t)$ is $-\frac{d}{ds}F(s)$. From the above, the general property is described as

$$\mathcal{L}\,[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}F(s)$$

From the above relation, we can now get the following results:

$$\text{Laplace transform of } tu(t) = \frac{1}{s^2}$$

$$\text{Laplace transform of } te^{-at} = \frac{1}{(s+a)^2}$$

(2) **Prove that the Laplace transform of** $e^{at}f(t)$, that is,

$$\mathcal{L}\,[e^{at} f(t)] = F(s-a)$$

where $F(s) = \mathcal{L}\,[f(t)]$.

Proof

$$\mathcal{L}\,[e^{at}f(t)] = \int_0^\infty e^{at} f(t)e^{-st}\,dt = \int_0^\infty f(t)e^{-(s-a)t}\,dt$$
$$= F(s-a)$$

(3) **Convolution property**

The Laplace transform of

$$\int_0^\infty f_1(t-\tau)\,f_2(\tau)\,d\tau = f_1(t)*f_2(t) = F_1(s)F_2(s)$$

where $F_1(s) = \mathcal{L}[f_1(t)]$ and $F_2(s) = \mathcal{L}[f_2(t)]$.

(4) **Initial-value theorem**

If $f(t)$ and $f'(t)$ have Laplace transforms and $\lim_{s\to\infty} sF(s)$ exists, then

$$\lim_{s\to\infty} sF(s) = \lim_{t\to 0^+} f(t)$$

EXAMPLE 2.3 Detemine the initial value of $f(t)$ from the corresponding $F(s)$ which is as follows:

$$F(s) = \frac{3s+2}{s(s^2+4s+5)}$$

Solution Applying the initial-value theorem, we get

$$f(0^+) = \lim_{t \to 0^+} f(t)$$

$$= \lim_{s \to \infty} sF(s)$$

$$= \lim_{s \to \infty} s \cdot \frac{3s+2}{s(s^2+4s+5)} = \lim_{s \to \infty} \frac{3s+2}{s^2+4s+5}$$

$$= \lim_{s \to \infty} \frac{\frac{3}{s}+\frac{2}{s^2}}{1+\frac{4}{s}+\frac{5}{s^2}} = 0$$

EXAMPLE 2.4 Calculate $f'(0^+)$ and $f''(0^+)$ for the function $F(s) = \dfrac{4s+1}{s(s^2+4s+5)}$.

Solution Here,

$$\mathcal{L}[f'(t)] = sF(s) - f(0^+) = s\frac{4s+1}{s(s^2+4s+5)} - 0 \qquad (\because f(0^+) = 0 \text{ from the initial-value theorem})$$

Again as per the initial-value theorem,

$$f'(0^+) = \lim_{t \to 0^+} f'(t) = \lim_{s \to \infty} s\mathcal{L}[f'(t)] = s^2 \cdot \frac{4s+1}{s(s^2+4s+5)} = \lim_{s \to \infty} \frac{4s^2+s}{s^2+4s+5}$$

$$= \lim_{s \to \infty} \frac{4+\frac{5}{s^2}}{1+\frac{4}{s}+\frac{5}{s^2}} = 4$$

Similarly,

$$\mathcal{L}[f''(t)] = s^2F(s) - sf(0^+) - f'(0^+)$$

$$= \frac{s^2(4s+1)}{s(s^2+4s+5)} - 0 - 4 = \frac{4s^2+s}{s^2+4s+5} - 4$$

$$= \frac{4s^2 + s - 4s^2 - 16s - 20}{s^2 + 4s + 5} = \frac{-15s - 20}{s^2 + 4s + 5}$$

$$= \frac{-(15s + 20)}{s^2 + 4s + 5}$$

Hence,

$$f''(0^+) = \lim_{t \to 0^+} f''(t) = \lim_{s \to \infty} s\mathcal{L}[f''(t)]$$

$$= \lim_{s \to \infty} s\left[\frac{-(15s + 20)}{s^2 + 4s + 5}\right] = \lim_{s \to \infty} \frac{-15s^2 - 20s}{s^2 + 4s + 5}$$

$$= \lim_{s \to \infty} \frac{-15 - \frac{20}{s}}{1 + \frac{4}{s} + \frac{5}{s^2}} = -15$$

(5) **Final-value theorem**

If $f(t)$ and $f'(t)$ have Laplace transforms and if all the poles of $sF(s)$ lie inside the left-half s plane, then

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$$

The need for restriction on $sF(s)$ that all its poles lie inside the left-half plane confirms that $f'(t)$ will approach zero asymptotically as t approaches infinity. The reason of this restriction is for the proof of the final-value theorem.

EXAMPLE 2.5 The Laplace transform of a function is $\frac{8s + 5}{s(s + 1)(s^2 + 4s + 5)}$. Determine its final value.

Solution Since the poles of $sF(s)$ $\left(= s\frac{8s + 5}{s(s + 1)(s^2 + 4s + 5)}\right)$ lie in the left-half plane, the final value of the function will be

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sf(s)$$

$$= \lim_{s \to 0} \frac{8s + 5}{(s + 1)(s^2 + 4s + 5)} = \frac{5}{5} = 1$$

EXAMPLE 2.6 Calculate the final value of $f(t)$ if $F(s) = \dfrac{5}{s(s^2+49)}$.

Solution Since

$$sF(s) = \frac{5}{s^2+49}$$

it has poles on the imaginary axis. Hence $f(t)$ has no final value.

(6) **Method of finding the Laplace inverse of** $F(s)$ that has a large number of same poles. That is

$$F(s) = \frac{A(s)}{(s+s_1)^N (s+s_2)\ldots(s+s_n)}$$

$F(s)$ can be expanded by the method of partial fractions. Thus,

$$F(s) = \frac{A_1}{s+s_1} + \frac{A_2}{(s+s_1)^2} + \ldots + \frac{A_N}{(s+s_1)^N} + \sum_{j=2}^{n} \frac{K_j}{s+s_j}$$

The procedure for determination of coefficients is:

$$K_j = (s+s_j)\, F(s)\big|_{s\to -s_j}$$

$$A_N = (s+s_1)^N F(s)\big|_{s\to -s_1}$$

$$A_{N-1} = \frac{d}{ds}(s+s_1)^N F(s)\Big|_{s\to -s_1}$$

$$A_1 = \frac{1}{(N-1)!}\frac{d^{N-1}}{ds^{N-1}}(s+s_1)^N F(s)\Big|_{s\to -s_1}$$

EXAMPLE 2.7 Calculate the Laplace inverse of $F(s) = \dfrac{2}{s^2(s+1)}$.

$$F(s) = \frac{A_1}{s} + \frac{A_2}{s^2} + \frac{K_1}{s+1}$$

Now,

$$K_1 = (s+1)F(s)\big|_{s\to -1} = \lim_{s\to -1}\frac{2}{s^2} = 2$$

$$A_2 = (s^2)F(s)\big|_{s\to 0} = \lim_{s\to 0}\frac{2}{s+1} = 2$$

$$A_1 = \frac{d}{ds}s^2F(s)\Big|_{s\to 0} = \lim_{s\to 0}\frac{d}{ds}\left(\frac{2}{s+1}\right)$$

$$= \lim_{s \to 0} [-2(s+1)^{-2}] = -2$$

Hence,

$$f(t) = -2 + 2t + 2e^{-t}$$

(7) **Real integration theorem**

If $f(t)$ is of exponential order, then the Laplace transform of $\int f(t)dt$ exists and is given by

$$\mathcal{L}\left[\int f(t)\,dt\right] = \frac{F(s)}{s} + \frac{f^{-1}(0)}{s}$$

where $F(s) = \mathcal{L}[f(t)]$ and $f^{-1}(0) = \int f(t)dt$, being evaluated at $t = 0$.

Proof

$$\left[\int f(t)dt\right] = \int_0^{\infty} \left[\int f(t)dt\right] e^{-st}\,dt$$

$$= \left[\int f(t)dt\right] \frac{e^{-st}}{-s}\Bigg|_0^{\infty} - \int_0^{\infty} f(t)\frac{e^{-st}}{-s}\,dt$$

$$= \frac{1}{s}\int f(t)dt\Bigg|_{t=0} + \frac{1}{s}\int_0^{\infty} f(t)e^{-st}\,dt$$

$$= \frac{f^{-1}(0)}{s} + \frac{F(s)}{s}$$

Hence the theorem is proved.

2.2 THE $\mathcal{Z}$-TRANSFORM

The $\mathcal{Z}$-transform is a mathematical tool used for dealing with the discrete-time control systems. In the case of discrete-time systems, continuous time t is usually replaced by the discrete time n. The $\mathcal{Z}$-transform of the same is described as

$$X(z) = \mathcal{Z}[f(n)] = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

Since the $\mathcal{Z}$-transform is an infinite power series, it exists for those values of z for which the series converges. The *region of convergence* is termed ROC. It is the set of all values of z for which $X(z)$ attains a finite value.

The $\mathcal{Z}$-transform of the finite expression $x(n) = [1, 2, 5, 7, 0, 1]$ is

$$X(z) = 1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + z^{-5}$$

The region of convergence will be on entire values of the z-plane except at $z = 0$.

EXAMPLE 2.8 Find the $\mathcal{Z}$-transform of the signal

$$x(n) = \left(\frac{1}{2}\right)^n u(n)$$

where $u(n)$ is the unit-step input.

Solution Since $u(n)$ is the unit-step input, its value is zero for $n < 0$. Hence the $\mathcal{Z}$-transform of $x(n)$ will be

$$X(z) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n u(n) z^{-n} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2} z^{-1}\right)^n$$

$$= 1 + \frac{1}{2} z^{-1} + \left(\frac{1}{2} z^{-1}\right)^2 + \ldots$$

$$= \lim_{n \to \infty} \left[\frac{1 - \left(\frac{1}{2} z^{-1}\right)^n}{1 - \frac{1}{2} z^{-1}} \right]$$

If $\left(\frac{1}{2} z^{-1}\right)$ is less than one, i.e. $\left|\frac{1}{2} z^{-1}\right| < 1$, then $\left(\frac{1}{2} z^{-1}\right)^n$ will be zero as $n \to \infty$.

Therefore,

$$X(z) = \frac{1}{1 - \frac{1}{2} z^{-1}}$$

where the region of convergence is $\left|\frac{1}{2} z^{-1}\right| < 1$ or $|z| > \frac{1}{2}$.

EXAMPLE 2.9 Find the $\mathcal{Z}$-transform of $x(n) = a^n u(n) + b^n u(-n-1)$, where $|b| > |a|$.

Solution Here,

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n} + \sum_{n=-\infty}^{\infty} b^n u(-n-1) z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n u(n) z^{-n} + \sum_{n=-\infty}^{-1} b^n u(-n-1) z^{-n} \qquad (\because u(n) = 0 \text{ when } n < 0)$$

$$= \sum_{n=0}^{\infty} a^n z^{-n} + \sum_{n=-\infty}^{-1} b^n z^{-n}$$

Now,

$$\sum_{n=-\infty}^{-1} b^n z^{-n} = \sum_{m=\infty}^{1} b^{-m} z^{m} \qquad \text{(Let } n = -m\text{)}$$

$$= \sum_{m=1}^{m=\infty} b^{-m} z^{m}$$

Since m is an arbitrary number, $\sum_{n=-\infty}^{-1} b^n z^{-n} = \sum_{n=1}^{n=\infty} b^{-n} z^{+n}$

Therefore,

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} + \sum_{n=1}^{\infty} b^{-n} z^{n}$$

$$= \sum_{n=0}^{\infty} a^n z^{-n} + \sum_{n=1}^{\infty} (b^{-1}z)^{n}$$

Now,

$$\sum_{n=0}^{\infty} a^n z^{-n} = \frac{1}{1 - az^{-1}}$$

where region of convergence (ROC) is $|az^{-1}| < 1$ or $|z| > |a|$.
Again,

$$\sum_{n=1}^{\infty} (b^{-1}z)^n = b^{-1}z + (b^{-1}z)^2 + \ldots$$

$$= \frac{(b^{-1}z)}{1 - b^{-1}z}$$

where $|b^{-1}z| < 1$ or $|z| < |b|$.

Hence the solution will be

$$\frac{1}{1 - az^{-1}} + \frac{b^{-1}z}{1 - b^{-1}z}$$

$$= \frac{1 - b^{-1}z + b^{-1}z(1 - az^{-1})}{(1 - az^{-1})(1 - b^{-1}z)}$$

$$= \frac{1 - b^{-1}z + b^{-1}z - ab^{-1}}{1 - b^{-1}z - az^{-1} + ab^{-1}}$$

$$= \frac{1 - \dfrac{a}{b}}{1 - \dfrac{z}{b} - \dfrac{a}{z} + \dfrac{a}{b}}$$

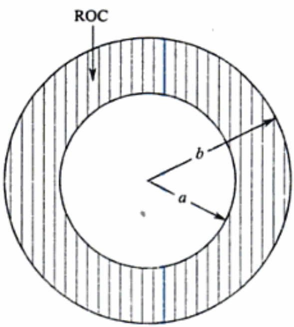


Fig. 2.1 Region of convergence (ROC)

$$= \frac{\frac{b-a}{b}}{\frac{b - z - abz^{-1} + a}{b}}$$

$$= \frac{b-a}{a + b - z - abz^{-1}}$$

As per Fig. 2.1, the shaded portion is the "region of convegence" The region of covergence will be $|a| < |z| < |b|$.

2.2.1 Inverse of $\mathcal{Z}$-transform

The inverse of the $\mathcal{Z}$-transform is the procedure for transforming from the z-domain to the time-domain.

EXAMPLE 2.10 Find the inverse $\mathcal{Z}$-transform of

$$X(z) = \frac{1}{1 - az^{-1}} \qquad |z| > |a|$$

Solution As per Cauchy residue theorem, if $f(z)$ be a function of the complex variable z and c be a closed path in the z-plane and the derivative $\frac{df(z)}{dz}$ exists on and inside the contour c and if $f(z)$ has no poles at $z = z_0$, then

$$\frac{1}{2\pi j}\oint_c \frac{f(z)}{z - z_0}\,dz = \begin{Bmatrix} f(z_0) & \text{if } z_0 \text{ is inside } c \\ 0 & \text{if } z_0 \text{ is outside } c \end{Bmatrix}$$

Again,

$$X(z) = \sum_{k=-\infty}^{\infty} x(k)\, z^{-k}$$

Now, integrating both sides by multiplying with z^{n-1}, we get

$$\oint_c X(z) z^{n-1} dz = \oint_c \sum_{k=-\infty}^{\infty} x(k) z^{n-1-k} dz$$

where c denotes the closed contour in the ROC of $X(z)$, being taken in the counterclockwise direction. Now,

$$\oint_c X(z)\, z^{n-1} dz = \sum_{k=-\infty}^{\infty} x(k) \oint_c z^{n-1-k} dz$$

As per Cauchy's integral theorem,

$$\frac{1}{2\pi j}\oint_c z^{n-1-k} dz = \begin{cases} 1, \text{ for } k = n \\ 0, \text{ for } k \neq n \end{cases}$$

Hence,

$$\frac{1}{2\pi j}\oint_c X(z) z^{n-1}\, dz = \sum_{k=-\infty}^{\infty} x(k) \frac{1}{2\pi j}\oint_c z^{n-1-k}\, dz$$

$$= x(n) \cdot 1 + 0 = x(n)$$

Therefore,

$$x(n) = \frac{1}{2\pi j}\oint_c \frac{z^{n-1}}{1}\cdot\frac{1}{1-az^{-1}}dz$$

as $X(z) = \dfrac{1}{1-az^{-1}}$ at $|z| > |a|$.

Thus,

$$x(n) = \frac{1}{2\pi j}\cdot\int\frac{z^n}{z-a}dz$$

when $n \geq 0$ and $f(z)$ has only zeros and no poles inside c. The only pole inside c will be the a.

Hence, as per Cauchy's residue theorem stated at the beginning of this example, $x(n) = f(z_0) = (a)^n$ when $n \geq 0$. When $n < 0$, $f(z) = z^n$ has nth order pole at $z = 0$, which is also inside the contour c.

When $n = -1$

$$x(-1) = \frac{1}{2\pi j}\cdot\oint_c\frac{1}{z(z-a)}dz.$$

$$= \left.\frac{1}{z-a}\right|_{z=0} + \left.\frac{1}{z}\right|_{z=a} = -\frac{1}{a}+\frac{1}{a} = 0$$

When $n = -2$

$$x(-2) = \frac{1}{2\pi j}\int\frac{1}{z^2(z-a)}dz$$

$$= \left.\frac{1}{z^2}\right|_{z=a} + \left|\frac{d}{dz}\left(\frac{1}{z-a}\right)\right|_{z=0} = \frac{1}{a^2} - (z-a)^{-2}\Big|_{z=0}$$

$$= \frac{1}{a^2} - (-a)^{-2} = \frac{1}{a^2} - \frac{1}{(-a)^2} = \frac{1}{a^2} - \frac{1}{a^2} = 0$$

Similarly, we will get for all other values of $n < 0$, $x(n) = 0$.

Hence it can be concluded that $x(n) = a^n u(n)$, whose inverse $\mathcal{Z}$-transform of $X(z) = \dfrac{1}{1-az^{-1}}$ where $|z| > a$.

An inverse of the $\mathcal{Z}$-transform can be obtained by two other methods:

1. Power series expansion
2. Partial fraction expansion

Inversion by power series expansion

EXAMPLE 2.11 Given the $\mathcal{Z}$-transform

$$X(z) = \frac{2}{2-3z^{-1}+z^{-2}}$$

where ROC is $|z| > 1$, determine by the power series expansion its inverse when ROC is $|z| > 1$ and when ROC is $|z| < \frac{1}{2}$.

Solution When the region of convergence is the exterior of a circle, i.e. $|z| > 1$), we have to develop power series expansion in negative powers of z. Thus,

$$X(z) = \frac{2}{2 - 3z^{-1} + z^{-2}}$$

$$= \frac{1}{1 - \frac{3}{2}z^{-1} + \frac{1}{z}z^{-2}}$$

Now,

$$\begin{array}{r|l}
 & 1 + \frac{3}{2}z^{-1} + \frac{7}{4}z^{-2} + \frac{15}{8}z^{-3} + \frac{31}{16}z^{-4} + \ldots \\
\hline
1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2} \rightarrow & 1 \\
 & 1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2} \\
\cline{2-2}
 & \frac{3}{2}z^{-1} - \frac{1}{2}z^{-2} \\
 & \frac{3}{2}z^{-1} - \frac{9}{4}z^{-2} + \frac{3}{4}z^{-3} \\
\cline{2-2}
 & \frac{7}{4}z^{-2} - \frac{3}{4}z^{-3} \\
 & \frac{7}{4}z^{-2} - \frac{21}{8}z^{-3} + \frac{7}{8}z^{-4} \\
\cline{2-2}
 & \frac{15}{8}z^{-3} - \frac{7}{8}z^{-4} \\
 & \frac{15}{8}z^{-3} - \frac{45}{16}z^{-4} + \frac{15}{16}z^{-5} \\
\cline{2-2}
 & \frac{31}{16}z^{-4} - \frac{15}{16}z^{-5} \\
 & \frac{31}{16}z^{-4} - \frac{93}{32}z^{-5} + \frac{31}{32}z^{-6} \\
\cline{2-2}
 & \frac{63}{32}z^{-5} - \frac{31}{32}z^{-6}
\end{array}$$

Hence $x(n)$ will be

$$1, \frac{3}{2}, \frac{7}{4}, \frac{15}{8}, \frac{31}{16}, \ldots$$

When the ROC is $|z| < 1/2$, it means that the ROC is the interior of a circle. The power series expansion will be in positive powers of z. Thus,

$$\begin{array}{r|l}
 & 2z^2 + 6z^3 + 14z^4 + 30z^5 + 62z^6 + \ldots \\
\hline
z^{-2} - 3z^{-1} + 2 & 2 \\
 & 2 - 6z + 4z^2 \\
\cline{2-2}
 & \quad 6z - 4z^2 \\
 & \quad 6z - 18z^2 + 12z^3 \\
\cline{2-2}
 & \qquad 14z^2 - 12z^3 \\
 & \qquad 14z^2 - 42z^3 + 28z^4 \\
\cline{2-2}
 & \qquad\quad 30z^3 - 28z^4 \\
 & \qquad\quad 30z^3 - 90z^4 + 60z^5 \\
\cline{2-2}
 & \qquad\quad 62z^4 - 60z^5 \\
 & \qquad\quad 62z^4 - 186z^5 + 124z^6 \\
\cline{2-2}
 & \qquad\qquad 126z^5 - 124z^6
\end{array}$$

Thus,

$$X(z) = 2z^2 + 6z^3 + 14z^4 + 30z^5 + 62z^6 + \ldots$$

Hence,

$$x(n) = \begin{bmatrix} \ldots 62, 30, 14, 6, 2, 0, 0 \\ \qquad\qquad\qquad\qquad\quad \uparrow \end{bmatrix}$$

Inversion by partial fraction method

EXAMPLE 2.12 Given the $\mathcal{Z}$-transform

$$X(z) = \frac{1}{(1 - z^{-1})^2 (1 + z^{-1})}$$

determine by the partial fraction method its inverse transform.

Solution

$$X(z) = \frac{z^3}{(z-1)^2 (z+1)}$$

or

$$\frac{X(z)}{z} = \frac{z^2}{(z-1)^2 (z+1)}$$

Now,

$$\frac{X(z)}{z} = \frac{A}{z+1} + \frac{B}{(z-1)} + \frac{C}{(z-1)^2}$$

$$A = \lim_{z \to -1} \frac{z^2}{(z-1)^2} = \frac{1}{4}$$

$$C = \lim_{z \to 1} \frac{z^2}{z+1} = \frac{1}{2}$$

$$B = \lim_{z \to 1} \frac{d}{dz}\left(\frac{z^2}{z+1}\right)$$

or

$$B = \lim_{z \to 1} \frac{2z(z+1) - z^2}{(z+1)^2}$$

$$= \frac{4-1}{4} = \frac{3}{4}$$

Therefore,

$$\frac{X(z)}{z} = \frac{\frac{1}{4}}{z+1} + \frac{\frac{3}{4}}{z-1} + \frac{\frac{1}{2}}{(z-1)^2}$$

or

$$X(z) = \frac{\frac{1}{4}z}{z+1} + \frac{\frac{3}{4}z}{z-1} + \frac{\frac{1}{2}z}{(z-1)^2}$$

$$= \frac{\frac{1}{4}}{1+z^{-1}} + \frac{\frac{3}{4}}{1-z^{-1}} + \frac{\frac{1}{2}z}{(z-1)^2}$$

Now,

$\dfrac{\frac{1}{4}}{1+z^{-1}}$ is the $\mathcal{Z}$-transform of $\dfrac{1}{4}(-1)^n u(n)$.

$\dfrac{\frac{3}{4}}{1-z^{-1}}$ is the $\mathcal{Z}$-transform of $\dfrac{3}{4}u(n)$.

We know that the $\mathcal{Z}$-transform of $u(n)$ is $\dfrac{1}{1-z^{-1}}$.

That means

$$X(z) = \frac{1}{1 - z^{-1}}$$

where $x(n) = u(n)$. Now,

$$\frac{dX(z)}{dz} = -z^{-2}(1 - z^{-1})^{-2}$$

or

$$z\frac{dX(z)}{dz} = -z^{-1}(1 - z^{-1})^{-2}$$

or

$$-z\frac{dX(z)}{dz} = \frac{z^{-1}}{(1 - z^{-1})^2} = \frac{z}{(z - 1)^2}$$

Again,

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

Therefore,

$$\frac{dX(z)}{dz} = \sum_{n=-\infty}^{\infty} x(n)[-nz^{-n-1}]$$

$$= -z^{-1}\ [\mathcal{Z}\text{–transform of } [nx(n)]]$$

or

$$-z\frac{dX(z)}{dz} = \mathcal{Z}\text{-transform of } nx(n)$$

Hence, the $\mathcal{Z}$-transform of $nu(n)$ is

$$-z\frac{dX(z)}{dz} = \frac{z}{(z - 1)^2}$$

when $x(n) = u(n)$.

Therefore, the $x(n)$ of the whole problem is

$$\frac{1}{4}(-1)^n u(n) + \frac{3}{4}u(n) + \frac{1}{2}nu(n)$$

2.2.2 Properties of $\mathcal{Z}$-Transform

Linearity

If the $\mathcal{Z}$-transform of $x_1(n)$ and $x_2(n)$ are $X_1(z)$ and $X_2(z)$ respectively, then the $\mathcal{Z}$-transform of $a_1x_1(n) + a_2x_2(n)$ will be $a_1X_1(z) + a_2X_2(z)$. This is the linearity property of $\mathcal{Z}$-transform.

Time-shifting

If the $\mathcal{Z}$-transform of $x(n)$ is $X(z)$, then the $\mathcal{Z}$-transform of $x(n-k)$ is $z^{-k}X(z)$. The ROC of $z^{-k}X(z)$ will be the same as $X(z)$, except for $z = 0$ if $k > 0$ and $z = \infty$ if $k < 0$.

Suppose the $\mathcal{Z}$-transform of $x(n)$ is

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

then the $\mathcal{Z}$-transform of $x(n-k)$ is

$$\sum_{n=-\infty}^{\infty} x(n-k)z^{-n}$$

Let $n - k = m$. Now, when $n = \infty$, $m = \infty$, when $n = -\infty$, $m = -\infty$.

Thus the $\mathcal{Z}$-transform of $x(n-k)$ becomes

$$\sum_{m=-\infty}^{\infty} x(m)z^{-(m+k)} \quad \text{or} \quad z^{-k}\sum_{m=-\infty}^{\infty} x(m)z^{-m} \quad \text{or} \quad z^{-k}X(z)$$

Scaling of the z-domain

If the $\mathcal{Z}$-transform of $x(n)$ is $X(z)$ at the ROC, $r_1 < |z| < r_2$, then $\mathcal{Z}$-transform of $a^n x(n)$ is

$$\sum_{n=-\infty}^{\infty} a^n x(n)z^{-n} \quad \text{or} \quad \sum_{n=-\infty}^{\infty} x(n)(a^{-1}z)^{-n} \quad \text{or} \quad X(a^{-1}z)$$

The ROC of $X(z)$ is $r_1 < |z| < r_2$, the ROC of $X(a^{-1}z)$ will be

$$r_1 < |a^{-1}z| < r_2 \quad \text{or} \quad |a|r_1 < |z| < |a|r_2$$

This property is termed the *scaling*, where a may be any constant real or complex.

Time reversal

If the $\mathcal{Z}$-transform of $x(n)$ is $X(z)$, having ROC, $r_1 < |z| < r_2$, then the $\mathcal{Z}$-transform of $x(-n)$ is $X(z^{-1})$ with the region of convergence $\frac{1}{r_2} < |z| < \frac{1}{r_1}$. This is termed the *time reversal property.*

Now,

$$\mathcal{Z}[x(-n)] = \sum_{n=-\infty}^{\infty} x(-n)z^{-n}$$

Let $m = -n$, then

$$\mathcal{Z}[x(-n)] = \sum_{m=-\infty}^{\infty} x(m)z^{m} = \sum_{m=-\infty}^{\infty} x(m)(z^{-1})^{-m}$$

$$= X(z^{-1})$$

The ROC of $X(z^{-1})$ is

$$r_1 < |z^{-1}z| < r_2 \quad \text{or} \quad r_1 < \left|\frac{1}{z}\right| < r_2 \quad \text{or} \quad \frac{1}{r_1} > |z| > \frac{1}{r_2}$$

Convolution of two sequences

If the $\mathcal{Z}$-transform of $x_1(n)$ is $X_1(z)$ and the $\mathcal{Z}$-transform of $x_2(n)$ is $X_2(z)$, then the $\mathcal{Z}$-transform of $x_1(n) * x_2(n)$ is $X(z)$, that is, equal to $X_1(z)\, X_2(z)$. Here $x_1(n) * x_2(n)$ means the convolution of the two sequences.

$$x(n) = x_1(n) * x_2(n) = \sum_{k=-\infty}^{\infty} x_1(k)x_2(n-k)$$

Hence,

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

$$= \sum_{n=-\infty}^{\infty}\left[\sum_{k=-\infty}^{\infty} x_1(k)x_2(n-k)\right]z^{-n}$$

$$= \sum_{k=-\infty}^{\infty} x_1(k)\left[\sum_{n=-\infty}^{\infty} x_2(n-k)z^{-n}\right]$$

Now,

$$\sum_{n=-\infty}^{\infty} x_2(n-k)z^{-n} = \sum_{m=-\infty}^{\infty} x_2(m)z^{-(m+k)} = z^{-k}X_2(z)$$

(By putting $n - k = m$, when $n = \infty$, $m = \infty$, and when $n = -\infty$, $m = -\infty$)

Therefore,

$$X(z) = \sum_{k=-\infty}^{\infty} x_1(k)z^{-k}X_2(z) = X_2(z)\, X_1(z)$$

Multiplication of two systems

If the $\mathcal{Z}$-transforms of $x_1(n)$ and $x_2(n)$ are $X_1(z)$ and $X_2(z)$ respectively, then the $\mathcal{Z}$-transform of $x_1(n)x_2(n)$ is

$$\frac{1}{2\pi j}\oint_c X_1(v)X_2\left(\frac{z}{v}\right)v^{-1}dv$$

where c is a closed contour which encloses the origin and lies within the common region of convergence of $X_1(v)$ and $X_2(1/v)$.

The inverse $\mathcal{Z}$-transform says

$$x_1(n) = \frac{1}{2\pi j}\oint_c X_1(v)v^{n-1}\,dv$$

The $\mathcal{Z}$-transform of $x(n)$, that is, $x_1(n)x_2(n)$ is

$$= \sum_{n=-\infty}^{\infty} \frac{1}{2\pi j} \oint_c X_1(v) v^{n-2} dv \cdot X_2(n) z^{-n}$$

$$= \frac{1}{2\pi j} \oint X_1(v) \left[\sum_{n=-\infty}^{\infty} x_2(n) \left(\frac{Z}{v} \right)^{-n} \right] v^{-1} dv$$

Therefore,

$$X(z) = \frac{1}{2\pi j} \oint X_1(v) X_2\left(\frac{z}{v} \right) v^{-1} dv$$

For obtaining the ROC of $X(z)$, it is understood that if $X_1(v)$ converges for $r_{1L} < |v| < r_{1U}$ and $X_2(z)$ converges for $r_{2L} < |z| < r_{2U}$, then the ROC of $X_2(z/v)$ is

$$r_{2L} < \left| \frac{z}{v} \right| < r_{2U}$$

Therefore, the ROC for $X(z)$ will be

$$r_{1L} r_{2L} < |z| < r_{1U} r_{2U}$$

Parseval's relation

If $x_1(n)$ and $x_2(n)$ are complex-valued sequences, then

$$\sum_{n=-\infty}^{n=\infty} x_1(n) x_2^*(n) = \frac{1}{2\pi j} \oint_c X_1(v) X_2^*\left(\frac{1}{v^*} \right) v^{-1} dv$$

If $r_{1L}\, r_{2L} < 1 < r_{1U}\, r_{2U}$ where $r_{1L} < |z| < r_{1U}$ and $r_{2L}\, |z| < r_{2U}$ are the regions of convergence of $X_1(z)$ and $X_2(z)$, we know that

$$X(z) = \frac{1}{2\pi j} \oint X_1(v) X_2\left(\frac{z}{v} \right) v^{-1} dv$$

If $x_1(n)$ and $x_2(n)$ are complex-valued sequences, then the $\mathcal{Z}$-transform of $x_1(n)\, x_2^*(n)$ is

$$X(z) = \frac{1}{2\pi j} \oint X_1(v) X_2^*\left(\frac{z^*}{v^*} \right) v^{-1} dv$$

When $z = 1$,

$$X(z) = \frac{1}{2\pi j} \oint X_1(v) X_2^*\left(\frac{1}{v^*} \right) v^{-1} dv$$

Initial-value theorem

If $x(n)$ is causal (i.e. $x(n) = 0$ for $n < 0$), then

$$x(0) = \lim_{z \to \infty} X(z)$$

$$X(z) = \sum_{n=0}^{\infty} x(n)z^{-n}$$

$$= x(0) + x(1)z^{-1} + x(2)z^{-2} + \ldots$$

Hence, for $z \to \infty$, $z^{-n} \to 0$.

One-sided $\mathcal{Z}$-transform

The one-sided or unilateral $\mathcal{Z}$-transform of a signal $x(n)$ is defined as follows:

$$X^{+}(z) = \sum_{n=0}^{\infty} x(n)z^{-n}$$

The results of the different properties of the $\mathcal{Z}$-transform will be changed as follows by the one-sided $\mathcal{Z}$-transform.

Shifting property. The $\mathcal{Z}$-transform of $x(n-k)$ is

$$X(z) = \sum_{n=0}^{\infty} x(n-k)z^{-n}$$

Let, $n - k = m$. Then,

$$X(z) = \sum_{m=-k}^{\infty} x(m)z^{-(m+k)} = \sum_{m=-k}^{\infty} x(m)z^{-m} \cdot z^{-k}$$

$$= \sum_{m=0}^{\infty} x(m)z^{-m}z^{-k} + \sum_{m=-k}^{-1} x(m)z^{-m}z^{-k}$$

$$= z^{-k}X^{+}(z) + \sum_{m=-k}^{-1} x(m)z^{-m}z^{-k}$$

Let, $m = -l$. Then,

$$X(z) = z^{-k}X^{+}(z) + \sum_{l=1}^{k} x(-l)z^{l}z^{-k}$$

$$= z^{-k}X^{+}(z) + z^{-k}\left[\sum_{l=1}^{k} x(-l)z^{l}\right]$$

Hence the general result will be

$$X(z) = z^{-k}\left[X^{+}(z) + \sum_{n=1}^{k} x(-n)z^{n}\right] \qquad \text{(replacing } l \text{ by } n\text{)}$$

Final-value theorem. If the one-sided $\mathcal{Z}$-transform of $x(n)$ is $X^+(z)$, then

$$\lim_{n\to\infty} x(n) = \lim_{z\to 1}(z-1)X^+(z)$$

The above limit exists, if the ROC of $(z-1)X^+(z)$ includes the unit circle. Here is the proof of the final-value theorem.

Proof

One-sided $\mathcal{Z}$-transform of $[x(n+1) - x(n)] = \lim_{n\to\infty}\sum_{n=0}^{n}[x(n+1)-x(n)]z^{-n}$

We know that $x(n+1)$ has the following $\mathcal{Z}$-transform:

$$\sum_{n=0}^{\infty} x(n+1)z^{-n}$$

Putting $n + 1 = m$, the above $\mathcal{Z}$-transform reduces to

$$\sum_{m=1}^{\infty} x(m)z^{-(m-1)} = \sum_{m=1}^{\infty} x(m)z^{-m}z$$

$$= \sum_{m=0}^{\infty} x(m)z^{-m}z - x(0)z = z\cdot X^+(z) - zx(0)$$

Therefore, the $\mathcal{Z}$-transform of $[x(n+1) - x(n)]$ is

$$zX^+(z) - zx(0) - X^+(z)$$

One-sided $\mathcal{Z}$-transform of $[x(n+1) - x(n)]$ is

$$zX^+(z) - zx(0) - X^+(z) = (z-1)X^+(z) - zx(0)$$

Hence,

$$(z-1)X^+(z) - zx(0) = \lim_{n\to\infty}\sum_{n=0}^{n}[x(n+1)-x(n)]z^{-n}$$

or

$$\lim_{z\to 1}(z-1)X^+(z) - \lim_{z\to 1} zx(0) = \lim_{z\to 1}\lim_{n\to\infty}\sum_{n=0}^{n}[x(n+1)-x(n)]z^{-n}$$

or

$$\lim_{z\to 1}(z-1)X^+(z) = x(0) + \lim_{n\to\infty}\left[\sum_{n=0}^{n} x(n+1)-x(n)\right]$$

or

$$\lim_{z\to 1}(z-1)X^+(z) = x(0)+x(1)+x(2)+x(3)+\ldots+x(\infty+1)-x(0)-x(1)-x(2)-x(3)-\cdots-x(\infty)$$

or

$$\lim_{z\to 1}(z-1)X^+(z) = x(\infty+1) = x(\infty)$$

$$= \lim_{n\to\infty} x(n)$$

2.3 CORRELATION OF TWO SEQUENCES

Correlation closely resembles convolution. The correlation measures the degree to which the two signals are similar. Suppose, we have two sequences $x(n)$ and $y(n)$. Each of them has finite energy. The cross-correlation of $x(n)$ and $y(n)$ is a sequence $r_{xy}(l)$, which is defined as

$$r_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n)\, y(n-l) \tag{2.1}$$

$$l = 0, \pm 1, \pm 2 \ldots$$

or

$$r_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n+l)\, y(n) \tag{2.2}$$

$$l = 0, \pm 1, \pm 2 \ldots$$

If the role of $x(n)$ and $y(n)$ is reversed, then

$$r_{yx}(l) = \sum_{n=-\infty}^{\infty} y(n)\, x(n-l) \tag{2.3}$$

or

$$r_{yx}(l) = \sum_{n=-\infty}^{\infty} y(n+l)\, x(n) \tag{2.4}$$

If (2.1) is compared with (2.4), and (2.2) is compared with (2.3), it can be concluded that

$$r_{xy}(l) = r_{yx}(-l)$$

Hence, $r_{yx}(l)$ is the folded version of $r_{xy}(l)$, where folding is done with respect to $l = 0$. Therefore, $r_{yx}(l)$ provides exactly the same information as $r_{xy}(l)$ about the similarity of $x(n)$ to $y(n)$.

The $\mathcal{Z}$-transform of $r_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n)y(n-l)$, will be $X(z)\, Y(z^{-1})$.

The proof of the above statement is as follows.

$$r_{xy}(l) = x(l) * y(-l)$$

Applying the convolution and time reversal properties, the above $\mathcal{Z}$-transform will be $X(z)Y(z^{-1})$. Where $y(n) = x(n)$, the above correlation is termed the *auto-correlation*. Thus,

$$r_{xx}(l) = \sum_{n=-\infty}^{\infty} x(n)x(n-l)$$

or

$$r_{xx}(l) = \sum_{n=-\infty}^{\infty} x(n+l)\,x(n)$$

Hence the $\mathcal{Z}$-transform of the *auto-correlation sequence* will be $X(z)X(z^{-1})$.

2.4 EIGENVALUES AND EIGENVECTORS

We know that a system H is linear if and only if $H[a_1x_1(n) + a_2x_2(n)] = a_1H[x_1(n)] + a_2H[x_2(n)]$ for arbitrary input sequences $x_1(n)$ and $x_2(n)$ and any arbitrary constants a_1 and a_2. The system H will be time-invariant when it is excited by an input signal $x(n)$ and that will produce an output signal $y(n)$ (Fig. 2.2). Thus,

$$y(n) = H[x(n)]$$

If an output signal is delayed by k units of time to yield $x(n-k)$, and again applied to the system, the characteristics of the system do not change with time, the output will be $y(n-k)$. This is the time-invariant system.

The input-output equation for the system is $y(n) = H[x(n)] = nx(n)$. The response of the system to $x(n)$ is, $y(n, k) = nx(n-k)$.

If we delay $y(n)$ by k units of time, we obtain

$$y(n-k) = (n-k)x(n-k)$$
$$= nx(n-k) - kx(n-k)$$

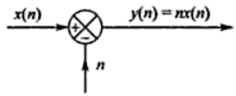


Fig. 2.2

Hence,

$$y(n, k) \neq y(n-k)$$

Therefore the system is time-variant.

Suppose a linear time-invariant system is described by the following differential equation

$$\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + \ldots + a_1\frac{dy(t)}{dt} + a_0y(t) = b_m\frac{d^m u(t)}{dt^m} + b_{m-1}\frac{d^{m-1}u(t)}{dt^{m-1}} + \ldots + b_1\frac{du(t)}{dt} + b_0u(t)$$

where $n > m$.

Utilizing the operator $S\left(S^k = \frac{d^k}{dt^k},\ k = 1, 2, \ldots, n\right)$, we get

$$(S^n + a_{n-1}S^{n-1} + \ldots + a_1S + a_0)\,y(S) = (b_mS^m + b_{m-1}S^{m-1} + \ldots + b_1S + b_0)\,u(S)$$

where the relation $S^n + a_{n-1}S^{n-1} + \ldots + a_1S + a_0 = 0$, is termed the *characteristic equation* of the system. If the transfer function of the system is described by

$$G(S) = \frac{b_m S^m + b_{m-1} S^{m-1} + \ldots + b_1 S + b_0}{S^n + a_{n-1} S^{n-1} + \ldots + a_1 S + a_0}$$

the characteristic equation is obtained by equating the denominator polynomial of the transfer function to zero.

Suppose $A = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$ is a matrix, and the transfer function of the system is

$$C[SI - A]^{-1}B + D$$

where C, B, D are the other matrices. Then the transfer function will be

$$C \cdot \frac{\text{Adjoint}\,(SI - A)}{|SI - A|} B + D$$

$$= \frac{C \cdot \{\text{Adjoint of}(SI - A)]B + |SI - A|D}{|SI - A|}$$

Therefore the characteristic equation is $|SI - A| = 0$. The roots of the characteristic equation are termed the *eigenvalues* of the matrix A, usually denoted by the symbol λ. Therefore,

$$|\lambda I - A| = 0$$

or

$$\left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} \right| = 0$$

or

$$\begin{vmatrix} \lambda - 1 & 1 \\ 0 & \lambda + 1 \end{vmatrix} = 0$$

or

$$(\lambda + 1)(\lambda - 1) = 0$$

or

$$\lambda = 1 \text{ or } -1$$

The eigenvector is defined as

$$[\lambda I - A]p = 0$$

where p is termed the eigenvector. When $\lambda = 1$ in the above example,

$$\lambda I - A = 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-1 & 0+1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$$

Let

$$p = \begin{bmatrix} p_{11} \\ p_{21} \end{bmatrix}$$

Therefore, the characteristic equation $[\lambda I - A]p = 0$ for $\lambda = 1$ becomes

$$\begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} p_{11} \\ p_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or

$$p_{21} = 0$$

$$2p_{21} = 0$$

Therefore, $p_{21} = 0$, and p_{11} may be any arbitrary number which may be assumed equal to 1.

Similarly for $\lambda = -1$, the equation $[\lambda I - A]p = 0$ becomes

$$\left\{-1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}\right\} \begin{bmatrix} p_{12} \\ p_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or

$$\begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p_{12} \\ p_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or

$$-2p_{12} + p_{22} = 0$$

In the above case, we have to assume arbitrarily that one value, say $p_{12} = 1$, then $p_{22} = 2$. Therefore, the two eigenvectors are

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Thus, from the above example, it is clear that any nonzero vector p_i that satisfies the matrix equation $(\lambda_i I - A)p_i = 0$ is termed the eigenvector where λ_i are the eigenvalues of A.

Suppose the matrix A is $\begin{bmatrix} 0 & 6 & -5 \\ 1 & 0 & 2 \\ 3 & 2 & 4 \end{bmatrix}$, then its eigenvalues can be found out from

$$|\lambda I - A| = 0$$

or

$$\left|\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 6 & -5 \\ 1 & 0 & 2 \\ 3 & 2 & 4 \end{bmatrix}\right| = 0$$

or

$$\begin{vmatrix} \lambda & -6 & 5 \\ -1 & \lambda & -2 \\ -3 & -2 & (\lambda - 4) \end{vmatrix} = 0$$

or

$$\lambda[\lambda(\lambda - 4) - 4] + 6(-\lambda + 4 - 6) + 5(2 + 3\lambda) = 0$$

or

$$\lambda^3 - 4\lambda^2 + 5\lambda - 2 = 0$$

Factorizing, we get

$$(\lambda - 2)(\lambda - 1)(\lambda - 1) = 0$$

For $\lambda = 2$, the eigenvector will be determined as follows:

$$\left\{2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 6 & -5 \\ 1 & 0 & 2 \\ 3 & 2 & 4 \end{bmatrix}\right\} \begin{bmatrix} p_{11} \\ p_{21} \\ p_{31} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or

$$\left\{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 6 & -5 \\ 1 & 0 & 2 \\ 3 & 2 & 4 \end{bmatrix}\right\} \begin{bmatrix} p_{11} \\ p_{21} \\ p_{31} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or

$$\begin{bmatrix} 2 & -6 & 5 \\ -1 & 2 & -2 \\ -3 & -2 & -2 \end{bmatrix} \begin{bmatrix} p_{11} \\ p_{21} \\ p_{31} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or

$$2p_{11} - 6p_{21} + 5p_{31} = 0 \tag{2.5}$$

$$-p_{11} + 2p_{21} - 2p_{31} = 0 \tag{2.6}$$

$$-3p_{11} - 2p_{21} - 2p_{31} = 0 \tag{2.7}$$

Let us take $p_{11} = 2$, as an arbitrary number. Therefore, subtracting (2.7) from (2.6), we get

$$2p_{11} + 4p_{21} = 0 \quad \text{or} \quad p_{11} + 2p_{21} = 0$$

or

$$2 = -2p_{21} \quad \text{or} \quad p_{21} = -1$$

Substituting for p_{21} in Eq. (2.5)

$$4 + 6 = -5p_{31} \quad \text{or} \quad p_{31} = -2$$

Hence,

$$\begin{bmatrix} p_{11} \\ p_{21} \\ p_{31} \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$$

For $\lambda = 1$, the eigenvector is assumed $\begin{bmatrix} p_{12} \\ p_{22} \\ p_{32} \end{bmatrix}$, then

$$\left\{1\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 6 & -5 \\ 1 & 0 & 2 \\ 3 & 2 & 4 \end{bmatrix}\right\}\begin{bmatrix} p_{12} \\ p_{22} \\ p_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or

$$\begin{bmatrix} 1 & -6 & 5 \\ -1 & 1 & -2 \\ -3 & -2 & -3 \end{bmatrix}\begin{bmatrix} p_{12} \\ p_{22} \\ p_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or

$$p_{12} - 6p_{22} + 5p_{32} = 0 \tag{2.8}$$

$$- p_{12} + p_{22} - 2p_{32} = 0 \tag{2.9}$$

$$-3p_{12} - 2p_{22} - 3p_{32} = 0 \tag{2.10}$$

Let p_{12} be taken arbitrarily as 1, multiplying (2.9) by 3 and (2.10) by 2, we get

$$- 3p_{12} + 3p_{22} - 6p_{32} = 0 \tag{2.11}$$

$$- 6p_{12} - 4p_{22} - 6p_{32} = 0 \tag{2.12}$$

Subtracting (2.12) from (2.11),

$$3p_{12} + 7p_{22} = 0$$

or

$$p_{22} = \frac{-3}{7}p_{12} = -\frac{3}{7}$$

Putting $p_{12} = 1$, $p_{22} = -(3/7)$ in Eq. (2.8), we get,

$$1 + \frac{18}{7} + 5p_{32} = 0 \qquad \text{or} \qquad p_{32} = -\frac{5}{7}$$

Therefore,

$$\begin{bmatrix} p_{12} \\ p_{22} \\ p_{32} \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{3}{7} \\ -\frac{5}{7} \end{bmatrix}$$

For the next eigenvalue $\lambda = 1$, the eigenvector $\begin{bmatrix} p_{13} \\ p_{23} \\ p_{33} \end{bmatrix}$ will be determined as follows:

$$\left\{1\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 6 & -5 \\ 1 & 0 & 2 \\ 3 & 2 & 4 \end{bmatrix}\right\}\begin{bmatrix} p_{13} \\ p_{23} \\ p_{33} \end{bmatrix} = \begin{bmatrix} -p_{12} \\ -p_{22} \\ -p_{32} \end{bmatrix}$$

or

$$\begin{bmatrix} 1 & -6 & 5 \\ -1 & 1 & -2 \\ -3 & -2 & -3 \end{bmatrix}\begin{bmatrix} p_{13} \\ p_{23} \\ p_{33} \end{bmatrix} = \begin{bmatrix} -1 \\ \frac{3}{7} \\ \frac{5}{7} \end{bmatrix}$$

or

$$p_{13} - 6p_{23} + 5p_{33} = -1 \tag{2.13}$$

$$-p_{13} + p_{23} - 2p_{33} = \frac{3}{7} \tag{2.14}$$

$$-3p_{13} - 2p_{23} - 3p_{33} = \frac{5}{7} \tag{2.15}$$

Multiplying (2.14) by 3 and (2.15) by 2, we get

$$-3p_{13} + 3p_{23} - 6p_{33} = \frac{9}{7} \tag{2.16}$$

$$-6p_{13} - 4p_{23} - 6p_{33} = \frac{10}{7} \tag{2.17}$$

Subtracting (2.17) from (2.16), we get

$$3p_{13} + 7p_{23} = \frac{9}{7} - \frac{10}{7} = \frac{-1}{7}$$

Taking $p_{13} = 1$ arbitrarily

$$3 + 7p_{23} = -\frac{1}{7} \quad \text{or} \quad 7p_{23} = \frac{-22}{7} \quad \text{or} \quad p_{23} = \frac{-22}{49}$$

Putting the values of p_{13} and p_{23} in Eq. (2.13), we get

$$1 + \frac{132}{49} + 1 = -5p_{33} \quad \text{or} \quad p_{33} = \frac{-46}{49}$$

Therefore,

$$\begin{bmatrix} p_{13} \\ p_{23} \\ p_{33} \end{bmatrix} = \begin{bmatrix} 1 \\ \dfrac{-22}{49} \\ \dfrac{-46}{49} \end{bmatrix}$$

Note: When the eigenvalues are of multiple-order type, then

$$(\lambda_i I - A)p_{n-q+1} = 0$$

$$(\lambda_i I - A)p_{n-q+2} = -p_{n-q+1}$$

where

n = number of eigenvalues

q = number of distinct eigenvalues.

For example, when the eigenvalues are 2, 1, 1, the number of eigenvalues will be 3 and the number of distinct eigenvalues will be 2.

That is why, for the second and third eigenvalues of 1, the equation of eigenvector stands as

$$(\lambda_2 I - A)\, p_{3-2+1} = 0 \quad \text{where } \lambda_2 = 1$$

$$(\lambda_3 I - 1)\, p_{3-2+2} = -p_{3-2+1} = -p_2$$

or

$$(\lambda_3 I - A)p_3 = -p_2 \quad \text{where } \lambda_3 = 1$$

2.5 HERMITIAN FORM

If X is a complex n-vector and P is a Hermitian matrix, then the complex quadratic form is called the *Hermitian form*. For example, consider the quadratic form

$$V(x) = 10x_1^2 + 4x_2^2 + x_3^2 + 2x_1x_2 - 2x_2x_3 - 4x_1x_3$$

which can be written as

$$V(x) = X^T P X = [x_1 \quad x_2 \quad x_3]\begin{bmatrix} 10 & 1 & -2 \\ 1 & 4 & -1 \\ -2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

where P is the Hermitian matrix. It is written with the help of following relation:

$$X^T P X = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j$$

$$[a_{ij} = a_{ji}]$$

Say, $n = 3$. Then,

$$X^TPX = \sum_{i=1}^{3} \sum_{j=1}^{3} a_{ij}x_ix_j$$

$$= \sum_{i=1}^{3} [a_{i1}x_ix_1 + a_{i2}x_ix_2 + a_{i3}x_ix_3]$$

$$= a_{11}x_1x_1 + a_{12}x_1x_2 + a_{13}x_1x_3 + a_{21}x_2x_1 + a_{22}x_2x_2 + a_{23}x_2x_3 + a_{31}x_3x_1 + a_{32}x_3x_2 + a_{33}x_3x_3$$

$$= a_{11}x_{12} + (a_{12} + a_{21})x_1x_2 + (a_{13} + a_{31})x_3x_1 + a_{22}x_2^2 + (a_{23} + a_{32})x_2x_3 + a_{33}x_3^2$$

$$= a_{11}x_1^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + a_{22}x_2^2 + 2a_{23}x_2x_3 + a_{33}x_3^2 \quad (\because\ a_{ij} = a_{ji})$$

Hence,

$$a_{11} = 10$$

$$a_{12} = a_{21} = 1$$

$$a_{13} = a_{31} = -2$$

$$a_{23} = a_{32} = -1$$

$$a_{22} = 4$$

$$a_{33} = 1$$

Therefore,

$$P = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 10 & 1 & -2 \\ 1 & 4 & -1 \\ -2 & -1 & 1 \end{bmatrix}$$

Again, if it is to be shown that $V(x)$ is positive definite, then all the minors of the matrix P are positive. This is also called the Sylvester's criterion. That means, in the above example:

$$10 > 0 \qquad \begin{vmatrix} 10 & 1 \\ 1 & 4 \end{vmatrix} > 0 \qquad \begin{vmatrix} 10 & 1 & -2 \\ 1 & 4 & -1 \\ -2 & -1 & 1 \end{vmatrix} > 0$$

The rules for positive and negative definiteness are as follows:

(a) If $V = x_1^2 + x_2^2$, then it is positive definite in a two-dimensional state space of x_1 and x_2.

(b) If $V = -\ a_1x_1^2 - a_2x_2^2 - a_3x_3^2$, then it is negative definite in a three-dimensional state space of x_1, x_2 and x_3.

(c) If $V = a_1(x_1 + 4)^2 + a_2(x_2 - 3)^2$ with $a_1 > 0$, $a_2 > 0$, then it is not positive definite in a two-dimensional state space.

(d) $V = x_1^2 + x_2^2$ is not positive definite in a three-dimensional state space of x_1, x_2 and x_3.

(e) If $V = (x_1 + x_2)^2$, then it is not positive definite in a two-dimensional state space because V has zero value at all points in V which statisfies $x_1 = -\,x_2$.

That means V has zero value not only at the origin but also at other points. That is why, it is termed positive semi-definite. Similarly, $x_1^2 + x_2^2$ is positive semi-definite in a three-dimensional state space. This is because it is not only zero at $x_1 = x_2 = x_3 = 0$ but also at all other points on the x_3 axis.

2.6 CALCULUS OF VARIATIONS

The calculus of variations is a powerful technique for solving the problems of control systems. The subject primarily concerns with finding maximum or minimum value of a definite integral involving a certain function. It is something beyond finding stationary values of a given function. An integral which assumes a definite value for functions is termed *functional*. For example, $\int_{x_1}^{x_2} f(x, y, y')dx$ is called functional. The necessary condition for $I = \int_{x_1}^{x_2} f(x, y, y')dx$ to be an extremum is that

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left\{\frac{\partial f}{\partial y'}\right\} = 0$$

where $y' = \dfrac{dy}{dx}$.

This condition is called the Euler's equation. We give below the proof of it.

Suppose $y = y(x)$ be the curve joining the points $A(x_1, y_1)$, $B(x_2, y_2)$ which makes an extremum. Let

$$y = y(x) + \varepsilon\eta(x) \qquad (2.18)$$

be a neighbouring curve joining these points so that,

$$\text{at A, } \eta(x_1) = 0 \text{ and at B, } \eta(x_2) = 0 \qquad (2.19)$$

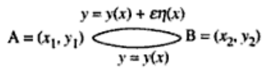


Fig. 2.3

The value of I along the curve (2.19) is

$$I = \int_{x_1}^{x_2} f[x, y(x) + \varepsilon\eta(x), y'(x) + \varepsilon\eta'(x)]\, dx \qquad (2.20)$$

This being a function of ε, is a maximum or minimum for $\varepsilon = 0$, when

$$\frac{dI}{d\varepsilon} = 0 \text{ at } \varepsilon = 0 \qquad (2.21)$$

Differentiating under the integral sign by thes Leibnitz's rule, if $f(x, \alpha)$ and $\dfrac{\partial f(x, a)}{\partial x}$ be continuous functions of x and α, then

$$\frac{d}{dx}\left[\int_a^b f(x, \alpha)\, dx\right] = \int_a^b \frac{\partial f(x, \alpha)}{\partial \alpha} dx$$

where a, b are constants independent of α. Therefore,

$$\frac{dI}{d\varepsilon} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial \varepsilon} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial \varepsilon} + \frac{\partial f}{\partial y'} \cdot \frac{\partial y'}{\partial \varepsilon} \right) dx \tag{2.22}$$

Since ε is independent of x, $\frac{dx}{d\varepsilon} = 0$.

Again from (2.20), $\frac{\partial y}{\partial \varepsilon} = \eta(x)$, $\frac{\partial y'}{\partial \varepsilon} = \eta'(x)$.

Substituting these values in (2.22), we get

$$\frac{dI}{d\varepsilon} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} \eta(x) + \frac{\partial f}{\partial y'} \eta'(x) \right] dx$$

Integrating the second term on the right by parts, we have

$$\frac{dI}{d\varepsilon} = \int_{x_1}^{x_2} \frac{\partial f}{\partial y} \eta(x) dx + \left[\frac{\partial f}{\partial y'} \eta(x) \Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) \eta(x) dx \right] \qquad \left(\because \eta'(x) = \frac{d\eta(x)}{dx} \right)$$

$$= \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) \right] \eta(x)\, dx \qquad \because \frac{dI}{d\varepsilon} = 0, \text{ for any value of } \eta(x), \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) \right] = 0$$

The above is called the Euler's equation.

Taylor's Series

If $f(Z)$ is analytic inside a circle C with centre at a, then for Z inside C,

$$f(Z) = f(a) + f'(a)(Z - a) + \frac{f''(a)}{2!}(Z - a)^2 + \ldots + \frac{f^n(a)}{n!}(Z - a)^n + \ldots$$

SUMMARY

The Laplace transform used for the solution of linear ordinary differential equations is described in detail with different examples of transformations. The $\mathcal{Z}$-transform used for the study of discrete control systems is also described in detail with different examples of $\mathcal{Z}$-transformations. The region of convergence of $\mathcal{Z}$-transforms is explained as well. The inverse Laplace and $\mathcal{Z}$-ttransforms are also discussed with examples. The properties of $\mathcal{Z}$-transform are covered too. The idea of one-sided $\mathcal{Z}$-transform is also given. The final-value theorem is also explained. In addition, the correlation of two sequences is covered. Eigenvalues and eigenvectors are presented in detail. The Hermitian form is explained with an example. The idea of calculus of variations is given as well. The Euler's equation and Taylor's series are reviewed.

QUESTIONS

1. Find the Laplace transforms of the following functions.

(a) $t \sin^2 t$ $\left[Ans. \; \dfrac{2(3s^2+4)}{s^2(s^2+4)^2}\right]$

(b) $\dfrac{\cos 2t - \cos 3t}{t}$ $\left[Ans. \; \dfrac{1}{2}\log\left(\dfrac{s^2+9}{s^2+4}\right)\right]$

(c) $\dfrac{1-e^t}{t}$ $\left[Ans. \; \log\left(\dfrac{s-1}{s}\right)\right]$

(d) $t \sinh at$ $\left[Ans. \; \dfrac{2as}{(s^2-a^2)^2}\right]$

2. Find the inverse Laplace transforms of the following functions.

(a) $\dfrac{s}{s^4+4a^4}$ $\left[Ans. \; \dfrac{1}{2a^2}\sin at \sinh at\right]$

(b) $\dfrac{s}{s^4+s^2+1}$ $\left[Ans. \; \dfrac{2}{\sqrt{3}}\sin\left(\dfrac{1}{2}t\right)\sin\left(\dfrac{1}{2}\sqrt{3}t\right)\right]$

(c) $\dfrac{1}{s(s^2+4)}$ $\left[Ans. \; \dfrac{1}{4}(1-\cos 2t)\right]$

(d) $\dfrac{s}{(s+1)^2(s^2+1)}$ $\left[Ans. \; \dfrac{1}{2}(\sin t - te^{-t})\right]$

3. Find the $\mathcal{Z}$-transforms of the following sequences.

(a) $\dfrac{a^n}{n!}e^{-a}$ $\left[Ans. \; e^{a(z^{-1}-1)}\right]$

(b) $(n-1)^2$ $\left[Ans. \; \dfrac{z^3-3z^2+4z}{(z-1)^3}\right]$

(c) $\sin (n+1)\theta$ $\left[Ans. \; \dfrac{z^2 \sin\theta}{z^2-2z\cos\theta+1}\right]$

(d) $(\cos\theta + i \sin\theta)^n$ $\left[Ans. \; \dfrac{z}{(z-e^{i\theta})}\right]$

4. Evaluate the inverse $\mathcal{Z}$-transforms of following functions.

   (a) $\log \dfrac{z}{z+1}$ $\left[\textit{Ans.}\ u_n = 0 \text{ for } n = 0, \dfrac{(-1)^n}{n} \text{ otherwise}\right]$

   (b) $\dfrac{z}{(z-1)^2}$ [*Ans.* n]

   (c) $\dfrac{8z - z^3}{(4-z)^3}$ [*Ans.* $(n^2 + 7n + 4)4^{n-1}$]

5. State whether the following statement is true or false: "If the Laplace transform of $f(t)$ is $F(s)$ and 'a' is a real number, then the Laplace transform of $e^{at}f(t - t_0)$ is $\dfrac{F(s - t_0)}{e^{at}}$". Give a brief justification for your answer.

6. What is the relation between a unit-step function and impulse function and from Laplace transform of unit-step function, how do you find the Laplace transform of impulse function?

Chapter 3 Transient and Steady State Behaviour of Systems

3.1 INTRODUCTION

The transient and steady-state behaviour of a system is usually termed the *time response* of the system. To analyze this behaviour, a mathematical model is developed. This chapter describes the time response of the first-order and second-order systems.

3.2 THE FIRST-ORDER SYSTEM

Figure 3.1 shows the block diagram of a first-order system. Here

$$[R(s) - O(s)]\frac{1}{Ts} = O(s)$$

or

$$\frac{R(s)}{Ts} = O(s)\left[1 + \frac{1}{Ts}\right]$$

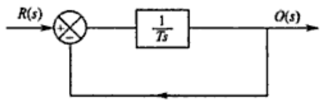


Fig. 3.1 First-order system.

or

$$\frac{O(s)}{R(s)} = \frac{1}{Ts}\cdot\frac{Ts}{Ts+1} = \frac{1}{Ts+1}$$

Depending on the type of input and initial conditions, the system response will be described. Suppose the input is a unit-step input and the initial conditions are zero.

For a unit-input step input, $R(s) = \frac{1}{s}$ and the output response will be

$$O(s) = \frac{1}{s(Ts+1)} = \frac{1}{s} - \frac{T}{Ts+1}$$

Taking the inverse Laplace transform,

$$o(t) = 1 - e^{-t/T}$$

Figure 3.2 shows the graphical representation of unit-step response of a first-order system. Now,

$$\left|\frac{do(t)}{dt} = \frac{1}{T}e^{-t/T}\right|_{t=0} = \frac{1}{T}$$

From the preceding expression, it is clear that at $t = 0$, the gradient of the output with respect to time will be $1/T$.

That means the base of the gradient at time $t = 0$ is the time constant for the steady-state value to reach unity, which is the steady-state value theoretically as t tends to infinity. From this it is clear that a large time constant will take more time to attain steady-state value and the system will be termed sluggish.

Figure 3.3 describes two systems I and II. Since the time constant of system I (T_1) is less than the time constant of system II (T_2), the system I reaches its steady-state value much earlier than the system II does, as per mathematical interpretation.

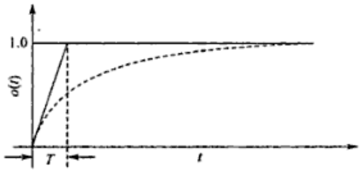


Fig. 3.2 *Unit-step response of first-order system.*

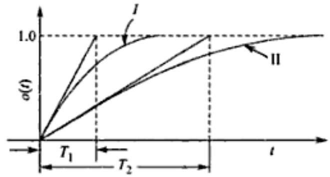


Fig. 3.3 *Comparison of two first-order systems.*

When the unit-ramp input is provided, then

$$R(s) = \frac{1}{s^2}$$

and

$$O(s) = \frac{1}{s^2(Ts+1)} = \frac{1}{Ts^2\left(s+\frac{1}{T}\right)}$$

Now,

$$\frac{1}{s^2\left(s+\frac{1}{T}\right)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+\frac{1}{T}}$$

where

$$A = \lim_{s\to 0}\frac{d}{ds}\left[s^2\frac{1}{s^2\left(s+\frac{1}{T}\right)}\right] = \lim_{s\to 0}\left[-\frac{1}{\left(s+\frac{1}{T}\right)^2}\right] = -T^2$$

$$B = \lim_{s\to 0}\frac{1}{s+\frac{1}{T}} = T$$

$$C = \lim_{s\to -\frac{1}{T}}\frac{1}{s^2} = T^2$$

Therefore,

$$O(s) = \frac{1}{Ts^2\left(s+\frac{1}{T}\right)} = \frac{1}{T}\left\{-\frac{T^2}{s}+\frac{T}{s^2}+\frac{T^2}{s+\frac{1}{T}}\right\}$$

$$= -\frac{T^2}{sT}+\frac{1}{s^2}+\frac{T}{s+\frac{1}{T}} = -\frac{T}{s}+\frac{1}{s^2}+\frac{T^2}{Ts+1}$$

$$= \frac{1}{s^2}-\frac{T}{s}+\frac{T^2}{Ts+1}$$

Taking the inverse Laplace of the above equation, we get

$$o(t) = t - T(1 - e^{-t/T})$$

The input was ramp input, i.e. $r(t) = t$. The output becomes $o(t) = t - T(1 - e^{-t/T})$.

Thus, it may be observed that when the input is unit-step input, i.e. the steady-state value is 1, then the transient response becomes $1 - e^{-t/T}$, and when the input is unit-ramp input, i.e. the steady-state value is t, then the transient response becomes $t - T(1 - e^{-t/T})$.

The error response is the difference between the input and the output. That means, in the case of unit-step input, the error response of the system will be

$$1 - (1 - e^{-t/T}) = e^{-t/T}$$

And in the case of unit-ramp input, the error response of the system will be

$$t - [t - T(1 - e^{-t/T})]$$
$$= T(1 - e^{-t/T})$$

The steady-state error in the case of unit-step input is

$$\lim_{t\to\infty}(\text{error response}) = \lim_{t\to\infty} e^{-t/T} = 0$$

The steady-state error in the case of unit-ramp input is

$$\lim_{t\to\infty} T(1 - e^{-t/T}) = T$$

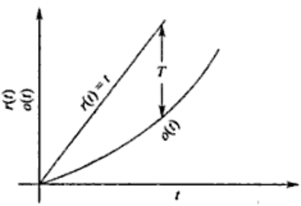


Fig. 3.4 Unit-ramp response of first-order system

Figure 3.4 describes the unit-ramp response of a first-order system where the steady-state error is T.

3.3 THE SECOND-ORDER SYSTEM

Figure 3.5 shows a position control system. This is nothing but a second-order system. When the error exists, then the motor develops a torque to rotate the output load in order to reduce the error to zero.

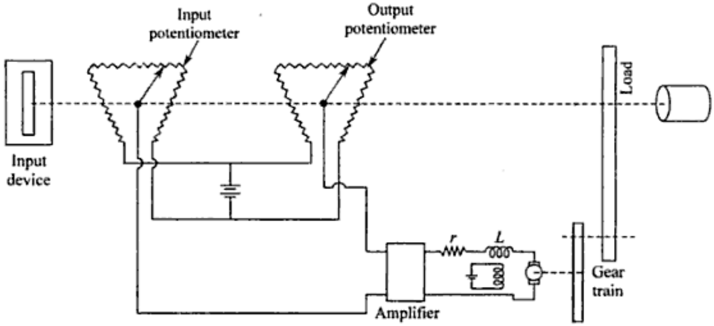


Fig. 3.5 Example of a second-order system.

For constant field current, the torque T developed by the motor is

$$T = K \cdot i \qquad \text{(where } i \text{ is the armature current)}$$

Therefore,

$$J \cdot \frac{d^2\theta}{dt^2} + F\frac{d\theta}{dt} = T = Ki$$

or

$$Js^2\theta(s) + Fs\theta(s) = KI(s) = T(s)$$

We assume that the gear ratio is made such that the output shaft rotates n times for each revolution of the motor shaft. Thus,

$$O(s) = n\theta(s)$$

Also,

$$[R(s) - O(s)]K_1 = E(s)$$

where

K_1 = voltage/angular displacement

$E(s)$ = error signal.

Now, $E(s)K_2$ = voltage applied to the motor where K_2 is the amplifier gain. Therefore,

$$K_2E(s) = E_b(s) + I(s)(r + Ls)$$

where $E_b(s)$ is the back emf. Now,

$$E_b(s) = K_3[\text{Laplace transform of } \dot{\theta}(t)]$$

$$= K_3s\theta(s)$$

Therefore, the block diagram will be as shown in Fig. 3.6.

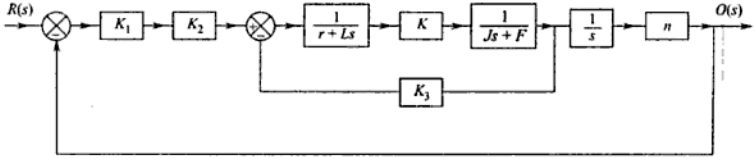


Fig. 3.6 Block diagram of Fig. 3.5.

From Fig. 3.6, the block diagram is converted to Fig. 3.7, where

$$G_1(s) = \frac{\dfrac{1}{r+Ls}\cdot K \cdot \dfrac{1}{Js+F}}{1+\dfrac{1}{r+Ls}\cdot K\dfrac{1}{Js+F}\cdot K_3}$$

$$= \frac{\dfrac{K}{(r+Ls)(Js+F)}}{\dfrac{(r+Ls)(Js+F)+K\,K_3}{(r+Ls)(Js+F)}}$$

$$= \frac{K}{(r+Ls)(Js+F)+KK_3}$$

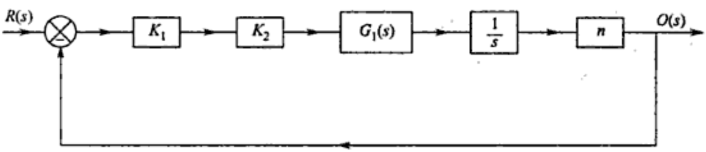


Fig. 3.7 Simplified block diagram of Fig. 3.6.

The final form of the block diagram will be as shown in Fig. 3.8.

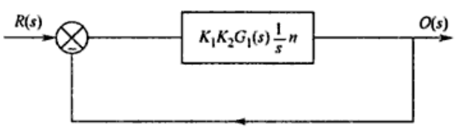


Fig. 3.8 Final form of the block diagram of Fig. 3.6.

Since the efect of Ls is small compared to mechanical system inertia, the same can be ignored. Therefore,

$$G_1(s) = \frac{K}{r(Js+F)+KK_3}$$

Forward path transfer function,

$$G(s) = K_1K_2G_1(s)\frac{1}{s}n$$

$$= \frac{K_1K_2Kn}{s[r(Js+F)+KK_3]} = \frac{\frac{K'}{r}}{\frac{s(r(Js+F)+KK_3]}{r}}$$

$$= \frac{\frac{K'}{r}}{s^2J + \frac{(Fr+KK_3)}{r}s}$$

where $K' = K_1K_1Kn$. Putting,

$$F' = \frac{Fr+KK_3}{r} \qquad \text{and} \qquad \frac{K'}{r} = K''$$

$$G(s) = \frac{K''}{s(Js+F')} = \frac{\frac{K''}{F'}}{s\left(\frac{J}{F'}s+1\right)} = \frac{K'''}{s(\tau s+1)}$$

where

$$\frac{K''}{F'} = K''' \qquad \text{and} \qquad \tau = \frac{J}{F'}$$

The above transfer function is that of the standard second-order system as shown in Fig. 3.9.

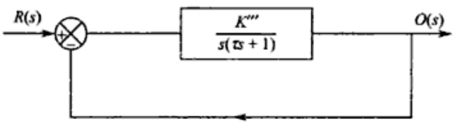


Fig. 3.9 Overall transfer function of the second-order system.

Therefore, the overall transfer function of the second-order system (Fig. 3.9) is

$$\frac{O(s)}{R(s)} = \frac{\frac{K'''}{s(\tau s+1)}}{1+\frac{K'''}{s(\tau s+1)}} = \frac{K'''}{s(\tau s+1)} \cdot \frac{s(\tau s+1)}{[s(\tau s+1)+K''']}$$

$$= \frac{K'''}{\tau s^2 + s + K'''}$$

$$= \frac{\frac{K'''}{\tau}}{s^2 + \frac{s}{\tau} + \frac{K'''}{\tau}}$$

The transfer function of a standard second-order system is expressed as

$$\frac{O(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

where

ζ = damping factor (or damping ratio)

ω_n = undamped natural frequency.

Comparing the transfer function of Fig. 3.9 with that of the standard second-order system, we have

$$2\zeta\omega_n = \frac{1}{\tau} \qquad \text{and} \qquad \omega_n^2 = \frac{K'''}{\tau}$$

or

$$2\zeta\sqrt{\frac{K'''}{\tau}} = \frac{1}{\tau}$$

or

$$\zeta = \frac{1}{2}\frac{1}{\sqrt{K'''\tau}} = \frac{\sqrt{F'}}{2\sqrt{K'''J}}$$

$$= \frac{\sqrt{F'}}{2\sqrt{\frac{K''}{F'}J}} = \frac{F'}{2\sqrt{K''J}}$$

and

$$\omega_n = \sqrt{\frac{K'''}{\tau}} = \sqrt{\frac{\frac{K''}{F'}}{\frac{J}{F'}}} = \sqrt{\frac{K''}{J}}$$

The equation $s^2 + 2\zeta\omega_n s + s^2 = 0$ is called the characteristic equation because the time response of any system is characterized by the roots of the denominator, which are the poles of the transfer function. Now the response of the second-order system to the unit-step function is

$$O(s) = \frac{1}{s} \cdot \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$= \frac{\omega_n^2}{s\left(s + \zeta\omega_n - j\omega_n\sqrt{1-\zeta^2}\right)\left(s + \zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}\right)}$$

Because, $s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$. The roots are

$$s = \frac{-2\zeta\omega_n \pm \sqrt{4\zeta^2\omega_n^2 - 4\omega_n^2}}{2}$$

$$= -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

Therefore,

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = (s + \zeta\omega_n - j\omega_n\sqrt{1-\zeta^2})(s + \zeta\omega_n + j\omega_n\sqrt{1-\zeta^2})$$

and $\zeta < 1$ indicates the underdamped condition. Now,

$$O(s) = \frac{A}{s} + \frac{B}{s + \zeta\omega_n - j\omega_n\sqrt{1-\zeta^2}} + \frac{B^*}{s + \zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}}$$

Hence,

$$A = \frac{\omega_n^2}{\omega_n^2} = 1$$

$$B = \frac{\omega_n^2}{s\left(s + \zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}\right)} \qquad \text{where } s = -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}$$

$$= \frac{\omega_n^2}{\left(-\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}\right)\left(2j\omega_n\sqrt{1-\zeta^2}\right)}$$

$$= \frac{\omega_n^2}{2\omega_n\sqrt{1-\zeta^2}\left(-\omega_n\sqrt{1-\zeta^2} - j\zeta\omega_n\right)}$$

$$= -\frac{1}{2\sqrt{1-\zeta^2}\left(\sqrt{1-\zeta^2} + j\zeta\right)}$$

Similarly,

$$B^* = -\frac{1}{2\sqrt{1-\zeta^2}\left(\sqrt{1-\zeta^2} - j\zeta\right)}$$

Therefore,

$$O(s) = \frac{1}{s} - \frac{\dfrac{1}{2\sqrt{1-\zeta^2}\left(\sqrt{1-\zeta^2}+j\zeta\right)}}{s+\zeta\omega_n - j\omega_n\sqrt{1-\zeta^2}} - \frac{\dfrac{1}{2\sqrt{1-\zeta^2}\left(\sqrt{1-\zeta^2}-j\zeta\right)}}{s+\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}}$$

Hence,

$$o(t) = 1 - \frac{e^{-\left(\zeta\omega_n - j\omega_n\sqrt{1-\zeta^2}\right)t}}{2\sqrt{1-\zeta^2}\left(\sqrt{1-\zeta^2}+j\zeta\right)} - \frac{e^{\left(-\zeta\omega_n - j\omega_n\sqrt{1-\zeta^2}\right)t}}{2\sqrt{1-\zeta^2}\left(\sqrt{1-\zeta^2}-j\zeta\right)}$$

$$= 1 - \frac{e^{-\zeta\omega_n t}\cdot e^{j\omega_n\sqrt{1-\zeta^2}\,t}}{2\sqrt{1-\zeta^2}\left(\sqrt{1-\zeta^2}+j\zeta\right)} - \frac{e^{-\zeta\omega_n t}\cdot e^{-j\omega_n\sqrt{1-\zeta^2}\,t}}{2\sqrt{1-\zeta^2}\left(\sqrt{1-\zeta^2}-j\zeta\right)}$$

$$= 1 - \frac{e^{-\zeta\omega_n t}}{2\sqrt{1-\zeta^2}}\left(\frac{e^{j\omega_n\sqrt{1-\zeta^2}\,t}}{\sqrt{1-\zeta^2}+j\zeta} + \frac{e^{-j\omega_n\sqrt{1-\zeta^2}\,t}}{\sqrt{1-\zeta^2}-j\zeta}\right)$$

$$= 1 - \frac{e^{-\zeta\omega_n t}}{2\sqrt{1-\zeta^2}}\left(\frac{e^{j\omega_n\sqrt{1-\zeta^2}\,t}\left(\sqrt{1-\zeta^2}-j\zeta\right) + e^{-j\omega_n\sqrt{1-\zeta^2}\,t}\left(\sqrt{1-\zeta^2}+j\zeta\right)}{1-\zeta^2+\zeta^2}\right)$$

$$= 1 - \frac{e^{-\zeta\omega_n t}}{2\sqrt{1-\zeta^2}}\left[\left(\sqrt{1-\zeta^2}-j\zeta\right)\left\{\cos\left(\omega_n\sqrt{1-\zeta^2}\right)t + j\sin\left(\omega_n\sqrt{1-\zeta^2}\right)t\right\}\right]$$

$$+\left(\sqrt{1-\zeta^2}+j\zeta\right)\left\{\cos\left(\omega_n\sqrt{1-\zeta^2}\right)t - j\sin\left(\omega_n\sqrt{1-\zeta^2}\right)t\right\}\Big]$$

$$= 1 - \frac{e^{-\zeta\omega_n t}}{2\sqrt{1-\zeta^2}}\left[2\sqrt{1-\zeta^2}\,\cos\omega_n\sqrt{1-\zeta^2}\,t + 2\zeta\sin\left(\omega_n\sqrt{1-\zeta^2}\right)t\right]$$

$$= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left[\sqrt{1-\zeta^2}\,\cos\omega_n\sqrt{1-\zeta^2}\,t + \zeta\sin\omega_n\sqrt{1-\zeta^2}\,t\right]$$

Now, let

$$A\sin\alpha = \sqrt{1-\zeta^2}$$

Then,

$$A\cos\alpha = \zeta$$

$$A^2\sin^2\alpha + A^2\cos^2\alpha = 1 - \zeta^2 + \zeta^2 = 1$$

or

$$A = 1$$

$$\tan\,\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{\sqrt{1-\zeta^2}}{\zeta}$$

Therefore,

$$o(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left[A\sin\alpha\cos\omega_n\sqrt{1-\zeta^2}\,t + A\cos\alpha\sin\omega_n\sqrt{1-\zeta^2}\,t\right]$$

$$= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\,A\sin\left(\omega_n\sqrt{1-\zeta^2}\,t + \alpha\right)$$

$$= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left(\omega_n\sqrt{1-\zeta^2}\,t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

The steady-state value of the output will be

$$o_{ss} = \lim_{t\to\infty} o(t) = 1$$

We know that in the underdamped condition the value of ζ will be less than one. The unit-step response of the underdamped second-order system will be as shown in Fig. 3.10.

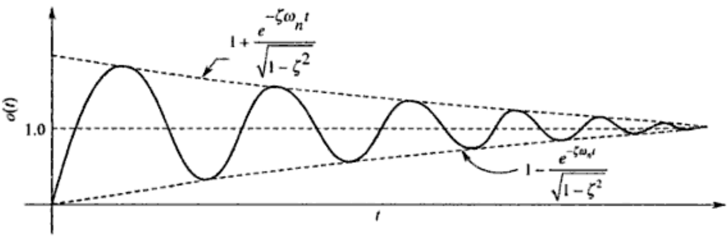


Fig. 3.10 *Unit-step response of the second-order system.*

The definitions of some terms specifying time response are (Fig. 3.11):

Delay time. The delay time t_d is the time required for the step response to reach 50 per cent of its final value in the first attempt, as indicated in Fig. 3.11.

Rise time. It is the time t_r required for the step response to rise from 0 to 100 per cent of its final value for underdamped systems, as shown in Fig. 3.11. But in the case of overdamped systems, the time required for the step response to rise from 10 per cent to 90 per cent of its final value is termed *rise time*.

Peak time. It is the time needed for the step response to reach its peak overshoot. It is denoted by t_p as shown in Fig. 3.11.

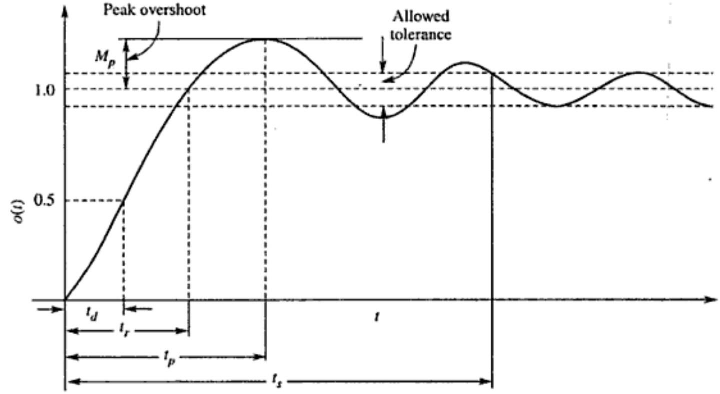


Fig. 3.11 Typical unit-step response of a control system.

Peak overshoot. This is the normalized difference between the peak overshoot and the steady output. The peak overshoot M_p in percentage

$$M_p = \frac{o(t_p) - o(\infty)}{o(\infty)} \times 100\%$$

where

$o(t_p)$ = output at peak time t_p.

$o(\infty)$ = output at steady state. The '∞' indicates the theoretical concept of attaining steady value at the mathematical value of time infinity.

Settling time. It is the time needed for the step response to reach and stay within a specified tolerance band. This is generally 2 to 5 per cent of the final value. It is shown in the Fig. 3.11 as t_s.

Calculation of rise time

$$o(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left(\omega_n \sqrt{1-\zeta^2}\, t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

Hence,

$$o(t_r) = 1 - \frac{e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta^2}} \sin\left(\omega_n \sqrt{1-\zeta^2}\, t_r + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

or

$$\frac{e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta^2}}\sin\left(\omega_n\sqrt{1-\zeta^2}\,t_r+\tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right)=0$$

or

$$\sin\left(\omega_n\sqrt{1-\zeta^2}\,t_r+\tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right)=\sin\pi$$

or

$$\omega_n\sqrt{1-\zeta^2}\,t_r+\tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}=\pi$$

or

$$\omega_n\sqrt{1-\zeta^2}\,t_r=\pi-\tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$$

or

$$t_r=\frac{\pi-\tan^{-1}\dfrac{\sqrt{1-\zeta^2}}{\zeta}}{\omega_n\sqrt{1-\zeta^2}}$$

Calculation of peak time

$\dfrac{do(t)}{dt}=0$ at peak time.

$$o(t)=1-\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left(\omega_n\sqrt{1-\zeta^2}\,t+\tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

$$\frac{do(t)}{dt}=\frac{\zeta\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}\,t+\tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right)-$$

$$\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\omega_n\sqrt{1-\zeta^2}\cos\left(\omega_n\sqrt{1-\zeta^2}\,t+\tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right)=0$$

or

$$\zeta\omega_n\sin\left(\omega_n\sqrt{1-\zeta^2}\,t+\tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right)-\omega_n\sqrt{1-\zeta^2}\cos\left(\omega_n\sqrt{1-\zeta^2}\,t+\tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right)=0$$

or

$$\zeta_n \sin\left[\omega_n\sqrt{1-\zeta^2}\,t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right] - \sqrt{1-\zeta^2}\cos\left(\omega_n\sqrt{1-\zeta^2}\,t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right) = 0$$

Let

$$A\cos\theta = \zeta \qquad \text{and} \qquad A\sin\theta = \sqrt{1-\zeta^2}$$

Therefore,

$$A^2\cos^2\theta + A^2\sin^2\theta = \zeta^2 + 1 - \zeta^2 = 1 \qquad \text{or} \qquad A = 1$$

Also,

$$\tan\theta = \frac{\sqrt{1-\zeta^2}}{\zeta} \qquad \text{or} \quad \theta = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$$

Therefore,

$$\sin\left(\omega_n\sqrt{1-\zeta^2}\,t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right)\cos\theta - \sin\theta\cos\left(\omega_n\sqrt{1-\zeta^2}\,t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right) = 0$$

or

$$\sin\left(\omega_n\sqrt{1-\zeta^2}\,t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta} - \theta\right) = 0$$

or

$$\sin\left(\omega_n\sqrt{1-\zeta^2}\,t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta} - \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right) = 0$$

or

$$\sin\,\omega_n\sqrt{1-\zeta^2}\,t = 0 = \sin 0^\circ = \sin\pi = \sin 2\pi = \sin 3\pi = \ldots$$

or

$$\omega_n\sqrt{1-\zeta^2}\,t = \pi$$

Therefore,

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

Calculation of peak overshoot

The peak overshoot is

$$M_p = o(t_p) - 1$$

$$= 1 - \frac{e^{-\zeta\omega_n t_p}}{\sqrt{1-\zeta^2}} \sin\left(\omega_n\sqrt{1-\zeta^2}\, t_p + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right) - 1$$

$$= -\frac{e^{-\zeta\omega_n t_p}}{\sqrt{1-\zeta^2}} \sin\left(\omega_n\sqrt{1-\zeta^2}\, t_p + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

$$= -\frac{e^{-\zeta\omega_n \pi/\omega_n\sqrt{1-\zeta^2}}}{\sqrt{1-\zeta^2}} \sin\left(\omega_n\sqrt{1-\zeta^2}\,\frac{\pi}{\omega_n\sqrt{1-\zeta^2}} + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

$$= +\frac{e^{-\zeta\pi/\sqrt{1-\zeta^2}}}{\sqrt{1-\zeta^2}} \sin\left(\tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

Now, let

$$\tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta} = \theta$$

or

$$\tan\theta = \frac{\sqrt{1-\zeta^2}}{\zeta} = \frac{\sin\theta}{\cos\theta} \quad \text{or} \quad \sin\theta = \sqrt{1-\zeta^2}$$

or

$$\theta = \sin^{-1}\sqrt{1-\zeta^2}$$

Therefore,

$$M_p = \frac{e^{-\zeta\pi/\sqrt{1-\zeta^2}}}{\sqrt{1-\zeta^2}} \sin\left(\sin^{-1}\sqrt{1-\zeta^2}\right)$$

$$= e^{-\zeta\pi/\sqrt{1-\zeta^2}}$$

Thus, the percentage overshoot

$$= e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100\%$$

Calculation of settling time

If the tolerance band is considered 2 per cent, then the envelope of the time response curve, i.e. $\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}$ will be 0.02. Therefore,

$$\frac{e^{-\zeta\omega_n t_s}}{\sqrt{1-\zeta^2}} = 0.02$$

where t_s is the settling time.

For low values of ζ,

$$e^{-\zeta\omega_n t_s} = 0.02$$

or

$$t_s = \frac{4}{\zeta\omega_n} = 4T$$

where T is the time constant of the exponential term $\dfrac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}$.

For tolerance band of 5 per cent, the time constant will be approximately $\dfrac{3}{\zeta\omega_n}$ or $3T$.

Steady-state error calculation

The steady-state error is represented by

$$\lim_{t\to\infty} [1 - o(t)] = \lim_{t\to\infty}\left[1 - 1 + \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left\{\omega_n\sqrt{1-\zeta^2}\, t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right\}\right] = 0$$

3.4 STEADY-STATE ERROR OF FEEDBACK CONTROL SYSTEM AND ITS TYPES

The unity feedback system shown in Fig. 3.12 is represented by

$$\frac{O(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

or

$$\frac{O(s)-R(s)}{R(s)} = \frac{G(s)-1-G(s)}{1+G(s)}$$

or

$$\frac{-E(s)}{R(s)} = \frac{-1}{1+G(s)}$$

or

$$E(s) = \frac{R(s)}{1+G(s)}$$

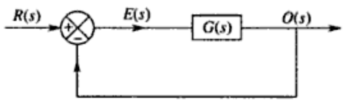


Fig. 3.12 Unity feedback system.

The steady-state error is represented by the final-value theorem,

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s)$$

$$= \lim_{s\to 0}\frac{sR(s)}{1+G(s)}$$

When the input $r(t) = u(t)$, then

$$R(s) = \frac{1}{s}$$

Therefore,

$$e_{ss} = sE(s) = \lim_{s\to 0} \frac{1}{1+G(s)} = \frac{1}{1+G(0)}$$

$$= \frac{1}{1+K_p}$$

where $K_p = G(0)$ is termed the position-error constant.

When the input is unit-ramp, then $r(t) = t$. Since $\frac{dr(t)}{dt} = 1$, this ramp input is also termed the velocity input, which is just like the unit-step input. The Laplace transform of $r(t) = t$ is $R(s) = \frac{1}{s^2}$. In this case, the steady-state error is

$$e_{ss} = \lim_{s\to 0} \frac{sR(s)}{1+G(s)} = \lim_{s\to 0} \frac{s\frac{1}{s^2}}{1+G(s)}$$

$$= \lim_{s\to 0} \frac{1}{s+sG(s)} = \lim_{s\to 0} \frac{1}{sG(s)}$$

$$= \frac{1}{K_v}$$

where $\lim_{s\to 0} sG(s) = K_v$ is termed the velocity-error constant.

When the input is unit-parabolic then $r(t) = \frac{t^2}{2}$. Therefore,

$$\frac{dr(t)}{dt} = \frac{2t}{2} = t \qquad \text{or} \qquad \frac{d^2r(t)}{dt^2} = 1$$

Since $\frac{d^2r(t)}{dt^2} = 1$, the unit-parabolic input $\frac{t^2}{2}$ is termed unit-acceleration input. The Laplace transform of $\frac{t^2}{2}$ is $\frac{1}{s^3}$. Hence $R(s) = \frac{1}{s^3}$. The steady-state error in the case of unit-parabolic input will be

$$e_{ss} = \lim_{s\to 0} \frac{sR(s)}{1+G(s)}$$

$$= \lim_{s\to 0} \frac{s\cdot\frac{1}{s^3}}{1+G(s)} = \frac{1}{s^2+s^2G(s)}$$

$$= \lim_{s \to 0} \frac{1}{s^2 G(s)} = \frac{1}{K_a}$$

where $\lim_{s \to 0} s^2 G(s)$ is termed the acceleration-error constant.

Here we have developed the following error constants during our study of steady-state error with different inputs on a unity feedback system.

Input	*Error constant*
Unit-step input	Position-error constant, $K_p = G(0)$
Unit-ramp or velocity input	Velocity-error constant, $K_v = \lim_{s \to 0} sG(s)$
Unit-parabolic or acceleration input	Acceleration-error constant, $K_a = \lim_{s \to 0} s^2 G(s)$

Type of feedback control systems

The open-loop transfer function $G(s)$ of the unity feedback system shown in Fig. 3.12 is expressed either in the time constant form or in the polar form:

$$G(s) = \frac{K(T_{z1}s + 1)(T_{z2}s + 1)\ldots}{s^n (T_{p1}s + 1)(T_{p2}s + 1)\ldots} \qquad \text{(Time-constant form)}$$

$$G(s) = \frac{K_1(s + z_1)(s + z_2)\ldots}{s^n (s + p_1)(s + p_2)\ldots} \qquad \text{(Pole-zero form)}$$

The term s^n in the denominator of both the above equations indicates the number of time integrations to be made. When $n = 0$, then the system is termed "type 0" system. When $n = 1$, then the system is termed "type 1" system. When $n = 2$, the system is termed "type 2" system.

Suppose, $G(s) = \dfrac{1}{Ts + 1}$ in the case of a unity feedback control system, then the system is a "type 0" system.

$$\frac{O(s)}{R(s)} = \frac{\dfrac{1}{Ts+1}}{1 + \dfrac{1}{Ts+1}} = \frac{\dfrac{1}{Ts+1}}{\dfrac{Ts+2}{Ts+1}}$$

$$= \frac{1}{Ts + 2}$$

indicates that it is a first-order system. Hence the above system is termed the first-order type 0 system.

If $G(s) = \dfrac{1}{s(s + a)}$ in a unity feedback system, then the system is a type 1 system, whereas

$$\frac{O(s)}{R(s)} = \frac{\dfrac{1}{s(s+a)}}{1+\dfrac{1}{s(s+a)}} = \frac{1}{s^2+sa+1}$$

indicates that the system will be of second-order type.

The steady-state errors of the type 0, type 1 and type 2 systems are tabulated below:

System	*Steady-state error for unit-step input*	*Steady-state error for unit-ramp input*	*Steady-state error for unit-parabolic input*
Type 0 system	$\frac{1}{1+G(0)} = \frac{1}{1+K_p}$	$\lim_{s\to 0}\frac{1}{sG(s)} = \infty$	$\lim_{s\to 0}\frac{1}{s^2G(s)} = \infty$
Type 1 system	$\frac{1}{1+G(0)} = \frac{1}{1+\infty} = 0$	$\lim_{s\to 0}\frac{1}{sG(s)} = \frac{1}{K_v}$	$\lim_{s\to 0}\frac{1}{s^2G(s)} = \infty$
Type 2 system	$\frac{1}{1+G(0)} = \frac{1}{1+\infty} = 0$	$\lim_{s\to 0}\frac{1}{sG(s)} = \frac{1}{\infty} = 0$	$\lim_{s\to 0}\frac{1}{s^2G(s)} = \frac{1}{K_a}$

For non-unity feedback systems, the steady-state error can be calculated in the following manner.

First of all, the actuating error signal is determined from Fig. 3.13.

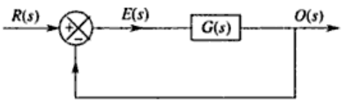


Fig. 3.13 *Non-unity feedback system.*

Now,

$$O(s) = \frac{G(s)}{1+G(s)H(s)} R(s)$$

or

$$\frac{O(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

Therefore,

$$E(s) = R(s) - O(s)H(s)$$

$$= R(s) - \frac{G(s)H(s)}{1+G(s)H(s)} R(s)$$

$$= \frac{R(s)[1 + G(s)H(s) - G(s)H(s)]}{1 + G(s)H(s)}$$

or

$$E(s) = R(s)\ \frac{1}{1 + G(s)H(s)}$$

The steady-state error will be

$$e_{ss} = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)H(s)}$$

Now the error constants can be determined in the same manner as it was done in the case of the unity feedback system by replacing $G(s)$ by $G(s)H(s)$.

EXAMPLE 3.1 What type of systems are the following block diagrams?

(a) Figure 3.14 has the open-loop transfer function, $G(s) = \dfrac{2}{s(s^2 + 2s + 2)}$

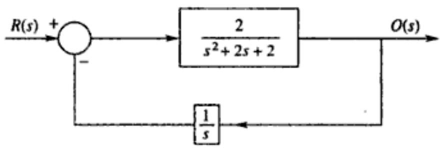


Fig. 3.14

Since here one integration is being observed, it is a type 1 system.

(b) Figure 3.15 has the open-loop transfer function, $G(s) = \dfrac{5}{(s + 2)(s^2 + 2s + 3)}$

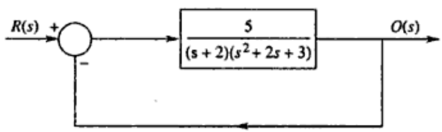


Fig. 3.15

Since here no integration is being observed, it is a type 0 system.

(c) Figure 3.16 has the open-loop transfer function, $G(s) = \dfrac{s+1}{s^2(s+2)}$

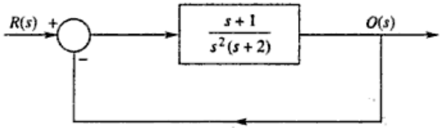


Fig. 3.16

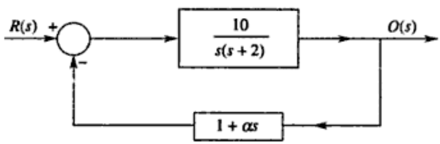
Since here double integration is being observed, it is a type 2 system.

SUMMARY

The time response of a first-order system has been dealt with in this chapter. The effects of unit-step input and ramp input (unit-velocity input) on the first-order system are explained. A second-order system example is taken and the effect of unit-step input is explained in detail. Delay time, rise time, peak time, peak-overshoot are defined. Calculations of rise time, peak time, peak overshoot and steady-state errors are shown. Steady-state errors of feedback control systems and their types are explained. Types of feedback control systems are explained with examples.

QUESTIONS

1. What do you mean by step input, ramp input, and impulse input? How do you represent the impulse function graphically?
2. Determine the step-input response of a second-order system.
3. Define 'settling time', 'rise time', and 'peak overshoot' of a control system.
4. The block diagram of a position control system with velocity feedback is shown below.



Determine the value of α so that the step response has maximum overshoot of 10 per cent. What is the steady-state error?

5. What is the difference between order and type of control systems? Explain clearly.
6. (a) Sketch the time domain response $c(t)$ of a typical underdamped, second-order system to a step input $r(t)$. On the sketch, indicate the following time domain specifications:
   - (i) Maximum peak overshoot, M
   - (ii) Rise time, t_r
   - (iii) Settling time, t_s
   - (iv) Steady-state error e_{ss} due to step input

(b) Derive expressions for (i) M_p and (ii) t_s for a unity feedback, second-order system whose open-loop transfer function is given as

$$G(s) = \frac{\omega_n^2}{s(s+2\zeta\,\omega_n)}$$

(c) For the above system determine the values of (i) the damping ratio ζ and (ii) the undamped natural frequency ω_n, so that the system responds to a step input with 5 per cent peak overshoot and a settling time of 2 seconds.

(d) What is the steady-state error e_{ss} of the above system to a unit ramp input.

7. Define the steady-state error and error constants with respect to unit-step, unit-velocity and unit-acceleration inputs. How can the steady-state error be reduced?

8. An ac-dc servo system is shown in the figure below. The transfer function of the demodulator is given as K_d (dc volts/ac volts/ac volts). The sensitivity of the synchro-error detector is K_s (volts/radian), the gain of the dc generator is K_g (volts/field ampere). The dc motor is separately excited and has a counteremf of K_b volts per radian/second and a torque constant K_T (newton-metre/ampere). The rotor inertia and friction are negligible.

(a) Draw the block diagram of the system indicating the transfer function of each block.

(b) Find the steady-state error for a velocity input θ_r or 1 radian/second given that the system parameters are:

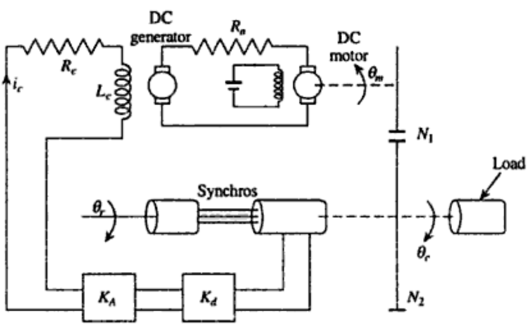
$K_s = 30$, $K_d = 4$, $K_A = 5$, $K_g = 100$

$K_b = 1$, $K_T = 0.5$

$R_c = 200\ \Omega$, $L_c = 2$ H

$R_a = 0.5\ \Omega$, $J_L = 0.5$ kg-m^2

$f_L = 1$ newton-metre per radian/second and $N_1/N_2 = 1$ = gear ratio coupling motor to load.



9. A servomechanism is used to control the angular position θ_0 of a mass through a command signal θ_i. The moment of inertia of the moving parts referred to the load shaft is 200kg-m^2 and the motor torque at the load is 6.88×10^4 N-m per rad of error. The damping torque coefficient referred to the load shaft is 5×10^3 N-m per rad/s.

(a) Find the time response of the servomechanism to a step input of 1 rad and determine the frequency of transient oscillation, the time to rise to the peak overshoot and the value of the peak overshoot.

(b) Determine the steady-state error when the command signal is a constant angular velocity of 1 rev./min.

(c) Determine the steady-state error which would exist when a steady torque of 1200 N-m is applied at the load shaft.

10. Find the type and order of the system with

$$G(s) = \frac{15}{s(s+5)} \quad \text{and} \quad H(s) = \frac{12}{s^2}$$

11. The closed-loop transfer function of a system is

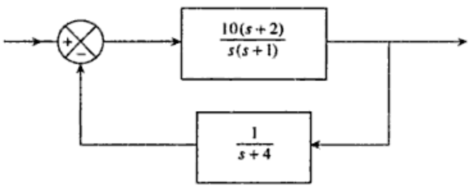
$$\frac{25\,K}{s^2 + (5 + 500K_t)s + 25K}.$$

Find the values of K and K_t so that the maximum overshoot of the output is approximately 20 per cent and the rise time is 0.05 s.

12. (a) A second-order servo system has poles at $-1 \pm j2$ and a zero at $-1 + j0$. Its steady-state output for a unit step input is 2. Determine its transfer function. What is its peak overshoot for a unit step input?

(b) Define Type 0, 1, and 2 systems. Determine the type, order of the system shown in the figure below. Determine the steady-state value of the error signal e for the following input:

$$V(t) = 0 \quad \text{for} \quad t < 0$$
$$V(t) = 2 + 3t \quad \text{for} \quad t \geq 0.$$



13. A second-order servo system has poles at $-1 \pm j2$ and a zero at -1. Its steady-state output for a unit step input is 3. Determine its transfer function. What is its peak overshoot for a unit-step input?

Chapter 4 State Variable Analysis

4.1 INTRODUCTION

We have so far modelled the physical system by the transfer function approach. Although the transfer function model provides us with simple and powerful analysis and design techniques, its demerit is that it is defined under zero initial conditions. Moreover, the transfer function approach is used to model only the linear time-invariant systems. The transfer function approach for multiple-input and multiple-output systems is really troublesome. To speak the truth, the transfer function modelling does not give any information regarding the internal state of the system. Even for a stable output system, it may so happen that some of the system elements may exceed their specified rating. Hence, information regarding the internal state of the system is really essential.

The above points are the main reasons for the development of the state-variable approach. It is nothing but a direct time-domain approach leading towards system optimization. Usually, we are familiar with two types of variables, e.g. input and output variables. In the case of state model, we have to deal with three types of variables—input, output, and state variables. For example, the displacement at any time can be determined if the applied force, initial velocity, and initial displacement are known. The displacement at any time in this case will be the output variable, the applied force at that time will be the input variable, and the initial velocity and the initial displacement will be the state of the system. The state of the system at any time is expressed by state variables. We know that the motion of a simple mechanical system is expressed mathematically by the equations

$$\frac{d}{dt}v(t) = \frac{1}{M}F(t) \qquad \text{and} \qquad \frac{d}{dt}x(t) = v(t)$$

Thus,

$$v(t) = \frac{1}{M}\int_{-\infty}^{t} F(t)dt$$

$$= \frac{1}{M}\int_{-\infty}^{t_0} F(t)dt + \frac{1}{M}\int_{t_0}^{t} F(t)dt$$

$$= v(t_0) + \frac{1}{M}\int_{t_0}^{t} F(t)dt \qquad (4.1)$$

Here $v(t_0)$ is the initial velocity and $F(t)$ is the input variable, that is, the force from $t = t_0$ onwards. Thus,

$$x(t) = \int_{-\infty}^{t} v(t)\,dt = \int_{-\infty}^{t_0} v(t)\,dt + \int_{t_0}^{t} v(t)\,dt$$

$$= x(t_0) + \int_{t_0}^{t} \left(v(t_0) + \frac{1}{M}\int_{t_0}^{t} F(t)\,dt \right) d\tau$$

$$= x(t_0) + v(t_0)(t - t_0) + \frac{1}{M}\int_{t_0}^{t} d\tau \int_{t_0}^{t} F(t)\,dt \tag{4.2}$$

Thus it may be observed that the output $x(t)$ can only be described if the states $x(t_0)$ and $v(t_0)$ are known in detail along with the input force variable.

All the above variables are usually represented by vectors termed input vector $u(t)$, output vector $y(t)$, and state vector $x(t)$. Thus,

$$u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_m(t) \end{bmatrix};\ y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_p(t) \end{bmatrix};\ x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

From Eqs. (4.1) and (4.2), it is clear that their solution will give two state variables $v(t)$ and $x(t)$ of the system. Hence, for an nth-order system, the state variable representations are arranged in the form of n first-order differential equations,

$$\frac{dx_1}{dt} = f_1(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_m)$$

$$\vdots$$

$$\frac{dx_n}{dt} = f_n(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_m)$$

and the integration of the above equation gives

$$x_i(t) = x_i(t_0) + \int_{t_0}^{t} f_i(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_m)\,dt$$

where $i = 1, 2, \ldots, n$.

Now we illustrate the method of state space representation with a system shown in Fig. 4.1.

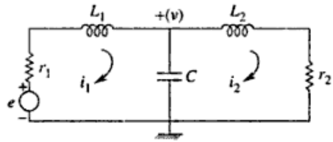


Fig. 4.1

From Fig. 4.1, we get

$$e - v = i_1 r_1 + L_1 \frac{di_1}{dt}$$

$$v = L_2 \frac{di_2}{dt} + i_2 r_2$$

$$i_1 = i_2 + C\frac{dv}{dt}$$

Let

$$v = x_1, \quad i_1 = x_2, \quad i_2 = x_3$$

$$\frac{dv}{dt} = \dot{x}_1 = \frac{1}{C}i_1 - \frac{1}{C}i_2 = \frac{1}{C}x_2 - \frac{1}{C}x_3$$

$$\frac{di_1}{dt} = \dot{x}_2 = -\frac{1}{L_1}i_1 r_1 + \frac{1}{L_1}(e - v)$$

$$= -\frac{1}{L_1}r_1 x_2 + \frac{1}{L_1}e - \frac{1}{L_1}x_1 \qquad (\because\ i_1 = x_2 \text{ and } v = x_1)$$

$$\frac{di_2}{dt} = \dot{x}_3 = \frac{1}{L_2}(v - i_2 r_2) = \frac{x_1}{L_2} - \frac{x_3 r_2}{L_2}$$

Here e is nothing but the input voltage $u(t)$. Thus, we have

$$\dot{x}_1 = \frac{1}{C}x_2 - \frac{1}{C}x_3$$

$$\dot{x}_2 = -\frac{1}{L_1}x_1 - \frac{r_1}{L_1}x_2 + \frac{u(t)}{L_1}$$

$$\dot{x}_3 = \frac{1}{L_2}x_1 - \frac{r_2}{L_2}x_3.$$

or

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{C} & -\frac{1}{C} \\ -\frac{1}{L_1} & -\frac{r_1}{L_1} & 0 \\ \frac{1}{L_2} & 0 & -\frac{r_2}{L_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{u(t)}{L_1} \\ 0 \end{bmatrix}$$

or

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{C} & -\frac{1}{C} \\ -\frac{1}{L_1} & -\frac{r_1}{L_1} & 0 \\ \frac{1}{L_2} & 0 & -\frac{r_2}{L_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L_1} \\ 0 \end{bmatrix} u(t)$$

The output can also be expressed as follows.

Let voltage across r_2 and current through r_2 be the outputs y_1 and y_2, then

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & r_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

since $y_1 = i_2 r_2$ and $y_2 = i_2$.

As another example, we find the state space representation of the circuit shown in Fig. 4.2. Here,

$$v(t) = ir + L\frac{di}{dt} + K_b \omega$$

$$T = \text{Torque} = Ki$$

$$T = J\frac{d^2\theta}{dt^2} + F\frac{d\theta}{dt}$$

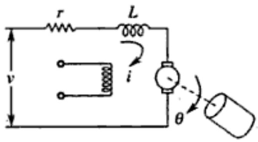


Fig. 4.2

Therefore,

$$V(s) = i(s)r + Lsi(s) + K_b\,\omega(s)$$

$$T(s) = Ki(s) = Js^2\theta(s) + Fs\theta(s)$$

$$v(t) = ir + L\frac{di}{dt} + K_b\frac{d\theta}{dt}$$

$$i(t) = \frac{J}{K}\frac{d^2\theta}{dt^2} + \frac{F}{K}\cdot\frac{d\theta}{dt}$$

Let $x_1 = \theta$, $x_2 = \dot{\theta}$, $x_3 = i$. Then,

$$v = u(t)$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{d\dot{\theta}}{dt} = \frac{d^2\theta}{dt^2} = -\frac{F}{J}x_2 + \frac{K}{J}x_3$$

$$\dot{x}_3 = \frac{di}{dt} = \frac{1}{L}u(t) - \frac{K_b}{L}x_2 - \frac{r}{L}x_3$$

Therefore,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\frac{F}{J} & \frac{K}{J} \\ 0 & -\frac{K_b}{L} & -\frac{r}{L} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{L} \end{bmatrix} u(t).$$

4.2 STATE MODEL OF LINEAR TIME-INVARIANT SYSTEM

A system is linear if and only if

$$H[a_1u_1(t) + a_2u_2(t)] = a_1H[u_1(t)] + a_2H[u_2(t)]$$

for arbitrary input sequences $u_1(t)$ and $u_2(t)$ and any arbitrary constants a_1 and a_2.

When a system H is excited by an input signal $u(t)$ and produces an output signal $y(t)$, then

$$y(t) = H[u(t)]$$

If an input signal is delayed by K units of time to yield $u(t - K)$, and is applied to the same system, and if the characteristics of the system do not change with the time, the output will be $y(t - K)$. Such a system is termed time-invariant system.

Suppose the system to be dealt with is as shown in Fig. 4.3.

The input-output equation for the system is $y(t) = H[(u(t)] = tu(t)$. The response of the above system for input $u(t - K)$ will be $tu(t - k)$. But in the case of a time-invariant system, it should be $y(t - K) = (t - K)u(t - K)$.

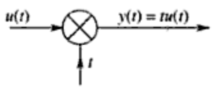


Fig. 4.3

Hence the system shown in Fig. 4.3 is time variant. Usually, for a linear time-invariant system the state equations are represented by

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t) + Ew(t) \tag{4.3}$$

where

$$x(t) = \text{State vector} = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} \underset{\text{matrix}}{(n \times 1)}$$

$$u(t) = \text{Input vector} = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_p(t) \end{bmatrix} \underset{\text{matrix}}{(p \times 1)}$$

$$w(t) = \text{Disturbance vector} = \begin{bmatrix} w_1(t) \\ w_2(t) \\ \vdots \\ w_v(t) \end{bmatrix} \underset{\text{matrix}}{(v \times 1)}$$

The output equations are represented by

$$y(t) = Cx(t) + Du(t) + Hw(t)$$

where

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} \underset{\text{matrix}}{(n \times 1)}$$

$$u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_p(t) \end{bmatrix} \underset{\text{matrix}}{(p \times 1)}$$

$$w(t) = \begin{bmatrix} w_1(t) \\ w_2(t) \\ \vdots \\ w_v(t) \end{bmatrix} \underset{\text{matrix}}{(v \times 1)}$$

The state equation coefficients are the matrices

$$A = [n \times n] \text{ matrix}$$

$$B = [n \times p] \text{ matrix}$$

$$C = [q \times n] \text{ matrix}$$

$$D = [q \times p] \text{ matrix}$$

$$E = [n \times v] \text{ matrix}$$

$$H = [q \times v] \text{ matrix}$$

The $Ax(t)$ is the homogeneous part of the state Eq. (4.3) and $u(t)$ and $w(t)$ are the forcing functions of the state equations.

The state-transition matrix is defined as the matrix which satisfies the linear homogeneous state equation

$$\frac{dx(t)}{dt} = Ax(t) \tag{4.4}$$

If $\phi(t)$ is the $n \times n$ matrix that represents the state-transition matrix, then it must satisfy the equation

$$\frac{d\phi(t)}{dt} = A\phi(t)$$

The solution of $\dfrac{dx(t)}{dt} = Ax(t)$ can be found out as follows. Let

$$x(t) = ce^{mt}$$

Then,

$$cme^{mt} = Ace^{mt}$$

or

$$m = A$$

Therefore,

$$x(t) = ce^{At}$$

Now, at $t = 0$, if $x(t) = x(0)$, then

$$x(t) = x(0) = c$$

Therefore, the solution of Eq. (4.4) is

$$x(t) = x(0)e^{At} \tag{4.5}$$

Suppose $\phi(t) = e^{At}$, then it must satisfy the equation

$$\frac{d\phi(t)}{dt} = A\phi(t)$$

because $\dfrac{d\phi(t)}{dt} = \dfrac{d}{dt}(e^{At}) = Ae^{At}$. Again, as per assumption, $A\phi(t) = Ae^{At}$

Therefore, the solution (4.5) already deduced, can also be written as

$$x(t) = \phi(t)x(0) \tag{4.6}$$

By applying the Laplace transform to Eq. (4.4), we can write

$$sX(s) - x(0) = AX(s)$$

or

$$X(s) = (sI - A)^{-1}x(0) \tag{4.7}$$

Since the above expressions are written in matrix form, in place of s, sI is to be taken where I is the unit matrix.

Taking the inverse Laplace transform of Eq. (4.7) yields

$$x(t) = \mathcal{L}^{-1}(sI - A)^{-1}\, x(0) \tag{4.8}$$

By comparing Eq. (4.6) with Eq. (4.8), the state-transition matrix is

$$\phi(t) = \mathcal{L}^{-1}(sI - A)^{-1}$$

where $\phi(t) = e^{At} = I + At + \dfrac{1}{2!}A^2t^2 + \dfrac{1}{3!}A^3t^3 + \cdots$

4.2.1 Properties of the State-transition Matrix

The properties of the state-transition matrix are:

$$\phi(0) = I$$

$$\phi^{-1}(t) = \phi(-t)$$

$$\phi(t_2 - t_1)\phi(t_1 - t_0) = \phi(t_2 - t_0) \qquad \text{for any } t_0, t_1 \text{ and } t_2$$

$$|\phi(t)|^K = \phi(Kt) \qquad \text{for } K = \text{positive integers}$$

Proof of the properties

We know, $\phi(t) = I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots$

When $t = 0$, $\phi(0) = I$ (Hence the first property is proved.)

Now,

$$\phi(t) = e^{At}$$

or

$$\phi(t)\, e^{-At} = e^{At} \cdot e^{-At} = I$$

or

$$e^{-At} = \phi^{-1}(t)$$

$$\phi(-t) = e^{A(-t)} = e^{-At} = \phi^{-1}(t) \qquad \text{(Hence the second property is proved.)}$$

Also,

$$\begin{aligned}\phi(t_2 - t_1)\,\phi(t_1 - t_0) &= e^{A(t_2 - t_1)} e^{A(t_1 - t_0)} \\ &= e^{At_2 - At_1 + At_1 - At_0} \\ &= e^{A(t_2 - t_0)} \\ &= \phi(t_2 - t_0) \qquad \text{(Hence the third property is proved.)}\end{aligned}$$

Lastly,

$$\begin{aligned}|\phi(t)|^K &= e^{At} \cdot e^{At} \ldots e^{At} \qquad (K \text{ terms}) \\ &= e^{KAt} = \phi(Kt) \qquad \text{(Hence the fourth property is proved.)}\end{aligned}$$

4.3 STATE-TRANSITION EQUATION

The state-transition equation is the solution of the linear state equation. The linear time-invariant state equation is

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t) + Ew(t).$$

Taking the Laplace transform of both sides, we get

$$sX(s) - x(0) = AX(s) + BU(s) + EW(s)$$

or

$$X(s)[sI - A] = x(0) + BU(s) + EW(s)$$

or

$$X(s) = [sI - A)^{-1} x(0) + [sI - A]^{-1} BU(s) + [sI - A]^{-1} EW(s)$$

Taking the Laplace inverse of the above equation, we get

$$x(t) = \mathcal{L}^{-1}[(sI - A)]^{-1}x(0) + \mathcal{L}^{-1}[(sI - A)]^{-1}[BU(s) + EW(s)]$$

or

$$x(t) = \phi(t)x(0) + \int_0^t \phi(t - \tau)[Bu(\tau) + Ew(\tau)]d\tau \qquad [\because \; F_1(s)F_2(s) = \mathcal{L}\left[\int_0^t F_2(\tau)F_1(t - \tau)\, d\tau\right]$$

EXAMPLE 4.1 If a state equation is expressed by

$$\begin{bmatrix} \dfrac{dx_1(t)}{dt} \\ \dfrac{dx_2(t)}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

then

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad E = 0$$

and

$$\text{for } u(t) = 1 \qquad t \geq 0$$

Therefore,

$$sI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$$

and

$$(sI - A)^{-1} = \frac{1}{s^2 + 3s + 2} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

The state-transition matrix will be

$$\phi(t) = \mathcal{L}^{-1}[(sI - A)^{-1}] = \begin{bmatrix} (2e^{-t} - e^{-2t}) & (e^{-t} - e^{-2t}) \\ (-2e^{-t} + 2e^{-2t}) & (-e^{-t} + 2e^{-2t}) \end{bmatrix}$$

Therefore,

$$x(t) = \phi(t)\, x(0) + \int_0^t \begin{bmatrix} (2e^{-(t-\tau)} - e^{-2(t-\tau)}) & (e^{-(t-\tau)} - e^{-2(t-\tau)}) \\ (-2e^{(-t-\tau)} + e^{-2(t-\tau)}) & (-e^{-(t-\tau)} + 2e^{-2(t-\tau)}) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} d\tau$$

or

$$x(t) = \begin{bmatrix} (2e^{-t} - e^{-2t}) & (e^{-t} - e^{-2t}) \\ (-2e^{-t} + 2e^{-2t}) & (-e^{-t} + 2e^{-2t}) \end{bmatrix} x(0) + \begin{bmatrix} 0.5 - e^{-t} + 0.5e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix} \qquad t \geq 0$$

Now, $\mathcal{L}^{-1}[(sI - A)^{-1}]BU(s)$ can also be calculated as follows:

$$\mathcal{L}^{-1}[(sI - A)^{-1}]BU(s) = \mathcal{L}^{-1}\left(\frac{1}{s^2 + 3s + 2} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{1}{s} \right)$$

$$= \mathcal{L}^{-1}\left(\frac{1}{s^2 + 3s + 2} \begin{bmatrix} \dfrac{1}{s} \\ 1 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 0.5 - e^{-t} + 0.5e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix} \qquad \text{when } t \geq 0$$

A state model can also be represented by a block diagram. Let us consider that the disturbance vector is zero. Then,

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

Figure 4.4 shows the block diagram representation of the state model of a linear multi-input multi-output system.

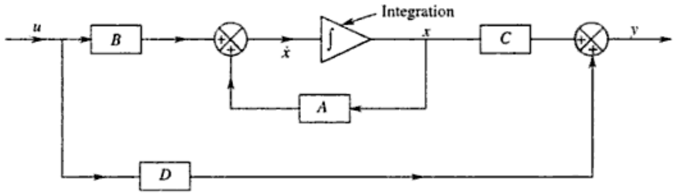


Fig. 4.4 Block diagram of the state model of a linear system.

The state diagram can also be developed by signal flow graph. The following illustration is given to develop the signal flow graph. Suppose, the state equation is

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

When $u(t) = 1$ for $t \geq 0$, we have

$$X(s) = (sI - A)^{-1}x(0) + (sI - A)^{-1}\, BU(s)$$

where

$$(sI - A)^{-1} = \frac{1}{s^2 + 3s + 2} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

$$= \frac{1}{\Delta} \begin{bmatrix} \dfrac{s+3}{s^2} & \dfrac{1}{s^2} \\ \dfrac{-2}{s^2} & \dfrac{s}{s^2} \end{bmatrix} \qquad [\text{Let } \Delta = 1 + 3s^{-1} + 2s^{-2}]$$

$$= \frac{1}{\Delta} \begin{bmatrix} s^{-1}(1 + 3s^{-1}) & s^{-2} \\ -2s^{-2} & s^{-1} \end{bmatrix}$$

Now,

$$X(s) = \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = (sI - A)^{-1}x(0) + (sI - A)^{-1}BU(s)$$

$$= \frac{1}{\Delta}\begin{bmatrix} s^{-1}(1+3s^{-1}) & s^{-2} \\ -2s^{-2} & s^{-1} \end{bmatrix}\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \frac{1}{\Delta}\begin{bmatrix} s^{-1}(1+3s^{-1}) & s^{-2} \\ -2s^{2} & s^{-1} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} U(s)$$

Therefore,

$$X_1(s) = \frac{s^{-1}(1+3s^{-1})}{\Delta} x_1(0) + \frac{s^{-2}}{\Delta} x_2(0) + \frac{s^{-2}}{\Delta} U(s)$$

$$X_2(s) = -\frac{2s^{-2}}{\Delta} x_1(0) + \frac{s^{-1}}{\Delta} x_2(0) + \frac{s^{-1}}{\Delta} U(s)$$

Again, if we draw, from the following equation, the signal flow graph will be as shown in Fig. 4.5.

$$\begin{bmatrix} \dfrac{dx_1(t)}{dt} \\ \dfrac{dx_2(t)}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

or

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ -2x_1(t) - 3x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ u(t) \end{bmatrix}$$

Therefore,

$$\dot{x}_1(t) = x_2(t) \qquad \text{or} \qquad sX_1(s) = X_2(s) + x_1(0)$$

$$\dot{x}_2(t) = -2x_1(t) - 3x_2(t) + u(t)$$

or

$$sX_2(s) = -2X_1(s) - 3X_2(s) + U(s) + x_2(0)$$

Using the Mason's gain formula,

$$X_1(s) = \frac{s^{-2}U(s)}{\Delta} + \frac{s^{-2}x_2(0)}{\Delta} + \frac{s^{-1}(1+3s^{-1})}{\Delta} x_1(0)$$

$$X_2(s) = \frac{s^{-1}U(s)}{\Delta} - \frac{2s^{-2}x_1(0)}{\Delta} + \frac{s^{-1}x_2(0)}{\Delta}$$

where $\Delta = 1 + 3s^{-1} + 2s^{2}$.

Thus, both from the state transition matrix and the signal flow diagram, the same result is achieved.

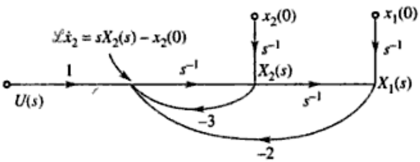


Fig. 4.5 Signal flow graph.

4.4 DEFINITION OF TRANSFER FUNCTION AND CHARACTERISTIC EQUATION

We know that $\frac{dx(t)}{dt} = Ax(t) + Bu(t) + Ew(t)$

$$y(t) = Cx(t) + Du(t) + Hw(t)$$

$$X(s) = (sI - A)^{-1}x(0) + (sI - A)^{-1}\,[BU(s) + EW(s)]$$

$$Y(s) = CX(s) + DU(s) + HW(s)$$

$$= C(sI - A)^{-1}x(0) + C(sI - A)^{-1}\,[BU(s) + EW(s)] + DU(s) + HW(s)$$

Therefore,

$$Y(s) = [C(sI - A)^{-1}\,B + D]U(s) + [C(sI - A)^{-1}E + H]W(s) \qquad [\because x(0) = 0]$$

Since the definition of a transfer function requires that the initial conditions be set to zero, the transfer function matrix between $u(t)$ and $y(t)$ when $w(t) = 0$, is

$$G_r(s) = C(sI - A]^{-1}\,B + D$$

The transfer function matrix between $w(t)$ and $y(t)$ when $u(t) = 0$ is

$$G_w(s) = C(sI - A)^{-1}\,E + H$$

Therefore,

$$Y(s) = G_r(s)U(s) + G_w(s)\,W(s)$$

A linear time-invariant system is described by the following differential equation,

$$\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + \cdots + a_1\frac{dy(t)}{dt} + a_0 y(t)$$

$$= b_m\frac{d^m u(t)}{dt^m} + b_{m-1}\frac{d^{m-1}u(t)}{dt^{m-1}} + \cdots + b_1\frac{du(t)}{dt} + b_0 u(t) \qquad \text{when } n > m$$

Utilizing the operator s

$$(s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0)Y(s) = (b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0)U(s)$$

Here $s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0 = 0$ is termed the characteristic equation. If the transfer function of the system is described by

$$G(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$

the characteristic equation is obtained by equating the denominator polynomial of the transfer function to zero. The characteristic equation from the state equations will be determined as follows.

$$G_r(s) = C(sI - A)^{-1}B + D$$

$$= C \cdot \frac{\text{Adj}(sI - A)}{|sI - A|}B + D$$

$$= \frac{C\{\text{Adj}(sI - A)\}B + |sI - A|D}{|sI - A|}$$

Hence, the characteristic equation will be $|sI - A| = 0$.

4.5 TRANSFORMATION FROM ONE SET OF DYNAMIC EQUATIONS TO ANOTHER SET OF DYNAMIC EQUATIONS

Consider the dynamic equations,

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) + Du(t)$$

Let the above dynamic equations be transformed into another set of equations with the help of the following transformations.

$$x(t) = P\bar{x}(t)$$
$$\bar{x}(t) = P^{-1}x(t)$$

The transformed dynamic equations will then be written as

$$\frac{d\bar{x}(t)}{dt} = A_1\bar{x}(t) + B_1u(t)$$

$$\bar{y}(t) = C_1\bar{x}(t) + D_1u(t)$$

Again,

$$\frac{d\bar{x}(t)}{dt} = P^{-1}\frac{dx(t)}{dt} = P^{-1}Ax(t) + P^{-1}Bu(t)$$

$$= P^{-1}AP\bar{x}(t) + P^{-1}Bu(t)$$

In the transformed dynamic equations, therefore,

$$A_1 = P^{-1}AP$$
$$B_1 = P^{-1}B$$

Also,

$$\bar{y}(t) = C_1\bar{x}(t) + D_1u(t)$$
$$= C_1P^{-1}x(t) + D_1u(t)$$

If $y(t)$ and $\bar{y}(t)$ are compared and made the same, then

$$C = C_1P^{-1} \quad \text{or} \quad CP = C_1 \quad \text{and} \quad D_1 = D$$

The transformation described here is also called *similarity transformation*.

4.6 DEFINITION OF CONTROLLABILITY AND OBSERVABILITY

A system is said to be completely state controllable if it is possible to transfer the system state from any initial state to any other desired state in a specified finite time by a control vector $u(t)$.

Knowing the output vector for a finite length of time, we can determine the initial state of the system. A system is said to be observable if every state can be completely identified by measurements of the output $y(t)$ over a finite time interval.

4.6.1 Controllability in Canonical Form

Suppose the dynamic equations of a system are expressed by

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) + Du(t)$$

and the characteristic equation of A is $|sI - A| = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 = 0$, then the dynamic equations are transformed into the controllability canonical form by the following relation.

$$P = SM$$

where $S = [B \;\; AB \;\; A^2B \ldots A^{n-1}B]$. The matrix S must have a valid inverse.

Also,

$$M = \begin{bmatrix} a_1 & a_2 & \ldots & a_{n-1} & 1 \\ a_2 & a_3 & \ldots & 1 & 0 \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ a_{n-1} & 1 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \end{bmatrix}$$

Finally, A will be transformed to A_1 and B to B_1 by the following relations:

$$A_1 = P^{-1}AP$$

$$B_1 = P^{-1}B$$

The matrix M will always have inverse because its determinant can never be equal to zero whatever may be the value of n. The result of the determinant will be either $+1$ or -1 according to the value of n.

This controllability in canonical form is termed CCF in short form. The result of the CCF always comes to a standard shape.

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -a_0 & -a_1 & -a_2 & \ldots & -a_{n-1} \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Example of the controllability canonical form (CCF)

Suppose the coefficient matrices of the state equation are:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Then,

$$|sI - A| = s\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} s-1 & -2 & -1 \\ 0 & s-1 & -3 \\ -1 & -1 & s-1 \end{bmatrix}$$

$$= (s-1)(s^2 - 2s + 1 - 3) - 1(6 + s - 1)$$

$$= s^3 - 3s^2 - s - 3$$

Now the characteristic equation $|sI - A| = s^3 - 3s^2 - s - 3 = 0$ is of the form $s^3 + a_2 s^2 + a_1 s + a_0 = 0$. Hence, $a_2 = -3$, $a_1 = -1$, $a_0 = -3$. Therefore,

$$M = \begin{bmatrix} a_1 & a_2 & 1 \\ a_1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -3 & 1 \\ -3 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

The controllability matrix is

$$S = [B \quad AB \quad A^2B]$$

where

$$B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad AB = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1\times1+2\times0+1\times1 \\ 0\times1+1\times0+3\times1 \\ 1\times1+1\times0+1\times1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$$

$$A^2B = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 5 & 8 \\ 3 & 4 & 6 \\ 2 & 4 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 9 \\ 7 \end{bmatrix}$$

Therefore,

$$S = [B \quad AB \quad A^2B] = \begin{bmatrix} 1 & 2 & 10 \\ 0 & 3 & 9 \\ 1 & 2 & 7 \end{bmatrix}$$

Since the determinant of S is not zero, it is non-singular. Hence the controllability canonical form can be determined as follows:

$$P = SM = \begin{bmatrix} 1 & 2 & 10 \\ 0 & 3 & 9 \\ 1 & 2 & 7 \end{bmatrix}\begin{bmatrix} -1 & -3 & 1 \\ -3 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ 0 & 3 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

Therefore,

$$A_1 = P^{-1}AP$$

$$= \frac{1}{9}\begin{bmatrix} 3 & 0 & -3 \\ 0 & 3 & 0 \\ 0 & 3 & 9 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & -1 & 1 \\ 0 & 3 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 0 & 3 & 0 \\ 0 & 3 & 9 \\ 9 & 12 & 18 \end{bmatrix}\begin{bmatrix} 3 & -1 & 1 \\ 0 & 3 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 0 & 9 & 0 \\ 0 & 0 & 9 \\ 27 & 9 & 27 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & 1 & 3 \end{bmatrix}$$

and

$$B_1 = P^{-1}B = \frac{1}{9}\begin{bmatrix} 3 & 0 & -3 \\ 0 & 3 & 0 \\ 0 & 3 & 9 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 0 \\ 0 \\ 9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Thus, $A_1 = P^{-1}AP$ comes into the form

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}$$

and $B_1 = P^{-1}B$ appears into the form

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

4.6.2 Observability in Canonical Form

The observability canonical form (OCF) is the dual form of CCF. Suppose the system is described by the equations

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) + Du(t)$$

where

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad C = [1 \quad 1 \quad 0]$$

The observability matrix is

$$V = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix}$$

The observability canonical form (OCF) matrix is given by $Q = (MV)^{-1}$, where

$$M = \begin{bmatrix} a_1 & a_2 & \cdots & a_{n-1} & 1 \\ a_2 & a_3 & \cdots & 1 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{n-1} & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

The OCF model of the system is described by

$$A_1 = Q^{-1}AQ \qquad C_1 = CQ \qquad B_1 = Q^{-1}B$$

When V is non-singular, then only the above transformation is possible. Now,

$$C = [1 \quad 1 \quad 0]$$

$$CA = [1 \quad 1 \quad 0]\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix} = [1 \quad 3 \quad 4]$$

$$CA^2 = [1 \quad 1 \quad 0]\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix} = [1 \quad 1 \quad 0]\begin{bmatrix} 2 & 5 & 8 \\ 3 & 4 & 6 \\ 2 & 4 & 5 \end{bmatrix} = [5 \quad 9 \quad 14]$$

Therefore,

$$V = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 4 \\ 5 & 9 & 14 \end{bmatrix}$$

Now $Q = (MV)^{-1}$, where

$$M = \begin{bmatrix} a_1 & a_2 & 1 \\ a_2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -3 & 1 \\ -3 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$MV = \begin{bmatrix} -1 & -3 & 1 \\ -3 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 4 \\ 5 & 9 & 14 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 0 & 4 \\ 1 & 1 & 0 \end{bmatrix}$$

Therefore,

$$Q = (MV)^{-1} = -\frac{1}{12}\begin{bmatrix} -4 & 2 & -4 \\ 4 & -2 & -8 \\ -2 & -2 & -2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & -\frac{1}{6} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{6} & \frac{2}{3} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{bmatrix}$$

The cofactor arrangement of the matrix Q will be

$$\begin{bmatrix} \left(\frac{1}{36} - \frac{2}{18}\right) & -\left(-\frac{1}{18} - \frac{2}{18}\right) & \left(-\frac{1}{18} - \frac{1}{36}\right) \\ -\left(-\frac{1}{36} - \frac{1}{18}\right) & \left(\frac{1}{18} - \frac{1}{18}\right) & -\left(\frac{1}{18} + \frac{1}{36}\right) \\ \left(-\frac{2}{18} - \frac{1}{18}\right) & -\left(\frac{2}{9} + \frac{1}{9}\right) & \left(\frac{1}{18} - \frac{1}{18}\right) \end{bmatrix} \cdot \begin{bmatrix} -\frac{3}{36} & \frac{3}{18} & -\frac{3}{36} \\ \frac{3}{36} & 0 & -\frac{3}{36} \\ -\frac{1}{6} & -\frac{3}{9} & 0 \end{bmatrix}$$

$$\text{Adjoint of } Q = \begin{bmatrix} -\frac{3}{36} & \frac{3}{36} & -\frac{1}{6} \\ \frac{3}{18} & 0 & -\frac{3}{9} \\ -\frac{3}{36} & -\frac{3}{36} & 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{12} & \frac{1}{12} & -\frac{1}{6} \\ \frac{1}{6} & 0 & -\frac{1}{3} \\ -\frac{1}{12} & -\frac{3}{36} & 0 \end{bmatrix}$$

$$\text{Inverse of } Q = Q^{-1} = \frac{1}{|Q|}\begin{bmatrix} -\frac{1}{12} & \frac{1}{12} & -\frac{1}{6} \\ \frac{1}{6} & 0 & -\frac{1}{3} \\ -\frac{1}{12} & -\frac{3}{36} & 0 \end{bmatrix} \quad \text{where } |Q| \text{ is the determinant of } Q.$$

$$= \frac{1}{-\frac{1}{12}}\begin{bmatrix} -\frac{1}{12} & \frac{1}{12} & -\frac{1}{6} \\ \frac{1}{6} & 0 & -\frac{1}{3} \\ -\frac{1}{12} & -\frac{3}{36} & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 0 & 4 \\ 1 & +1 & 0 \end{bmatrix}$$

Thus,

$$A_1 = Q^{-1}AQ = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 0 & 4 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}\left(-\frac{1}{12}\right)\begin{bmatrix} -4 & 2 & -4 \\ 4 & -2 & -8 \\ -2 & -2 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 2 \\ -2 & 0 & 4 \\ 1 & 1 & 0 \end{bmatrix}\left(-\frac{1}{12}\right)\begin{bmatrix} 2 & -4 & -22 \\ -2 & -8 & -14 \\ -2 & -2 & -14 \end{bmatrix}$$

$$= -\frac{1}{12}\begin{bmatrix} 0 & 0 & -36 \\ -12 & 0 & -12 \\ 0 & -12 & -36 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 3 \\ 1 & 0 & 1 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -a_0 \\ 1 & 0 & -a_1 \\ 0 & 1 & -a_2 \end{bmatrix}$$

since $|sI - A| = s^3 - 3s^2 - s - 3 = 0$ gives $a_0 = -3$, $a_1 = -1$, $a_2 = -3$.

$$C_1 = CQ = [1 \quad 1 \quad 0]\left(-\frac{1}{12}\right)\begin{bmatrix} -4 & 2 & -4 \\ 4 & -2 & -8 \\ -2 & -2 & -2 \end{bmatrix}$$

$$= -\frac{1}{12}[0 \quad 0 \quad -12] = [0 \quad 0 \quad 1]$$

$$B_1 = Q^{-1} B = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 0 & 4 \\ 1 & +1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

Thus, it is observed that B_1 does not conform to any particular form in OCF, whereas A_1 and C_1 are of the OCF form of standard shapes:

$$A_1 = \begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & \cdots & 0 & -a_2 \\ \vdots & \vdots & \cdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & 1 & -a_{n-1} \end{bmatrix}$$

$$C_1 = [0 \quad 0 \quad 0 \quad \cdots \quad 1]$$

Thus it can be concluded that in the case of CCF, s must be non-singular and in the case of OCF, V must be non-singular. For CCF, the matrices A_1 and B_1 will come to a standard form and for OCF the matrices A_1 and C_1 will come to a standard form.

4.7 DIAGONAL CANONICAL FORM

If the state and the output equations are written in the form

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) + Du(t)$$

and A has distinct eigenvalues, then there is the possibility of non-singular transformation.

With the transformation, $x(t) = Tx_1(t)$, the dynamic equations will be transformed to

$$\frac{d\overline{x}(t)}{dt} = A_1\overline{x}(t) + B_1u(t)$$

$$\overline{y}(t) = C_1\overline{x}(t) + D_1u(t)$$

where

$$A_1 = T^{-1}AT, \qquad B_1 = T^{-1}B, \qquad C_1 = CT, \qquad D_1 = D$$

The matrix A_1 will be the diagonal matrix,

$$A_1 = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

where $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ are the n-distinct eigenvalues. The coefficients B_1, C_1, D_1 will not follow any particular form.

EXAMPLE 4.2 The diagonal canonical transformation matrix T will be

$$T = [\, p_1 \quad p_2 \quad p_3 \ldots p_n]$$

where $p_1, p_2, p_3, \ldots, p_n$ are the eigenvectors for the eigenvalues $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$.

We know that

$$(\lambda_i I - A)p_i = 0 \qquad \text{or} \qquad \lambda_i p_i = Ap_i$$

Thus,

$$[\lambda_1 p_1 \quad \lambda_2 p_2 \quad \ldots \quad \lambda_n p_n] = [Ap_1 \quad Ap_2 \quad \ldots \quad Ap_n] = A[p_1 \quad p_2 \quad \ldots \quad p_n]$$

Again,

$$[p_1 \quad p_2 \quad \ldots \quad p_n]\begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix} = [\lambda p_1 \quad \lambda p_2 \quad \ldots \quad \lambda p_n] = A[p_1 \quad p_2 \quad \ldots \quad p_n]$$

or

$$[p_1 \quad p_2 \quad \ldots \quad p_n]A_1 = A[p_1 \quad p_2 \quad \ldots \quad p_n]$$

or

$$TA_1 = AT$$

or

$$A_1 = T^{-1} AT$$

It may also be observed that if A is of the controllability canonical form and it has distinct eigenvalues, then T will be

$$T = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \cdots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \cdots & \lambda_n^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \lambda_3^{n-1} & \cdots & \lambda_n^{n-1} \end{bmatrix}$$

The matrix T is also termed the *Vandermode matrix*.

EXAMPLE 4.3 Suppose

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$$

and $|\lambda I - A| = 0$ is the characteristic equation. Then,

$$\lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} = \lambda I - A$$

or

$$\lambda I - A = \begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 6 & 11 & \lambda + 6 \end{bmatrix}$$

Now the determinant of $\lambda I - A$, that is,

$$|\lambda I - A| = 0 = \begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 6 & 11 & \lambda + 6 \end{bmatrix}$$

or

$$\lambda[\lambda(\lambda + 6) + 11] + 1(6) = 0$$

or

$$\lambda^3 + 6\lambda^2 + 11\lambda + 6 = 0$$

or

$$(\lambda + 1)(\lambda + 2)(\lambda + 3) = 0$$

Since $\lambda = -1, -2, -3$ are the distinct eigenvalues and the matrix A is of controllability canonical form, therefore, the Vandermode matrix will be

$$\begin{bmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{bmatrix} = T$$

where

$$\lambda_1 = -1; \qquad \lambda_2 = -2; \qquad \lambda_3 = -3$$

Thus,

$$T = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix}$$

Hence the diagonal canonical form will be

$$A_1 = T^{-1}A\,T$$

where T^{-1} can be calculated as follows:

$$\text{Cofactor arrangement} = \begin{bmatrix} -6 & 6 & -2 \\ -5 & 8 & -3 \\ -1 & 2 & -1 \end{bmatrix}$$

$$\text{Adjoint of } T = \begin{bmatrix} -6 & -5 & -1 \\ 6 & 8 & 2 \\ -2 & -3 & -1 \end{bmatrix}$$

$$\text{Determinant of } T = 1(-18 + 12) - 1(-9 + 3) + 1(-4 + 2)$$

$$= -6 + 6 - 2 = -2$$

$$\text{Inverse of } T\text{, that is, } T^{-1} = -\frac{1}{2}\begin{bmatrix} -6 & -5 & -1 \\ 6 & 8 & 2 \\ -2 & -3 & -1 \end{bmatrix}$$

Therefore,

$$T^{-1}AT = -\frac{1}{2}\begin{bmatrix} -6 & -5 & -1 \\ 6 & 8 & 2 \\ -2 & -3 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix}$$

$$= -\frac{1}{2}\begin{bmatrix} -6 & -5 & -1 \\ 6 & 8 & 2 \\ -2 & -3 & -1 \end{bmatrix}\begin{bmatrix} -1 & -2 & -3 \\ 1 & 4 & 9 \\ -1 & -8 & -27 \end{bmatrix}$$

$$= -\frac{1}{2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

It can also be shown that

$$T = \begin{bmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{bmatrix} = [p_1 \quad p_2 \quad p_3]$$

where p_1, p_2, p_3 are eigenvectors for the eigenvalues λ_1, λ_2, and λ_3, that is,

$$p_1 = \begin{bmatrix} 1 \\ \lambda_1 \\ \lambda_1^2 \end{bmatrix} \qquad p_2 = \begin{bmatrix} 1 \\ \lambda_2 \\ \lambda_2^2 \end{bmatrix} \qquad p_3 = \begin{bmatrix} 1 \\ \lambda_3 \\ \lambda_3^2 \end{bmatrix}$$

Since $[\lambda_i I - A]\,[p_i] = [0]$, for $\lambda_0 = -1$, we have

$$\left\{-1\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}\right\}\begin{bmatrix} p_{11} \\ p_{21} \\ p_{31} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or

$$\begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ 6 & 11 & 5 \end{bmatrix}\begin{bmatrix} p_{11} \\ p_{21} \\ p_{31} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or

$$-p_{11} - p_{21} = 0$$

$$-p_{21} - p_{31} = 0$$

$$6p_{11} + 11p_{21} + 5p_{31} = 0$$

Let us assume $p_{11} = 1$. Then,

$$p_{21} = -p_{11} = -1$$

$$p_{31} = -p_{21} = +1$$

Hence,

$$p_1 = \begin{bmatrix} p_{11} \\ p_{21} \\ p_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

This satisfies the equation

$$6p_{11} + 11p_{21} + 5p_{31} = 0$$

where

$$\text{LHS} = 6p_{11} + 11p_{21} + 5p_{31} = 6 - 11 + 5 = 0 = \text{RHS}$$

Again for $\lambda_2 = -2$, we get

$$\left\{-2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}\right\}\begin{bmatrix} p_{12} \\ p_{22} \\ p_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or

$$\begin{bmatrix} -2 & -1 & 0 \\ 0 & -2 & -1 \\ 6 & 11 & 4 \end{bmatrix} \begin{bmatrix} p_{12} \\ p_{22} \\ p_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or

$$-2p_{12} - p_{22} = 0$$

$$-2p_{22} - p_{32} = 0$$

$$6p_{12} + 11p_{22} + 4p_{32} = 0$$

If p_{12} is assumed 1, then $p_{22} = -2$ and $p_{32} = -2p_{22} = 4$. Therefore,

$$p_2 = \begin{bmatrix} p_{12} \\ p_{22} \\ p_{32} \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$$

The third equation $6p_{12} + 11p_{22} + 4p_{32} = 0$ is also satisfied by the above values of p_{12}, p_{22}, and p_{32}. That is,

$$\text{LHS} = 6(1) + 11(-2) + 4(4) = 6 - 22 + 16 = 0 = \text{RHS}$$

When $\lambda = -3$,

$$\left\{ -3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \right\} \begin{bmatrix} p_{13} \\ p_{23} \\ p_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or

$$\begin{bmatrix} -3 & -1 & 0 \\ 0 & -3 & -1 \\ 6 & 11 & 3 \end{bmatrix} \begin{bmatrix} p_{13} \\ p_{23} \\ p_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or

$$-3p_{13} - p_{23} = 0$$

$$-3p_{23} - p_{33} = 0$$

$$6p_{13} + 11p_{23} + 3p_{33} = 0$$

If $p_{13} = 1$, then

$$p_{23} = -3p_{13} = -3$$

$$p_{33} = -3p_{23} = 9$$

The equation, $6p_{13} + 11p_{23} + 3p_{33} = 0$ is also satisfied with the above values of p_{13}, p_{23}, and p_{33}. That is,

$$\text{LHS} = 6p_{13} + 11p_{23} + 3p_{33} = 6 \times 1 + 11(-3) + 3 \times (9) = 6 - 33 + 27 = 0 = \text{RHS}$$

Therefore,

$$p_3 = \begin{bmatrix} p_{13} \\ p_{23} \\ p_{33} \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 9 \end{bmatrix}$$

Thus it is proved by example that

$$T = \begin{bmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{bmatrix} = [p_1 \quad p_2 \quad p_3]$$

4.8 JORDAN CANONICAL FORM

When the eigenvalues of A are not distinct, i.e. they are of multiple order, the diagonal matrix transformation is not possible. When the matrix A cannot be transformed into a diagonal matrix, then it can be transformed into almost diagonal and that is called the Jordan canonical form. This form is as follows:

$$A_1 = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & 0 & \lambda_3 \end{bmatrix}$$

where the eigenvalues are λ_1, λ_1, λ_1, λ_2, and λ_3. It means that the matrix A has a third-order eigenvalue having magnitude λ_1 and distinct eigenvalues λ_2 and λ_3.

Example of Jordan canonical form

Suppose the given matrix, $A = \begin{bmatrix} 0 & 6 & -5 \\ 1 & 0 & 2 \\ 3 & 2 & 4 \end{bmatrix}$

The characteristic equation is, $|\lambda I - A| = 0$. Therefore,

$$\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 6 & -5 \\ 1 & 0 & 2 \\ 3 & 2 & 4 \end{bmatrix} = 0$$

or

$$\begin{bmatrix} \lambda & -6 & 5 \\ -1 & \lambda & -2 \\ -3 & -2 & \lambda - 4 \end{bmatrix} = 0$$

or

$$\lambda[\lambda(\lambda - 4) - 4] + 6[-\lambda + 4 - 6] + 5[2 + 3\lambda] = 0$$

or

$$\lambda^3 - 4\lambda^2 + 5\lambda - 2 = 0$$

or

$$(\lambda - 2)(\lambda - 1)(\lambda - 1) = 0$$

Therefore the matrix A has eigenvalues 2, 1, and 1.

For $\lambda = 2$, the eigenvector $\begin{bmatrix} p_{11} \\ p_{21} \\ p_{31} \end{bmatrix}$ will be determined as follows:

$$\left\{2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 6 & -5 \\ 1 & 0 & 2 \\ 3 & 2 & 4 \end{bmatrix}\right\}\begin{bmatrix} p_{11} \\ p_{21} \\ p_{31} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or

$$\left\{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 6 & -5 \\ 1 & 0 & 2 \\ 3 & 2 & 4 \end{bmatrix}\right\}\begin{bmatrix} p_{11} \\ p_{21} \\ p_{31} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or

$$\begin{bmatrix} 2 & -6 & 5 \\ -1 & 2 & -2 \\ -3 & -2 & -2 \end{bmatrix}\begin{bmatrix} p_{11} \\ p_{21} \\ p_{31} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or

$$2p_{11} - 6p_{21} + 5p_{31} = 0 \qquad (4.9)$$

$$-p_{11} + 2p_{21} - 2p_{31} = 0 \qquad (4.10)$$

$$-3p_{11} - 2p_{21} - 2p_{31} = 0 \qquad (4.11)$$

Let us take $p_{11} = 2$, as an arbitrary number, then subtracting (4.11) from (4.10), we get

$$p_{21} = -1$$

Substituting for p_{21} in (4.9), we get

$$p_{31} = -2$$

Hence,

$$p_1 = \begin{bmatrix} p_{11} \\ p_{21} \\ p_{31} \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$$

Similarly, for $\lambda = 1$, the eigenvector $\begin{bmatrix} p_{12} \\ p_{22} \\ p_{32} \end{bmatrix}$ is given by

$$\left\{1\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 6 & -5 \\ 1 & 0 & 2 \\ 3 & 2 & 4 \end{bmatrix}\right\}\begin{bmatrix} p_{12} \\ p_{22} \\ p_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or

$$\begin{bmatrix} 1 & -6 & 5 \\ -1 & 1 & -2 \\ -3 & -2 & -3 \end{bmatrix}\begin{bmatrix} p_{12} \\ p_{22} \\ p_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or

$$p_{12} - 6p_{22} + 5p_{32} = 0 \tag{4.12}$$

$$-p_{12} + p_{22} - 2p_{32} = 0 \tag{4.13}$$

$$-3p_{12} - 2p_{22} - 3p_{32} = 0 \tag{4.14}$$

When p_{12} is arbitrarily taken as 1, solving Eqs. (4.12) to (4.14), we get

$$p_{22} = -\frac{3}{7} \quad \text{and} \quad p_{32} = -\frac{5}{7}$$

Therefore,

$$p_2 = \begin{bmatrix} p_{12} \\ p_{22} \\ p_{32} \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{3}{7} \\ -\frac{5}{7} \end{bmatrix}$$

For the next eigenvalue $\lambda = 1$, the eigenvector $\begin{bmatrix} p_{13} \\ p_{23} \\ p_{33} \end{bmatrix}$ is given by

$$\left\{1\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 6 & -5 \\ 1 & 0 & 2 \\ 3 & 2 & 4 \end{bmatrix}\right\}\begin{bmatrix} p_{13} \\ p_{23} \\ p_{33} \end{bmatrix} = -\begin{bmatrix} p_{12} \\ p_{22} \\ p_{32} \end{bmatrix}$$

or

$$\begin{bmatrix} 1 & -6 & 5 \\ -1 & 1 & -2 \\ -3 & -2 & -3 \end{bmatrix} \begin{bmatrix} p_{13} \\ p_{23} \\ p_{33} \end{bmatrix} = \begin{bmatrix} -1 \\ \frac{3}{7} \\ \frac{5}{7} \end{bmatrix}$$

or

$$p_{13} - 6p_{23} + 5p_{33} = -1 \tag{4.15}$$

$$-p_{13} + p_{23} - 2p_{33} = \frac{3}{7} \tag{4.16}$$

$$-3p_{13} - 2p_{23} - 3p_{33} = \frac{5}{7} \tag{4.17}$$

Assuming $p_{13} = 1$, arbitrarily, and solving Eqs. (4.15) to (4.17), we get

$$p_{23} = -\frac{22}{49} \quad \text{and} \quad p_{33} = -\frac{46}{49}$$

Therefore,

$$p_3 = \begin{bmatrix} p_{13} \\ p_{23} \\ p_{33} \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{22}{49} \\ -\frac{46}{49} \end{bmatrix}$$

Hence,

$$T = [p_1 \quad p_2 \quad p_3] = \begin{bmatrix} 2 & 1 & 1 \\ -1 & -\frac{3}{7} & -\frac{22}{49} \\ -2 & -\frac{5}{7} & -\frac{46}{49} \end{bmatrix}$$

The inverse of T will be calculated as follows:

$$\text{Cofactor arrangement} = \begin{bmatrix} \frac{28}{343} & -\frac{2}{49} & -\frac{1}{7} \\ \frac{11}{49} & \frac{6}{49} & -\frac{4}{7} \\ -\frac{1}{49} & -\frac{5}{49} & \frac{1}{7} \end{bmatrix}$$

$$\text{Determinant of } T = 2\left(\frac{138}{343} - \frac{110}{343}\right) - 1\left(\frac{46}{49} - \frac{44}{49}\right) + 1\left(\frac{5}{7} - \frac{6}{7}\right) = -\frac{1}{49}$$

$$\text{Inverse of } T = \frac{1}{-\frac{1}{49}}\begin{bmatrix} \frac{28}{343} & \frac{11}{49} & -\frac{1}{49} \\ -\frac{2}{49} & \frac{6}{49} & -\frac{5}{49} \\ -\frac{1}{7} & -\frac{4}{7} & \frac{1}{7} \end{bmatrix} = \begin{bmatrix} -4 & -11 & 1 \\ 2 & -6 & 5 \\ 7 & 28 & -7 \end{bmatrix}$$

Therefore,

$$A_1 = T^{-1}AT$$

$$= \begin{bmatrix} -4 & -11 & 1 \\ 2 & -6 & 5 \\ 7 & 28 & -7 \end{bmatrix}\begin{bmatrix} 0 & 6 & -5 \\ 1 & 0 & 2 \\ 3 & 2 & 4 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ -1 & -\frac{3}{7} & -\frac{22}{49} \\ -2 & -\frac{5}{7} & -\frac{46}{49} \end{bmatrix}$$

$$= \begin{bmatrix} -8 & -22 & 2 \\ 9 & 22 & -2 \\ 7 & 28 & -7 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ -1 & -\frac{3}{7} & -\frac{22}{49} \\ -2 & -\frac{5}{7} & -\frac{46}{49} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

The above form is the Jordan canonical form.

4.9 SENSITIVITY ANALYSIS

It is necessary to study the deviation of a system from its nominal behaviour on account of the changes in its parameters. That is why sensitivity analysis is very important. Three types of sensitivities have been developed.

(a) Eigenvalue sensitivity
(b) Performance sensitivity
(c) Trajectory sensitivity

The 'eigenvalue sensitivity' is not so impressive from the point of view of direct measurement of system performance. In the case of 'performance sensitivity,' the important variations in system behaviour caused by plant parameter changes may not be reflected in the *performance index*. The 'trajectory sensitivity' on the other hand is a very important measure of sensitivity.

4.9.1 Trajectory Sensitivity

Suppose the parameters of a system are represented by a vector, $K = (K_1, K_2, \ldots, K_r]^T$. The state of the system is expressed as follows:

$$\dot{x} = f(x, K, t, u)$$

where $x(t_0)$ is equal to, by definition, x^0. The nominal parameter is K_0. The parameter of the actual system is $K = K_0 + \Delta K$.

Let us assume that parameter variation ΔK from the nominal value does not affect the order of the system. Say $x = x(t, K)$ is the solution of $\dot{x} = f(x, K, t, u)$ and $x(t_0) = x^0$

The parameter change of the vector may provide the following outcome:

$$\Delta x(t, K) = x(t, K) - x(t, K_0)$$

According to Taylor series,

$$\Delta x(t, K) = \frac{\delta x}{\delta K_1}\Delta K_1 + \frac{\delta x}{\delta K_2}\Delta K_2 + \cdots$$

where all the derivatives are taken at the nominal value of K_0. Hence

$$\Delta x(t, K) = \sum_{j=1}^{r} \left.\frac{\delta x}{\delta K_j}\right|_{K_0} \Delta K_j$$

$$K_0 = [K_{10} \quad K_{20} \quad K_{30} \quad \ldots \quad K_{r0}]^T$$

where $\left.\dfrac{\delta x(t, K)}{\delta K_j}\right|_{K_0}$ is called the *trajectory sensitivity vector* and is denoted by $\sigma_j(t, K_0)$.

Again,

$$\dot{x} = f(x, K, t, u)$$

Therefore,

$$\frac{\delta \dot{x}}{\delta K_j} = \frac{\delta f}{\delta x}\cdot\frac{\delta x}{\delta K_j} + \frac{\delta f}{\delta K_j}$$

and

$$\frac{\delta x^0}{\delta K_j} = 0 \qquad (\because x(t_0) = x^0)$$

$$[j = 1, 2, \ldots, r]$$

or

$$\dot{\sigma}_j = \left.\frac{\delta f}{\delta x}\right|_{K_0} \sigma_j + \left.\frac{\delta f}{\delta K_j}\right|_{K_0}$$

and

$$\sigma_j(0) = 0 \qquad \text{for } j = 1, 2, \ldots r.$$

The above equation is termed the *trajectory sensitivity equation*. Suppose solving the above equation, we get

$$\sigma(t, K_0) = [\sigma_1 \quad \sigma_2 \quad \ldots \quad \sigma_r]$$

Then this is called the *trajectory sensitivity matrix.*

$$\Delta x(t, K) = \sigma(t, K_0)\Delta K$$

The above represents the parameter induced change of trajectory.

The sensitivity functions for the output can be obtained in the following manner. Suppose

$$y = g(x, t, u, k)$$

The sensitivity equation will be

$$\mu_j = \left.\frac{\delta y}{\delta K_j}\right|_{K_0} = \left.\frac{\delta g}{\delta x}\right|_{K_0} \sigma_j + \left.\frac{\delta g}{\delta K_j}\right|_{K_0}$$

In the case of linear time-invariant system,

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

and

$$x(t_0) = x^0$$

where

$$x = [n \times 1] \text{ matrix}$$

$$u = [p \times 1] \text{ matrix}$$

$$y = [q \times 1] \text{ matrix}$$

$$A = A(K) = [n \times n] \text{ matrix}$$

$$B = B(K) = [n \times p] \text{ matrix}$$

$$C = C(K) = [q \times n] \text{ matrix}$$

$$D = D(K) = [q \times p] \text{ matrix}$$

The following outcome will be found

$$\frac{\delta \dot{x}}{\delta K_j} = \dot{\sigma}_j = A(K_0)\sigma_j + \left.\frac{\delta A}{\delta K_j}\right|_{K_0} x(t, K_0) + \left.\frac{\delta B}{\delta K_j}\right|_{K_0} u(t)$$

where $\sigma_j(t_0) = 0,\ j = 1, 2, \ldots, r.$

$$\mu_j = \left.\frac{\delta y}{\delta K_j}\right|_{K_0} = C(K_0)\sigma_j + \left.\frac{\delta C}{\delta K_j}\right|_{K_0} x(t, K_0) + \left.\frac{\delta D}{\delta K_j}\right|_{K_0} u(t)$$

EXAMPLE 4.4 A linear system is expressed as follows:

$$\dot{x} = Ax + Bu, \qquad x(0) = 0$$
$$y = Cx$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -K_1 & -K_2 & -K_3 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

$$\frac{\delta A}{\delta K_1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \qquad \frac{\delta A}{\delta K_2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \qquad \frac{\delta A}{\delta K_3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\frac{\delta B}{\delta K_1} = \frac{\delta B}{\delta K_2} = \frac{\delta B}{\delta K_3} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{\delta C}{\delta K_1} = \frac{\delta C}{\delta K_2} = \frac{\delta C}{\delta K_3} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

Now, we know

$$\dot{\sigma}_j = A(K_0)\sigma_j + \left.\frac{\delta A}{\delta K_j}\right|_{K_0} x(t, K_0) + \left.\frac{\delta B}{\delta K_j}\right|_{K_0} u(t)$$

$$\sigma_j(t_0) = 0 \qquad j = 1, 2, \ldots, r$$

$$\mu_j = C(K_0)\sigma_j + \left.\frac{\delta C}{\delta K_j}\right|_{K_0} x(t, K_0) + \left.\frac{\delta D}{\delta K_j}\right|_{K_0} u(t)$$

$$\begin{bmatrix} \dot{\sigma}_{11} \\ \dot{\sigma}_{12} \\ \dot{\sigma}_{13} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -K_{10} & -K_{20} & -K_{30} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{12} \\ \sigma_{13} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} \dot{\sigma}_{21} \\ \dot{\sigma}_{22} \\ \dot{\sigma}_{23} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -K_{10} & -K_{20} & -K_{30} \end{bmatrix} \begin{bmatrix} \sigma_{21} \\ \sigma_{22} \\ \sigma_{23} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} \dot{\sigma}_{31} \\ \dot{\sigma}_{32} \\ \dot{\sigma}_{33} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -K_{10} & -K_{20} & -K_{30} \end{bmatrix} \begin{bmatrix} \sigma_{31} \\ \sigma_{32} \\ \sigma_{33} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Therefore,

$$\dot{\sigma}_{11} = \sigma_{12}$$

$$\dot{\sigma}_{12} = \sigma_{13}$$

$$\dot{\sigma}_{13} = -K_{10}\sigma_{11} - K_{20}\sigma_{12} - K_{30}\sigma_{13} - x_1$$

$$\dot{\sigma}_{21} = \sigma_{22}$$

$$\dot{\sigma}_{22} = \sigma_{23}$$

$$\dot{\sigma}_{23} = -K_{10}\sigma_{21} - K_{20}\sigma_{22} - K_{30}\sigma_{23} - x_2$$

$$\dot{\sigma}_{31} = \sigma_{32}$$

$$\dot{\sigma}_{32} = \sigma_{33}$$

$$\dot{\sigma}_{33} = -K_{10}\sigma_{31} - K_{20}\sigma_{32} - K_{30}\sigma_{33} - x_3$$

$$\mu_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{12} \\ \sigma_{13} \end{bmatrix} = \sigma_{11}$$

$$\mu_2 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \sigma_{21} \\ \sigma_{22} \\ \sigma_{23} \end{bmatrix} = \sigma_{21}$$

$$\mu_3 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \sigma_{31} \\ \sigma_{32} \\ \sigma_{33} \end{bmatrix} = \sigma_{31}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -K_{10} & -K_{20} & -K_{30} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

or

$$\dot{x}_1 = x_2, \; \dot{x}_2 = x_3$$

$$\dot{x}_3 = -K_{10}x_1 - K_{20}x_2 - K_{30}x_3 + u$$

The signal flow graph of the sensitivity model will be as shown in Fig. 4.6.

The block diagram of the sensitivity model can also be developed with the equations already established (see Fig. 4.7). Now

$$\dot{\sigma}_j = A(K_0)\sigma_j + \left.\frac{\delta A}{\delta K_j}\right|_{K_0} x(t, K_0) + \left.\frac{\delta B}{\delta K_j}\right|_{K_0} u(t)$$

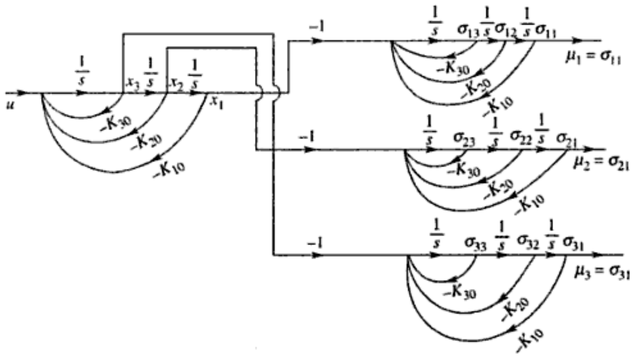


Fig. 4.6 Signal flow graph.

$$\sigma_j(t_0) = 0, \quad j = 1, 2, \ldots, r$$

$$\mu_j = C(K_0)\,\sigma_j + \left.\frac{\delta C}{\delta K_j}\right|_{K_0} x(t, K_0) + \left.\frac{\delta D}{\delta K_j}\right|_{K_0} u(t)$$

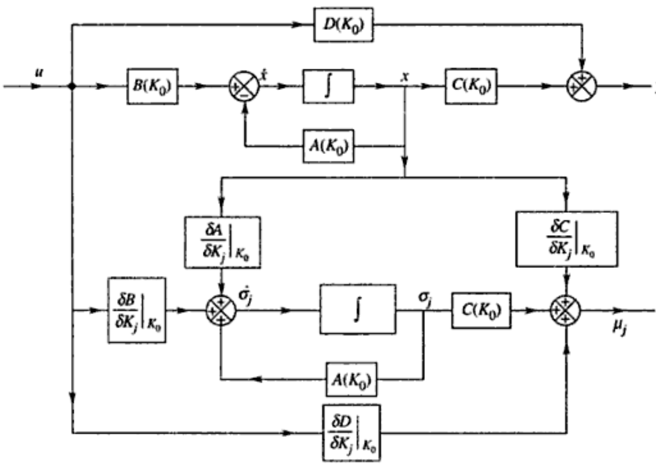


Fig. 4.7 Block diagram of the sensitivity model.

4.10 CONTROLLABLE COMPANION FORM

If a system in the state form is controllable/observable, it can be reduced through a non-singular transformation to an equivalent controllable observable system in a certain structured form which can be termed a controllable companion form/observable companion form. Suppose a system is

$$\dot{x} = Ax + Bu$$

where

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} = [b_1 \quad b_2]$$

The system is found controllable. The reason of controllability is already discussed. Hence the controllability matrix

$$u = [B \quad AB \quad A^2 \quad B]$$

$$= \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 & 4 \\ 1 & 1 & 3 & 3 & 9 & 9 \end{bmatrix}$$

Let us find out three linearly independent columns of the u matrix. That means, we have to select the inputs which are mutually independent. Let these inputs be designated as

$$u^* = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 3 \end{bmatrix}$$

Now, this matrix u^* is to be arranged in the form

$$[b_1 \quad A^{v_1-1} \cdot b_1 \quad A^{v_2-1} \cdot b_2]$$

where v_1 and v_2 are the controllability indices. In this example, the form of the linearly independent columns matrix will be

$$[b_1 \quad Ab_1 \quad b_2]$$

where $v_1 = 2$, $v_2 = 1$ and b_1 and b_2 are the first and second columns of B. Now, we have to define another term σ_K, where

$$\sigma_K = \sum_{i=1}^{K} v_i \quad \text{when } K = 1, 2, \ldots, p \ \ (p \text{ is the number of inputs})$$

Since, $B = 2 \times 2$ matrix, therefore, u will be also a 2×1 matrix. That means, $p = 2$. The modified u^* matrix will be after arranging as per order $[b_1 \quad Ab_1 \quad b_2]$

$$\bar{u} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 3 & 1 \end{bmatrix}$$

$$\sigma_1 = v_1 = 2$$

$$\sigma_2 = v_1 + v_2 = 2 + 1 = 3$$

For the transformation matrix P which will be equal to

$$\begin{bmatrix} p_1 \\ p_1 A \\ p_2 \end{bmatrix}$$

p_1 will be the σ_1th row of $\bar{u}_1^{-1}$ (inverse of $\bar{u}_1$) and p_2 will be the σ_2th row of $\bar{u}_1^{-1}$ (inverse of $\bar{u}_1$).
Determinant of $\bar{u}_1$ is

$$= 1(0 - 3) - 1(0 - 1) = -2$$

Co-factor arrangement of $\bar{u}_1$ is

$$\begin{bmatrix} -3 & 1 & 0 \\ -1 & 1 & -2 \\ 1 & -1 & 0 \end{bmatrix}$$

Re-arrangement of the co-factor, i.e. transposition is

$$\begin{bmatrix} -3 & -1 & 1 \\ 1 & 1 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

Hence the inverse of u_1 will be

$$-\frac{1}{2}\begin{bmatrix} -3 & -1 & 1 \\ 1 & 1 & -1 \\ 0 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 1.5 & 0.5 & -0.5 \\ -0.5 & -0.5 & 0.5 \\ 0 & 1 & 0 \end{bmatrix}$$

Therefore, $p_1 = \sigma_1$th row, i.e. the second row is

$$= [-0.5 \quad -0.5 \quad 0.5]$$

and $p_2 = \sigma_2$th row i.e. the third row is

$$= [0 \quad 1 \quad 0]$$

Hence the transformation matrix

$$P = \begin{bmatrix} p_1 \\ p_1 A \\ p_2 \end{bmatrix} = \begin{bmatrix} -0.5 & -0.5 & 0.5 \\ -0.5 & -1 & 1.5 \\ 0 & 1 & 0 \end{bmatrix}$$

since

$$p_1 A = [-0.5 \quad -0.5 \quad 0.5]\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = [-0.5 \quad -1 \quad 1.5]$$

If $\bar{\dot{x}} = \bar{A}\bar{x} + \bar{B}u$ is the companion form of $\dot{x} = Ax + Bu$, then

$$\bar{A} = PAP^{-1}$$
$$\bar{B} = PB$$

$$\bar{A} = \begin{bmatrix} -0.5 & -0.5 & 0.5 \\ -0.5 & -1 & 1.5 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} -0.5 & -0.5 & 0.5 \\ -0.5 & -1 & 1.5 \\ 0 & 1 & 0 \end{bmatrix}^{-1}$$

Now,

$$\begin{bmatrix} -0.5 & -0.5 & 0.5 \\ -0.5 & -1 & 1.5 \\ 0 & 1 & 0 \end{bmatrix}^{-1} = \frac{1}{-0.5(-1.5)+0.5(-0.5)} \begin{bmatrix} -1.5 & 0.5 & -0.25 \\ 0 & 0 & 0.5 \\ -0.5 & 0.5 & 0.25 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 1 & -0.5 \\ 0 & 0 & 1 \\ -1 & 1 & 0.5 \end{bmatrix}$$

Therefore,

$$\bar{A} = \begin{bmatrix} -0.5 & -0.5 & 0.5 \\ -0.5 & -1 & 1.5 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} -3 & 1 & -0.5 \\ 0 & 0 & 1 \\ -1 & 1 & 0.5 \end{bmatrix}$$

$$= \begin{bmatrix} -0.5 & -1 & 1.5 \\ -0.5 & -2 & 4.5 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} -3 & 1 & -0.5 \\ 0 & 0 & 1 \\ -1 & 1 & 0.5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ -3 & 4 & 0.5 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\bar{B} = \begin{bmatrix} -0.5 & -0.5 & 0.5 \\ -0.5 & -1 & 1.5 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 1 & 0.5 \\ 0 & 1 \end{bmatrix}$$

This companion form will help us to design a linear state variable having the specified closed loop poles. For example, we want the closed loop poles at –1, –2, –3.

If we take the characteristic equation of the calculated companion form, we get

$$|sI - \bar{A}| = 0$$

or

$$\left| s\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ -3 & 4 & 0.5 \\ 0 & 0 & 2 \end{bmatrix} \right| = 0$$

or

$$\begin{bmatrix} s & -1 & 0 \\ 3 & s-4 & -0.5 \\ 0 & 0 & s-2 \end{bmatrix} = 0$$

or

$$s[(s-4)(s-2)] + 1[3(s-2)] = 0$$

or

$$s^3 - 6s^2 + 11s - 6 = 0$$

This equation clearly indicates that the system is unstable since the signs are +, –, +. Moreover if we write the Routh's criterion, it will be

$$\begin{array}{lrr} s^3 & 1 & 11 \\ s^2 & -6 & -6 \\ s^1 & 10 & 0 \\ s^0 & -6 & \end{array}$$

Hence, in the first column, there are changes of signs. But if the characteristic equation is taken,

$$s^3 + 6s^2 + 11s + 6 = 0 \qquad \text{or} \qquad (s+1)(s+3)(s+2) = 0$$

we get our desired poles and the system is found to be stable.

Therefore some modification needs to be made to $\bar{A}$. Practically, we have to provide some feedback input with the actual input.

Now, to get the characteristic equation $s^3 + 6s^2 + 11s + 6 = 0$, the matrix A will have to be

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$$

because then $|[sI - A]|$ will be

$$\left| \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \right| = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 6 & 11 & s+6 \end{bmatrix}$$

$$= s(s^2 + 6s + 11) + 6 = s^3 + 6s^2 + 11s + 6$$

If we say, $A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$ then, $A_1 = \bar{A} + \bar{M}$

where $\bar{M}$ will be matrix developed due to the feedback input.

That means, $\bar{M} = \bar{B}\bar{N}$, where $\bar{N}$ is the feedback matrix. Now,

$$\bar{B} = \begin{bmatrix} 0 & 0 \\ 1 & 0.5 \\ 0 & 1 \end{bmatrix}$$

Let

$$\bar{N} = \begin{bmatrix} \bar{N}_{11} & \bar{N}_{12} & \bar{N}_{13} \\ \bar{N}_{21} & \bar{N}_{22} & \bar{N}_{23} \end{bmatrix}$$

Now $\bar{N}$ will be a (2×3) matrix because $\bar{B}$ is a (3×2) matrix. Therefore, from $\bar{M} = A_1 - \bar{A}$, we get

$$\begin{bmatrix} 0 & 0 \\ 1 & 0.5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{N}_{11} & \bar{N}_{12} & \bar{N}_{13} \\ \bar{N}_{21} & \bar{N}_{22} & \bar{N}_{23} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ -3 & 4 & 0.5 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 3 & -4 & 0.5 \\ -6 & -11 & -8 \end{bmatrix}$$

or

$$\begin{bmatrix} 0 & 0 & 0 \\ \bar{N}_{11} + 0.5\bar{N}_{21} & \bar{N}_{12} + 0.5\bar{N}_{22} & \bar{N}_{13} + 0.5\bar{N}_{23} \\ \bar{N}_{21} & \bar{N}_{22} & \bar{N}_{23} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 3 & -4 & 0.5 \\ -6 & -11 & -8 \end{bmatrix}$$

or

$$\bar{N}_{21} = -6, \quad \bar{N}_{22} = -11, \quad \bar{N}_{23} = -8$$

$$\bar{N}_{11} + 0.5\bar{N}_{21} = 3, \quad \bar{N}_{12} + 0.5\bar{N}_{22} = -4$$

$$\bar{N}_{13} + 0.5\bar{N}_{23} = 0.5$$

or

$$\bar{N}_{11} = 3 + 3 = 6, \quad \bar{N}_{12} = -4 + 5.5 = 1.5, \quad \bar{N}_{13} = 0.5 + 4 = 4.5$$

Therefore,

$$\bar{N} = \begin{bmatrix} 6 & 1.5 & 4.5 \\ -6 & -11 & -8 \end{bmatrix}$$

The value of N corresponding to the actual system

$$\dot{x} = Ax + Bu$$

will be

$$N = \bar{N}P$$

$$= \begin{bmatrix} 6 & 1.5 & 4.5 \\ -6 & -11 & -8 \end{bmatrix} \begin{bmatrix} -0.5 & -0.5 & 0.5 \\ -0.5 & -1 & 1.5 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -3.75 & 0 & 5.25 \\ 8.5 & 6 & -19.5 \end{bmatrix}$$

4.11 STATE FEEDBACK DECOUPLING

In linear multivariable systems, a change in any input will, in general, result in changes in all output variables. Such systems are characterized by coupling or interaction. In certain applications such interactions between controls are not desirable. The design of multivariable systems should be such that they become noninteracting or decoupled. The design objective of noninteracting or decoupled systems is to obtain a system in which each input affects only one output. The main advantage of such a design is that once noninteraction is achieved, the system is reduced to a number of single input/single output subsystems.

Suppose $u(t) = Fx(t) + Gw(t)$ where $w(t)$ is the new control input and F is the state feedback matrix. The main state equation is $\dot{x} = Ax + Bu$ and the output equation is $y = Cx$. The block diagram will be as shown in Fig. 4.8.

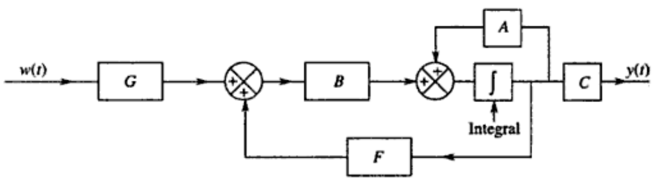


Fig. 4.8 Block diagram.

Let the differential equations of a linear multivariable system that is decoupled into first-order subsystems be expressed as

$$\dot{y}(t) = My(t) + Kw(t) = MCx(t) + Kw(t)$$

where K is a diagonal $[v_i]$ matrix such that $\prod_{i=1}^{m} v_i \neq 0$ where m is the number of subsystems. Here, M is also a diagonal matrix. Now

$$sY(s) = MY(s) + KW(s)$$

or

$$Y(s)\,[sI - M] = KW(s)$$

or

$$Y(s) = [sI - M]^{-1}\, KW(s)$$

or

$$Y(s) = \frac{AD_j[sI - M]}{|sI - M|} \cdot KW(s)$$

where $|sI - M|$ is the characteristic polynomial of the decoupled system. Again,

$$\begin{aligned}\dot{y}(t) &= C\dot{x}(t)\\ &= C[Ax(t) + Bu(t)]\\ &= C[Ax(t) + B\{Fx(t) + Gw(t)\}]\\ &= C(A + BF)x(t) + CBGw(t).\end{aligned}$$

Also,

$$\dot{y}(t) = MCx(t) + Kw(t)$$

Therefore,

$$MC = C(A + BF) = CA + CBF$$

or

$$CBF = MC - CA$$

or

$$F = (CB)^{-1}(MC - CA)$$

Also,

$$K = CBG$$

or

$$G = (CB)^{-1}K$$

Now suppose,

$$\dot{x}(t) = \begin{bmatrix}1 & 1 & 0\\ 0 & -2 & 0\\ 0 & 1 & 3\end{bmatrix}x(t) + \begin{bmatrix}1 & 1\\ -1 & 1\\ -1 & 1\end{bmatrix}u(t)$$

$$y(t) = \begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}x(t)$$

Therefore,

$$[CB] = \begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\ -1 & 1\\ -1 & 1\end{bmatrix} = \begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}$$

or

$$[CB]^{-1} = \frac{1}{2}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}$$

Therefore,

$$\begin{aligned}G &= [CB]^{-1}K\\ &= \frac{1}{2}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}v_1 & 0\\ 0 & v_2\end{bmatrix} = \frac{1}{2}\begin{bmatrix}v_1 & -v_2\\ v_1 & v_2\end{bmatrix}\end{aligned}$$

Thus,

$$F = (CB)^{-1}(MC - CA)$$

$$= \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}(MC - CA)$$

$$= \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\left\{\begin{bmatrix} m_{01} & 0 \\ 0 & m_{02} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 1 & 3 \end{bmatrix}\right\}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\left\{\begin{bmatrix} m_{01} & 0 & 0 \\ 0 & 0 & m_{02} \end{bmatrix} - \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}\right\}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} m_{01} - 1 & -1 & 0 \\ 0 & -1 & m_{02} - 3 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} m_{01} - 1 & 0 & -(m_{02} - 3) \\ m_{01} - 1 & -2 & (m_{02} - 3) \end{bmatrix}$$

If the values of F and G are made as shown above, then automatically it is found that

$$\begin{bmatrix} \dot{y}_1(t) \\ \dot{y}_2(t) \end{bmatrix} = \begin{bmatrix} m_{01} & 0 \\ 0 & m_{02} \end{bmatrix}\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} + \begin{bmatrix} v_{01} & 0 \\ 0 & v_{02} \end{bmatrix}\begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix}$$

or

$$\dot{y}_1(t) = m_{01}y_1(t) + v_{01}w_1(t)$$

$$\dot{y}_2(t) = m_{02}y_2(t) + v_{02}w_2(t)$$

Thus $\dot{y}_1(t)$ is totally decoupled with $y(t)$ and $w(t)$ and $\dot{y}_2(t)$ is totally decoupled with $y_2(t)$ and $w_2(t)$.

SUMMARY

The concept of state variable is explained. The state model of a linear time-invariant system is derived. The properties of the state transition matrix are then enumerated. The state transition equation, the transfer function, and the characteristic equation are defined. Transformation from one set of dynamic equations to another set of dynamic equations is then dealt with. Controllability and observability are defined and illustrated with examples. The diagonal canonical form is explained with examples. The Jordan canonical form is also elucidated with an example. Trajectory sensitivity and controllable campanion form are explained with examples. The idea of state feedback decoupling is also given with an example.

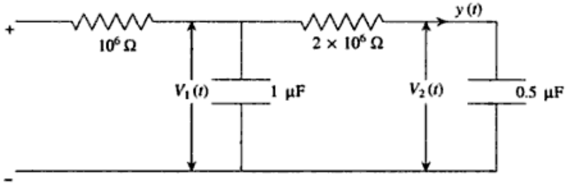
QUESTIONS

1. What is the utility of state variable characterization of a system?
2. The transfer function of a control system is given by

$$\frac{C(s)}{R(s)} = \frac{6(s+1)}{s(s+2)(s+3)}$$

Draw the state diagram and obtain the state equation.

3. Write the differential equations characterizing the network shown in the figure below and hence obtain the state equation $\dot{x} = Ax + Bu$ and the output equation $y = Cx + Du$. Take the voltages across the capacitors $V_1(t)$ and $V_2(t)$ as the two state variables and the current through the $2 \times 10^6\ \Omega$ resistor as the output variable.



4. For the system described by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Determine the state transition matrix.

5. Write the state equations for a system described by the differential equation.

$$\frac{d^3c(t)}{dt^3} + 6\frac{dc(t)}{dt} + 5c(t) = r(t)$$

6. Outline the Laplace transform method of determining the state transition matrix that is required in the solution of the state equation.
7. Obtain the state transition matrix of the system represented by the following state equations and using the same, determine the time response for $t \geq 0$.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [r]$$

The initial conditins are $x_1(0) = 1$ and $x_2(0) = -1$ and the input $r(t)$ is a unit-step function at $t = 0$.

8. Obtain the solution of a state equation $\dot{x}(t) = Ax(t) + Bu(t)$ in the state transition matrix.
9. Explain the controllability and observability of a system.

10. Predict the controllability and observability for the system

$$\dot{x}(t) = Ax(t) + Bu(t) \text{ and } y(t) = Cx(t)$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 4 & 5 & 1 \end{bmatrix}$$

11. Obtain the response of the following system using (i) the canonical transformation method and (ii) the Laplace transform method.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix}[u]$$

where $u(t) = 0 \quad \text{for } t < 0$

$= e^{-t} \quad \text{for } t \geq 0$

with $x_1(0) = x_2(0) = 0$

12. What do you mean by controllable companion form?

13. What do you mean by a multivariable control system? Explain with suitable examples.

14. Explain the decoupling of the multivariable control systems. In which respect is decoupling useful?

15. For a given system, find the plant transfer function matrix and the system poles. Also discuss the decoupling of this system. Given that:

$$A = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

16. Obtain the equivalent system in controllable companion form given that:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Chapter 5

Stability of Linear Control Systems

5.1 INTRODUCTION

This chapter describes the stability of a linear time-invariant system. In general, a stable system means that there will not be a large change in the system output when there occurs a small change in the system input, in initial conditions, or in system parameters. Usually, a linear time-invariant system is stable if the following conditions are satisfied:

(a) When the system is excited by a bounded input, the output is bounded.
(b) If the input does not exist, then the output tends towards zero irrespective of initial conditions. This is also termed *asymptotic stability.*

Suppose we take a function Ae^{-at}, where a is positive. Then it is clear that the function tends to zero as t tends to infinity. This is a clear case of asymptotic stability.

The curve shown in Fig. 5.1 (*i.e.* Ae^{-at}) is also a bounded curve. If we take the Laplace transform of Ae^{-at}, it will be $\dfrac{A}{s+a}$. The characteristic equation is $s + a = 0$. The root of the characteristic equation is $s = -a$.

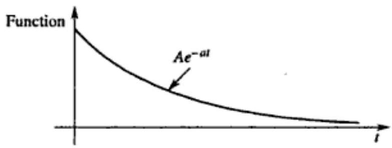


Fig. 5.1 Example of a bounded system.

Suppose the transfer function in the time response is $\dfrac{o(t)}{r(t)} = Ae^{-at}$, where $o(t)$ is the output and $r(t)$ the input. If the input $r(t)$ is the unit impulse input, then

$$o(t) = Ae^{-at}\,\delta(t)$$

The Laplace transform of the above relation is

$$O(s) = \frac{A}{s+a}$$

Thus, with the help of Laplace transformation, we arrive at a very common rule of stability, that is, if all the roots of the characteristic equation have negative real parts then the impulse response is bounded and decreases to zero. The system is termed *bounded input* with bounded stable output, and the asymptotic stability is maintained.

If, on the other hand, the function is Ae^{at} where a is a positive integer, then the function tends to infinity as t tends to infinity. This is totally an unstable system as shown in Fig. 5.2. The Laplace transform of $\frac{o(t)}{r(t)} = Ae^{at}$ with unit impulse input will be $O(s) = \frac{A}{s-a}$. The characteristic equation is $s - a = 0$, or $s = a$ is the positive root. Therefore, a system with the root of the characteristic equation having a positive real part is unbounded and unstable.

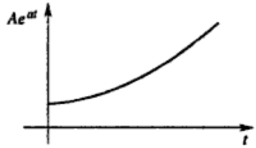


Fig. 5.2 Example of an unbounded system

If the impulse response of the system is

$$O(s) = \frac{A}{\{s-(-a+jb)\}\{s-(-a-jb)\}}$$

$$= \frac{A_1}{s-(-a+jb)} + \frac{A_2}{s-(-a-jb)}$$

then,

$$o(t) = A_1 e^{(-a+jb)t} + A_2 e^{(-a-jb)t}$$

$$= A_1 e^{-at}(\cos bt + j \sin bt) + A_2 e^{-at}(\cos bt - j \sin bt)$$

Figure 5.3 will be the curve of the above function $o(t)$. Obviously this system is bounded input, bounded output, and asymptotically stable. It is also clear that the roots of the characteristic equations are $-a + jb$ and $-a - jb$, when a and b are positive integers. The negative real parts of the roots of the characteristic equation indicate a stable system. On the other hand, if

$$O(s) = \frac{A}{\{s-(a+jb)\}\{s-(a-jb)\}} = \frac{A_1}{s-(a+jb)} + \frac{A_2}{s-(a-jb)}$$

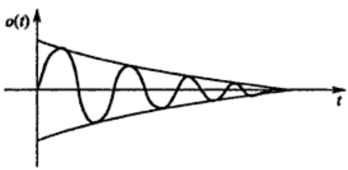


Fig: 5.3 Example of an asymptotically stable system.

then,

$$o(t) = A_1 e^{(a+jb)t} + A_2 e^{(a-jb)t}$$

$$= A_1 e^{at}(\cos bt + j \sin bt) + A_2 e^{at}(\cos bt - j \sin bt)$$

and the system is unstable. Figure 5.4 clearly indicates that the system is unbounded and unstable. When

$$\frac{O(s)}{R(s)} = \frac{1}{s^2 + a^2}$$

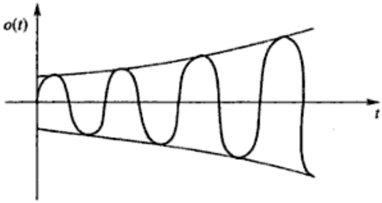


Fig. 5.4 *Example of an unbounded and unstable system.*

and

$$r(t) = \delta(t)$$

then

$$O(s) = \frac{1}{s^2 + a^2}$$

$$= \frac{1}{(s + ja)(s - ja)} = \frac{A_1}{s + ja} + \frac{A_2}{s - ja}$$

Therefore,

$$o(t) = A_1 e^{-jat} + A_2 e^{jat}$$

$$= A_1(\cos at - j \sin at) + A_2(\cos at + j \sin at)$$

$$= (A_1 + A_2) \cos at + j(A_2 - A_1) \sin at$$

$$A_1 = \lim_{s \to -ja} \frac{1}{s - ja} = \frac{1}{-2ja}$$

$$A_2 = \lim_{s \to ja} \frac{1}{s + ja} = \frac{1}{2ja}$$

Thus,

$$o(t) = j\left(\frac{1}{2ja} + \frac{1}{2ja}\right) \sin at = \frac{1}{a} \sin at$$

Figure 5.5 shows the curve of $o(t) = \frac{1}{a} \sin at$. From the curve it is quite clear that $o(t)$ does not tend to zero as $t \to \infty$ for impulse input response, although to make the system asymptotically

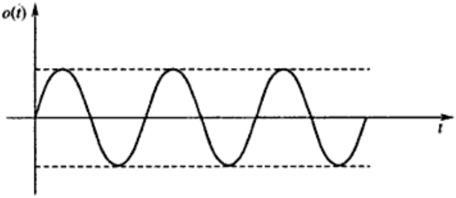


Fig. 5.5 Example of a marginally stable system.

stable, $o(t)$ should tend towards zero in the absence of the input. On the other hand, it is also quite clear that the output is bounded. Now therein lies a dilemma whether the system is acceptable or not from the point of view of stability. This case is termed ***marginally stable***.

Suppose now,

$$\frac{O(s)}{R(s)} = \frac{1}{(s^2 + a^2)^2}$$

where

$$r(t) = \delta(t)$$

then

$$O(s) = \frac{1}{(s + ja)(s - ja)(s + ja)(s - ja)}$$

$$= \frac{A_1}{s + ja} + \frac{A_2}{s + ja} + \frac{A_3}{s - ja} + \frac{A_4}{s - ja}$$

or

$$o(t) = \frac{1}{2a^3}(\sin at - at\cos at)$$

In this case, the roots of the characteristic equation are, ja, ja, $-ja$, $-ja$. Thus it is a case of repeated roots. Figure 5.6 is the curve of the above equation. It is totally unbounded and hence the system is unstable.

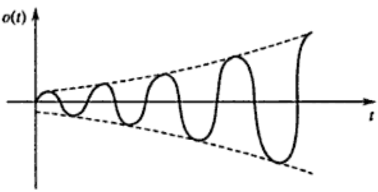


Fig. 5.6 Example of an unbounded and unstable system.

Now in the case of $\frac{O(s)}{R(s)} = \frac{A}{s}$ with $r(t) = \delta(t)$, we have

$$O(s) = \frac{A}{s} \quad \text{or} \quad o(t) = A$$

The root of the characteristic equation is zero and the curve $o(t)$ will be as shown in Fig. 5.7.

The system is found bounded but not asymptotic since $o(t)$ does not tend to zero as t tends to infinity in the absence of input. Since the asymptotic stability condition is not satisfied but the bounded output condition is, this system is termed not asymptotically stable but marginally stable.

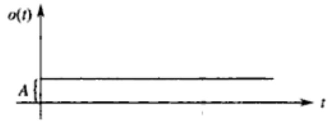


Fig. 5.7 *Example of a marginally stable system.*

If we know that a system is stable, then that generally does not serve our purpose fully. We have also to know how fast the transients die out in the system. The settling time of a pair of complex conjugate poles is inversely proportional to the real part (negative) of the roots. In other words, in the s-plane, the root of the characteristic equation moves further away from the imaginary axis towards the left-hand side and the **relative stability** of the system improves.

5.2 METHODS OF DETERMINING STABILITY BY STUDYING THE ROOTS OF THE CHARACTERISTIC EQUATION

Suppose the characteristic equation is $a_0s^n + a_1s^{n-1} + a_2s^{n-2} + \ldots + a_{n-1}s + a_n = 0$.

Sometimes it is really very difficult to find out the roots from large complicated algebraic equations. That is why the following methods have been developed for studying the stability of linear control systems.

1. If any of the coefficients of the characteristic equation is absent or if any of the coefficients of the characteristic equation is negative with the value of $a_0 > 0$, then it is observed that the system is either unstable or marginally stable.
2. The positiveness of the coefficients of the characteristic equation is the necessary and sufficient condition for stability of systems of first and second orders.
3. In the case of the characteristic equation having a higher degree than the second, it is not possible to predict the system stability when all the coefficients of the characteristic equation are positive.

A. Hurwitz and E.J. Routh have independently developed the necessary and sufficient conditions of stability of the system. The Hurwitz criterion is based on determinants and the Routh criterion is based on array formulation.

5.2.1 Hurwitz Stability Criterion

Suppose the characteristic equation of the nth order system is

$$a_0s^n + a_1s^{n-1} + \ldots + a_{n-1}s + a_n = 0$$

The Hurwitz determinant is

$$\begin{vmatrix} a_1 & a_0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ a_3 & a_2 & a_1 & a_0 & 0 & 0 & \cdots & 0 & 0 \\ a_5 & a_4 & a_3 & a_2 & a_1 & a_0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_{2n-1} & a_{2n-2} & a_{2n-3} & \cdots & \cdots & \cdots & \cdots & a_{n+1} & a_n \end{vmatrix}$$

Here the coefficients with indices larger than n are taken zero. Similarly, the coefficients with negative indices are replaced with zeros.

The condition of the stability is that the n determinants formed from the principal minors of the Hurwitz determinant will be greater than zero. That is,

$$\Delta_1 = a_1 > 0$$

$$\Delta_2 = \begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix} > 0$$

$$\Delta_3 = \begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{vmatrix} > 0$$

$$\vdots$$

$$\Delta_n = \begin{vmatrix} a_1 & a_0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ a_3 & a_2 & a_1 & a_0 & 0 & 0 & \cdots & 0 & 0 \\ a_5 & a_4 & a_3 & a_2 & a_1 & a_0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_{2n-1} & a_{2n-2} & a_{2n-3} & \cdots & \cdots & \cdots & \cdots & a_{n+1} & a_n \end{vmatrix} > 0$$

Moreover, when $\Delta_{n-1} = 0$, the system is marginally stable.

EXAMPLE 5.1 Suppose a fourth-order system has the characteristic equation

$$s^4 + 8s^3 + 18s^2 + 16s + 4 = 0$$

That is,

$$a_0 s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4 = 0$$

where

$$a_0 = 1,\ a_1 = 8,\ a_2 = 18,\ a_3 = 16,\ a_4 = 4$$

Therefore, the Hurwitz determinant is

$$\begin{vmatrix} a_1 & a_0 & 0 & 0 \\ a_3 & a_2 & a_1 & a_0 \\ a_5 & a_4 & a_3 & a_2 \\ a_7 & a_6 & a_5 & a_4 \end{vmatrix}$$

The minors of the Hurwitz determinant are:

$$\Delta_1 = a_1 = 8 > 0$$

$$\Delta_2 = \begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix} = \begin{vmatrix} 8 & 1 \\ 16 & 18 \end{vmatrix} = 144 - 16 = 128 > 0$$

$$\Delta_3 = \begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{vmatrix} = \begin{vmatrix} 8 & 1 & 0 \\ 16 & 18 & 8 \\ 0 & 4 & 16 \end{vmatrix}$$

$$= 8(288 - 32) - 1(256 - 0) = 1792 > 0$$

$$\Delta_4 = \begin{vmatrix} a_1 & a_0 & 0 & 0 \\ a_3 & a_2 & a_1 & a_0 \\ a_5 & a_4 & a_3 & a_2 \\ a_7 & a_6 & a_5 & a_4 \end{vmatrix}$$

$$= \begin{vmatrix} 8 & 1 & 0 & 0 \\ 16 & 18 & 8 & 1 \\ 0 & 4 & 16 & 18 \\ 0 & 0 & 0 & 4 \end{vmatrix}$$

$$= 4\begin{vmatrix} 8 & 1 & 0 \\ 16 & 18 & 8 \\ 0 & 4 & 16 \end{vmatrix} = 4\{8(288 - 32) - 1(256)\} = (4 \times 7 \times 256) > 0$$

Hence the system is stable.

5.2.2 Routh Stability Criterion

The Routh stability criterion is based on ordering the coefficients of the characteristic equation in the form of an array called the **Routh's array**. Let the characteristic equation be

$$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_{n-1} s + a_n = 0$$

The Routh's array is:

$$\begin{array}{lllllll}
s^n & a_0 & a_2 & a_4 & a_6 & - & - \\
s^{n-1} & a_1 & a_3 & a_5 & - & - & - \\
s^{n-2} & b_1 & b_2 & b_3 & - & - & - \\
s^{n-3} & c_1 & c_2 & - & - & - & - \\
s^{n-4} & d_1 & d_2 & - & - & - & - \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \\
s^2 & e_1 & a_n & & & & \\
s^1 & f_1 & & & & & \\
s^0 & a_n & & & & &
\end{array}$$

The coefficients

$$b_1 = \frac{a_1a_2 - a_0a_3}{a_1}$$

$$b_2 = \frac{a_1a_4 - a_0a_5}{a_1}$$

In this way, the coefficients of the third row are developed until the last coefficient of the same row is zero.

Similarly, the coefficients of the fourth, fifth , . . ., nth and $(n + 1)$th rows are evaluated as:

$$c_1 = \frac{b_1a_3 - a_1b_2}{b_1}$$

$$c_2 = \frac{b_1a_5 - a_1b_3}{b_1}$$

$$d_1 = \frac{c_1b_2 - b_1c_2}{c_1}$$

$$d_2 = \frac{c_1b_3 - b_1c_3}{c_1}$$

Arranging the array in the above pattern, the stability of the system is defined as follows. For a system to be stable, it is necessary and sufficient that each term of the first column of the Routh's array of the characteristic equation be positive when $a_0 > 0$. Moreover, if the system is unstable, then the number of sign changes of the terms of the first column of the Routh's array represents the number of the roots of the characteristic equation in the right-half of the s-plane.

5.2.3 Relation between the Routh and Hurwitz Criteria

In the case of the Routh criterion, $a_0 > 0$, $a_1 > 0$, $b_1 > 0$, $c_1 > 0$, $d > 0$, . . .

$$b_1 = \frac{a_1a_2 - a_0a_3}{a_1} = \frac{\begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix}}{a_1}$$

Again, in the case of the Hurwitz criterion,

$$a_1 > 0, \begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix} > 0$$

From the Routh criterion, we also observe that $b_1 > 0$ and $a_1 > 0$. Therefore,

$$b_1 = \frac{\begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix}}{a_1} > 0$$

since

$$a_1 > 0, \quad \begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix} > 0$$

This is also true for c_1, d_1 for the Routh criterion. Hence, the Hurwitz criterion and the Routh criterion are basically the same and draw the same conclusion.

EXAMPLE 5.2 Let the characteristic equation be

$$3s^4 + 10s^3 + 5s^2 + 5s + 1 = 0$$

The Routh array is:

$$\begin{array}{llll}
s^4 & 3 & 5 & 1 \\
s^3 & 10 & 5 & \\
s^2 & \dfrac{50-15}{10} = 3.5 & \dfrac{10\times 1 - 3\times 0}{10} = 1 & \\
s^1 & \dfrac{3.5\times 5 - 1\times 10}{3.5} = \dfrac{7.5}{3.5} & & \\
s^0 & 1 & &
\end{array}$$

Since all the coefficients in the first column are positive, the system is stable. Usually, in any row if there is any common multiple it should be withdrawn for avoiding the computational complications. For example, in the above example, the integer 5 is the common multiple in the second row. The problem can therefore also be solved as follows:

$$\begin{array}{llll}
s^4 & 3 & 5 & 1 \\
s^3 & 2 & 1 & \\
s^2 & \dfrac{2\times 5 - 1\times 3}{2} = 3.5 & \dfrac{2\times 1 - 0}{2} = 1 & \\
s^1 & \dfrac{3.5\times 1 - 2\times 1}{3.5} = \dfrac{1.5}{3.5} & & \\
s^0 & 1 & &
\end{array}$$

Here, too, it is observed that each term of the first column is positive. Hence the system is stable.

Critical examples

The following special cases are generally encountered while applying the Routh stability criterion:

1. It may so happen that the first term of any row of the Routh array is zero, then in that case the method of studying the Routh criterion is as follows.

A very small positive number ε is used in place of zero and the process is continued as per the usual procedure.

Let us consider the characteristic equation

$$s^5 + s^4 + 2s^3 + 2s^2 + 4s + 6 = 0$$

The Routh's array is:

$$\begin{array}{lll} s^5 & 1 & 2 \quad 4 \\ s^4 & 1 & 2 \quad 6 \\ s^3 & \varepsilon & -2 \\ s^2 & \dfrac{2\varepsilon+2}{\varepsilon} & 6 \\ s^1 & \dfrac{-4\varepsilon-4-6\varepsilon^2}{2\varepsilon+2} & \\ s^0 & 6 & \end{array}$$

In the fifth row, the first term is $\dfrac{-4\varepsilon-4-6\varepsilon^2}{2\varepsilon+2}$. As $\varepsilon \to 0$, it will be –2.

Hence the system is unstable because there are two changes of sign. One from the fourth row to the fifth row and the other from the fifth row to the sixth row. From this, it can also be concluded there will be two poles in the right-half of the s-plane.

2. It may so happen that all the elements of any one row of the Routh array are zero. We illustrate this through an example:

$$s^5 + 2s^4 + 6s^3 + 12s^2 + 8s + 16 = 0$$

The Routh's array is:

$$\begin{array}{llll} s^5 & 1 & 6 & 8 \\ s^4 & 1 & 6 & 8 \\ s^3 & 0 & 0 & \\ s^2 & & & \\ s & & & \\ s^0 & & & \end{array}$$

Here in the third row all the terms are zero. The auxiliary polynomial is formed from the (3 – 1) i.e. the second row. That is,

$$P(s) = s^4 + 6s^2 + 8$$

The above polynomial is differentiated with respect to s. Thus,

$$\frac{dP(s)}{ds} = 4s^3 + 12s$$

The zeros in the third row are replaced by 4 12, or 1 3.

Hence the Routh's array will be:

$$\begin{array}{llll} s^5 & 1 & 6 & 8 \\ s^4 & 1 & 6 & 8 \\ s^3 & 1 & 3 & \\ s^2 & 3 & 8 & \\ s & \frac{1}{3} & 0 & \\ s^0 & 8 & & \end{array}$$

Since there is no change in the sign in the first column, it can be demanded that there will be no root which has positive real part. But if all the terms are zero in one row, it will indicate that the roots lie on the imaginary axis which can be calculated by making

$$P(s) = 0 \quad \text{or} \quad s^4 + 6s^2 + 8 = 0$$

That is,

$$s^2 = \frac{-6 \pm \sqrt{36 - 32}}{2} = \frac{-6 \pm 2}{2} = -2, -4$$

Thus,

$$s = \pm j\sqrt{2} \qquad \text{or} \qquad \pm j2$$

Two pairs of roots lying on the imaginary axis will indicate a marginal stable system when there is no other root which has positive real part. The reason behind this is as follows:

Suppose $\frac{1}{(s^2 + a^2)(s^2 + b^2)}$ is the transfer function of a system where a and b are greater than zero and $a \neq b$. For unit impulse input, the output will be

$$\frac{1}{(s^2 + a^2)(s^2 + b^2)}$$

Now,

$$\frac{1}{(s^2 + a^2)(s^2 + b^2)} = \frac{A}{(s^2 + a^2)} + \frac{B}{(s^2 + b^2)}$$

$$= -\frac{1}{(a^2 - b^2)} \frac{1}{(s^2 + a^2)} + \frac{1}{(a^2 - b^2)} \frac{1}{(s^2 + b^2)}$$

$$= -\frac{1}{a(a^2 - b^2)} \sin at + \frac{1}{b(a^2 - b^2)} \sin bt$$

When $t \to \infty$, the above value will not tend to infinity as $a \neq b$. Therefore the system will be marginally stable.

3. If the first term of any row becomes zero, then there is the following alternative method for assessing stability.

Suppose the characteristic equation is

$$s^5 + s^4 + 2s^3 + 2s^2 + 3s + 5 = 0$$

The Routh's array is:

$$\begin{array}{llll} s^5 & 1 & 2 & 3 \\ s^4 & 1 & 2 & 5 \\ s^3 & 0 & -2 & \\ s^2 & & & \\ s & & & \\ s^0 & & & \end{array}$$

The first term of the third row is zero. Let us replace s by $\frac{1}{p}$. Thus, we get

$$\left(\frac{1}{p}\right)^5 + \left(\frac{1}{p}\right)^4 + 2\left(\frac{1}{p}\right)^3 + 2\left(\frac{1}{p}\right)^2 + 3\left(\frac{1}{p}\right) + 5 = 0$$

or

$$1 + p + 2p^2 + 2p^3 + 3p^4 + 5p^5 = 0$$

or

$$5p^5 + 3p^4 + 2p^3 + 2p^2 + p + 1 = 0$$

The Routh's array is:

$$\begin{array}{llll} p^5 & 5 & 2 & 1 \\ p^4 & 3 & 2 & 1 \\ p^3 & -\frac{4}{3} & -\frac{2}{3} & \\ p^2 & \frac{1}{2} & 1 & \\ p^1 & 2 & & \\ p^0 & 1 & & \end{array}$$

Thus there are two p roots which are in the right-hand side of the p-plane. Automatically, the two s roots will be in the right-hand side of the s-plane since s was replaced by $\frac{1}{p}$. Hence, the system is unstable.

4. If zeros are obtained in the first column twice, then the method of solution will be as follows:

The first zero is replaced by a very small number ε and the procedure is repeated. Again, if a zero is obtained as $\varepsilon \to 0$, then the auxiliary equation in the second case is determined and the original polynomial of the characteristic equation is divided by the auxiliary polynomial of the auxiliary equation to get rid of the above problem and the rest of the polynomial is then tested by the Routh's array. As an example, let the characteristic equation be:

$$2s^6 + 2s^5 + 3s^4 + 3s^3 + 2s^2 + s + 1 = 0$$

The Routh's array is:

$$\begin{array}{lcccc}
s^6 & 2 & 3 & 2 & 1 \\
s^5 & 2 & 3 & 1 & \\
s^4 & \varepsilon & 1 & 1 & \\
s^3 & \dfrac{3\varepsilon - 2}{\varepsilon} & \dfrac{\varepsilon - 2}{\varepsilon} & & \\
s^2 & \dfrac{5\varepsilon - 2 - \varepsilon^2}{3\varepsilon - 2} & 1 & & \\
s^1 & \dfrac{-2\varepsilon - \varepsilon^2}{5\varepsilon - 2 - \varepsilon^2} & & & \\
s^0 & 1 & & &
\end{array}$$

As $\varepsilon \to 0$, in the sixth row, the first term becomes zero again. Therefore, the auxiliary equation in the s^2 row will be $s^2 + 1 = 0$.

Dividing the main polynomial of the characteristic equation by the auxiliary polynomial $s^2 + 1$, we get

$$\begin{array}{r|l}
 & 2s^4 + 2s^3 + s^2 + s + 1 \\ \hline
s^2 + 1 & 2s^6 + 2s^5 + 3s^4 + 3s^3 + 2s^2 + s + 1 \\
 & 2s^6 + \quad + 2s^4 \\ \hline
 & \quad 2s^5 + s^4 + 3s^3 + 2s^2 + s + 1 \\
 & \quad 2s^5 \quad + 2s^3 \\ \hline
 & \qquad s^4 + s^3 + 2s^2 + s + 1 \\
 & \qquad s^4 \quad + s^2 \\ \hline
 & \qquad\quad s^3 + s^2 + s + 1 \\
 & \qquad\quad s^3 \quad + s \\ \hline
 & \qquad\qquad s^2 + 1 \\
 & \qquad\qquad s^2 + 1 \\ \hline
 & \qquad\qquad \times
\end{array}$$

Now, the Routh's array is developed on: $2s^4 + 2s^3 + s^2 + s + 1 = 0$. Thus, we have

s^4	2	1	1
s^3	2	1	0
s^2	ε	1	0
s	$\dfrac{\varepsilon - 2}{\varepsilon}$	0	0
s^0	1	0	0

As $\varepsilon \to 0$, $\dfrac{\varepsilon - 2}{\varepsilon} = 1 - \dfrac{2}{\varepsilon}$ will be negative. Therefore there are two sign changes, i.e. from the third row (s^2 row) to the fourth row (s-row) and from the fourth row to the fifth row (s^0 row). This indicates that the system is unstable since there are two roots in the right-half s-plane.

5.2.4 Applications of the Routh-stability Criterion

EXAMPLE 5.3 In the linear feedback system as shown in Fig. 5.8, determine the value of K for which the system is stable.

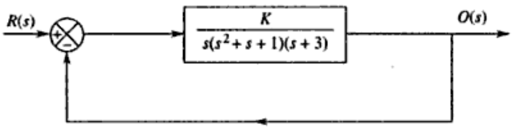


Fig. 5.8 Example 5.3.

Solution For the given system:

$$[R(s) - O(s)]\,\frac{K}{s(s^2 + s + 1)\,(s + 3)} = O(s)$$

or

$$R(s)\frac{K}{s(s^2 + s + 1)\,(s + 3)} = O(s)\left[1 + \frac{K}{s(s^2 + s + 1)(s + 3)}\right]$$

or

$$R(s)K = O(s)[s(s^2 + s + 1)(s + 3) + K]$$

or

$$\frac{O(s)}{R(s)} = \frac{K}{s(s^2 + s + 1)(s + 3) + K}$$

Therefore the characteristic equation is $s(s^2 + s + 1)(s + 3) + K = 0$
or

$$(s^3 + s^2 + s)(s + 3) + K = 0$$

or

$$s^4 + 4s^3 + 4s^2 + 3s + K = 0$$

The Routh's array is:

$$\begin{array}{llll} s^4 & 1 & 4 & K \\ s^3 & 4 & 3 & \\ s^2 & \dfrac{13}{4} & K & \\ s^1 & \dfrac{\dfrac{39}{4} - 4K}{\dfrac{13}{4}} & & \\ s^0 & K & & \end{array}$$

To make the system stable, $\dfrac{\dfrac{39}{4} - 4K}{\dfrac{13}{4}} > 0$ and $K > 0$. Thus, $\dfrac{39}{4} - 4K > 0$ and $K > 0$.

or $\quad \dfrac{39}{4} > 4K$ and $K > 0 \quad$ or $\quad \dfrac{39}{16} > K$ and $K > 0 \quad$ or $\quad \dfrac{39}{16} > K > 0$

The system will therefore be stable when the value of K lies between 0 and $\dfrac{39}{16}$. When $K = \dfrac{39}{16}$, the system will be *marginally stable* and at the time of the marginal stable condition, we have

$$\frac{13}{4}s^2 + \frac{39}{16} = 0 \quad \text{or} \quad s^2 + \frac{3}{4} = 0 \quad \text{or} \quad s = \pm j\frac{\sqrt{3}}{2}$$

EXAMPLE 5.4 In the characteristic equation $s^3 + 8s^2 + 26s + 40 = 0$, determine whether all the roots have real parts more negative than -1.

Solution Let us consider a plane p where the p-plane axis is on the left-side of the s-plane axis and the horizontal distance between the axes is 1.

Hence compared to s-plane, the origin of the p-plane will be $(-1, 0)$. If the origin is shifted from the s-plane to the p-plane, the value of p will be $s + 1$, or $s = p - 1$.

Hence the given characteristic equation will be changed to

$$(p - 1)^3 + 8(p - 1)^2 + 26(p - 1) + 40 = 0$$

or

$$p^3 + 5p^2 + 13p + 21 = 0$$

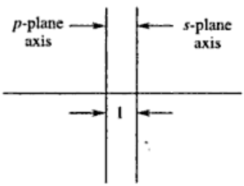


Fig. 5.9 Example 5.4.

The Routh's array is:

$$\begin{array}{lll} p^3 & 1 & 13 \\ p^2 & 5 & 21 \\ p^1 & \frac{44}{5} & \\ p^0 & 21 & \end{array}$$

Since all the terms of the first column are positive, all the roots have real parts more negative than −1. This is an example of analysis of relative stability.

EXAMPLE 5.5 How many roots with positive real parts, does each of the following polynominal possess?

(a) $s^3 + s^2 - s + 1$

(b) $s^4 - s^2 - 2s + 2$

Solution The Routh's array for the polynomial $s^3 + s^2 - s + 1$ is:

$$\begin{array}{lll} s^3 & 1 & -1 \\ s^2 & 1 & 1 \\ s^1 & -2 & \\ s^0 & 1 & \end{array}$$

Since there are two changes of sign from the, 'second row first term' to the 'third row first term, and 'third row first term' to the 'fourth row first term', there are two roots with positive real parts.

The Routh array for the polynomial $s^4 - s^2 - 2s + 2$ or $s^4 + 0.s^3 - s^2 - 2s + 2$ is:

$$\begin{array}{llll} s^4 & 1 & -1 & 2 \\ s^3 & \varepsilon & -2 & \\ s^2 & \dfrac{-\varepsilon + 2}{\varepsilon} & 2 & \\ s^1 & \dfrac{2\varepsilon - 4 - 2\varepsilon^2}{2 - \varepsilon} & & \\ s^0 & 2 & & \end{array}$$

where ε is a very small positive number which tends towards zero.

As ε tends to zero

$$\frac{-\varepsilon + 2}{\varepsilon} = -1 + \frac{2}{\varepsilon} \text{ tends to } + \infty \quad \text{and} \quad \frac{2\varepsilon - 4 - 2\varepsilon^2}{2 - \varepsilon} \text{ tends to } -2.$$

Therefore, there are two sign changes from the 'third row first term' to the 'fourth row first term' (+ to −) and from the 'fourth row first term' to the 'fifth row first term' (− to +). Hence, there are two roots with positive real parts.

EXAMPLE 5.6 Determine for what positive value of K does the polynomial $s^4 + 8s^3 + 24s^2 + 32s + K$ have roots with zero real parts.

Solution The Routh's array is:

$$\begin{array}{llll} s^4 & 1 & 24 & K \\ s^3 & 8 & 32 & \\ s^2 & 20 & K & \\ s^1 & \dfrac{640 - 8K}{20} & & \\ s^0 & K & & \end{array}$$

If $\dfrac{640 - 8K}{20} = 0$, then the s^1 row will have zero as the first term for $K = 80$, for $s = \pm j2$. Hence for $K = 80$, the polynomial will have roots with zero real parts.

SUMMARY

In this chapter, the definitions of stable system are presented. The methods of determining the stability criterion by studying the roots of the characteristic equation are described. The Hurwitz stability criterion and Routh stability criterion are explained in detail. The relation between the Routh and the Hurwitz criterion is also explained. Different examples are solved. The application of Routh-stability criterion is also shown.

QUESTIONS

1. What are the necessary and the sufficient conditions of stability for linear time-invariant systems? For what class of systems, the necessary conditions are also the sufficient ones?
2. The output angle, θ_o, of a position control system is given by
   $$\{s^3 + (4 - \mu)s^2 + (\mu + 6)s + 9\}\theta_o = 9\theta_i$$
   where θ_i is the input angle. Find the necessary and sufficient conditions on the range(s) of values of μ for which the system is stable.
3. The output angle of a position control servo system is given by the equation
   $$[D^2 + (4 - \mu)D^2 + (\mu + 6)D + 9]\theta_o = 9\theta_i$$
   Determine the necessary and sufficient conditions on the ranges of values of μ for which the system is stable.
4. Determine if the system with the following characteristic equation represents a stable system: $Y(s) = s^4 + 8s^3 + 18s^2 + 16s + 5$.
5. Determine the value(s) of the parameter K such that the unity feedback system having the open-loop transfer function
   $$G(s) = \frac{4}{s^4 + Ks^3 + (K + 4)s^2 + (K + 3)s}$$
   will have sustained oscillations.

6. State the limitations of Routh's criteria of stability.

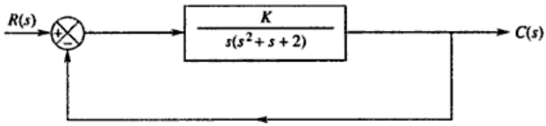
7. What do you understand by absolute stability and relative stability? Which method indicates what type of stability?

8. The characteristic equation for a feedback control system is given by

$$s^3 + 20Ks^2 + 5s^2 + 10s + 15 = 0$$

Determine the range of values of K for which the system is stable.

9. In the system shown in the figure below, find the value of K that just results in instability.

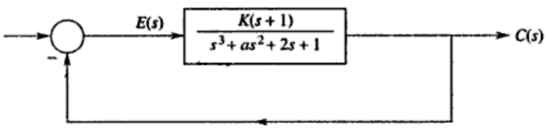


10. A feedback system has the open-loop transfer function of

$$G(s) + H(s) = \frac{Ke^{-s}}{s(s^2 + 5s + 10)}$$

Find the limiting value of K for maintaining stability.

11. A system shown in the figure below oscillates with frequency ω, if it has poles at $s = \pm j\omega$ and no poles in the right-half s-plane, determine the values of K and a so that the system shown in the figure oscillates at a frequency of 2 rad/s.



12. The open-loop transfer function of a unity feedback control system is given by

$$G(s) = \frac{K}{(s+2)(s+4)(s^2+6s+25)}$$

By applying the Routh-Hurwitz criterion, determine

(a) the range of K for which the closed-loop system will be stable

(b) the values of K which will cause sustained oscillations in the closed-loop system.

What are the corresponding oscillation frequencies?

13. A unity feedback control system has the open-loop transfer function

$$G(s) = \frac{K}{s^3 + s^2 + s - 3}$$

(a) Determine the range of values of K for which

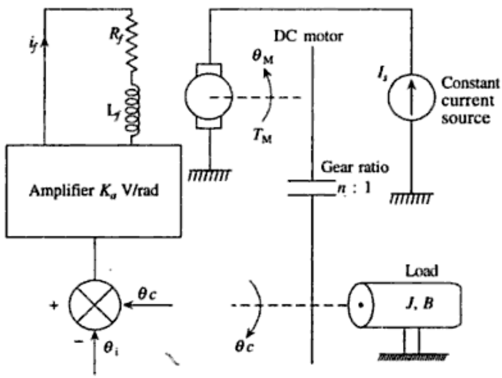
(i) the open-loop system is stable

(ii) the closed-loop system is stable

Comment on the results obtained.

(b) Find the roots of the characteristic equation of the system corresponding to the limiting values of K determined in (a) above.

14. The figure below shows a closed-loop position control system. The armature of the dc motor is fed from a constant current source and develops a torque $T_M = K_t$, which is applied to the load through a speed-reduction gear of ratio $n : 1$. The effective moment of inertia and the viscous friction damping constant of the load (including the effect of the motor and gears) are J and B, respectively. The following data are given:



$J = 6$ kg-m^2

$B = 6$ N-m/rad per second

$K_a = 2$ V/rad

$R_f = 80\ \Omega$

$L_f = 8$ H

$K_t = 20$ N-m/amp

$n = 50$

(a) Obtain the open-loop and the closed-loop transfer functions of the position control scheme.

(b) Calculate the steady-state errors in the closed-loop scheme for (i) a step-input of 1 radian (ii) constant angular velocity input of 1 radian/second.

(c) What are the effects on the performance of the closed-loop system, if the amplifier gain K_a is varied, keeping the other parameters as specified earlier? For what range of values of K_a is the system stable?

15. The open-loop transfer function of a unity feedback system is given by

$$G(s) = \frac{K}{s(1+s\tau_1)(1+s\tau_2)}$$

Using the Routh-Hurwitz criterion, determine the necessary conditions for the system to be stable.

16. By means of the Routh criterion, determine the stability of the systems represented by the following characteristic equations. Wherever necessary, find the number of roots in the right-half of s-plane and on the imaginary axis.

(i) $s^5 + s^4 + 3s^3 + 9s^2 + 16s + 10 = 0$

(ii) $s^6 + 3s^5 + 5s^4 + 9s^3 + 8s^2 + 6s + 4 = 0$

Chapter

6 Study of the Locus of the Roots of Characteristic Equation

6.1 INTRODUCTION

The determination of the roots of a characteristic equation is a tiresome job, if the degree of the characteristic polynomial is three or higher. That is why, a simple technique, i.e. the root locus technique as established by W.R. Evans is most widely used. This method is nothing but a graphical method of plotting the locus of the roots in the s-plane as the system parameter is varied over the complete range of values starting from zero to infinity.

6.2 FUNDAMENTAL IDEA OF THE ROOT LOCUS

In Fig. 6.1, the open-loop transfer function of the second-order system is

$$G(s) = \frac{K}{s(s+b)}$$

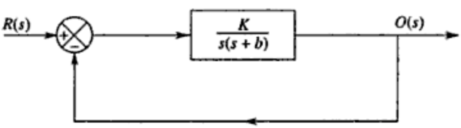


Fig. 6.1 Second-order system.

The closed-loop transfer function of the system is calculated as follows:

$$[R(s) - O(s)]\frac{K}{s(s+b)} = O(s)$$

or

$$\left[R(s)\frac{K}{s(s+b)}\right] = O(s)\left[1 + \frac{K}{s(s+b)}\right] = O(s)\left[\frac{s(s+b)+K}{s(s+b)}\right]$$

or

$$\frac{O(s)}{R(s)} = \frac{K}{s(s+b)+K} = \frac{K}{s^2+bs+K}$$

Hence the characteristic equation is $s^2 + bs + K = 0$, whose roots are

$$\frac{-b \pm \sqrt{b^2-4K}}{2}, \quad \text{or} \quad -\frac{b}{2} + \sqrt{\left(\frac{b}{2}\right)^2 - K} \quad \text{and} \quad -\frac{b}{2} - \sqrt{\left(\frac{b}{2}\right)^2 - K}$$

The system parameters are b and K which depend on natural frequency ω_n and damping coefficient ζ since $\omega_n^2 = K$ and $2\zeta\omega_n = b$.

From the above roots, the following observations can be made:

(a) For $0 \le K \le \frac{b^2}{4}$, the roots are real and distinct.

When $K = 0$, the roots are 0 and $-b$. Again 0 and $-b$ are the open-loop poles of the system since the open-loop transfer function is $\frac{K}{s(s+b)}$ and the open-loop characteristic equation is $s(s + b) = 0$.

(b) For $\frac{b^2}{4} < K < \infty$, the roots are complex conjugate with real part $-\frac{b}{2}$.

From the above data, a root locus can be drawn as shown in Fig. 6.2.

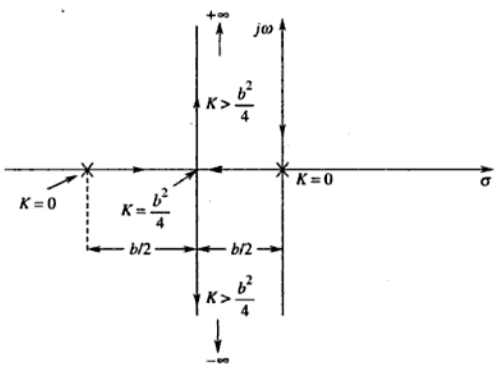


Fig. 6.2 Root locus.

The root locus starts from two points, i.e. 0 and $-b$ and approaches towards $-\frac{b}{2}$ on the horizontal axis.

Since the roots during this time lie on the real axis, the system is termed overdamped. When the two roots meet at $-\frac{b}{2}$ on the real axis, the system is critically damped. From here, one of the roots will move upwards towards $+\infty$ on the imaginary axis and the other downwards towards $-\infty$ on the imaginary axis maintaining their real value constant.

6.2.1 Correlation with Mason's Gain Formula

From the Mason's gain formula, the overall system gain is defined as

$$T = \frac{\sum_K P_K \Delta_K}{\Delta}$$

where

P_K = path gain of the Kth forward path

Δ = determinant of the graph

= 1 − (sum of the loop gains of all individual loops) + (sum of the gain products of all possible combinations of two non-touching loops) − (sum of the gain products of all possible combination of three non-touching loops) + . . .

When the above is represented in Laplace transformation, $\Delta(s) = 0$ will be the characteristic equation.

If in the generalized form, P_{mr} is taken as the gain product of the mth possible combinations of r non-touching loops of the signal flow graph, then

$$\Delta(s) = 1 - \sum_m P_{m1} + \sum_m P_{m2} - \sum_m P_{m3} + \ldots$$

$$= 1 + P(s) = 0$$

or

$P(s) = G(s)\,H(s)$ $\quad$ [$\because \Delta(s) = 1 + G(s)\,H(s)$ in the case of $T(s) = \dfrac{G(s)}{1 + G(s)H(s)}$ for a single-loop feedback system.]

Therefore,

$$P(s) = -1$$

Again s is nothing but a complex variable, therefore,

$$|P(s)| = 1$$

and

$$\angle P(s) = \pm 180°(2q + 1)$$

where

$$q = 0, 1, 2, \ldots, \text{etc.}$$

6.2.2 Method of Drawing the Root Locus

Let us consider a single-loop feedback system as shown in Fig. 6.3.

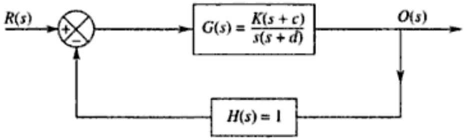


Fig. 6.3 Single-loop feedback control system.

The characteristic equation is

$$1 + G(s) = 0$$

or

$$1 + \frac{K(s+c)}{s(s+d)} = 0$$

Therefore,

$$\left|\frac{K(s+c)}{s(s+d)}\right| = 1$$

and

$$\angle\left[\frac{K(s+c)}{s(s+d)}\right] = \pm\,180°(2q+1), \text{ where } q = 0, 1, 2, \ldots$$

Since K is constant,

$$\angle\left[\frac{(s+c)}{s(s+d)}\right] = \pm\,180°(2q+1), \text{ where } q = 0, 1, 2, \ldots$$

We know that $s = \sigma + j\omega$. Therefore,

$$\angle\left[\frac{\sigma + j\omega + c}{(\sigma + jw)(\sigma + jw + d)}\right] = \pm\,180°(2q+1)$$

Let us assume

$$\tan^{-1}\frac{\omega}{\sigma+c} - \tan^{-1}\frac{\omega}{\sigma} - \tan^{-1}\frac{\omega}{\sigma+d} = -\pi$$

or

$$\tan^{-1}\frac{\omega}{\sigma} + \tan^{-1}\frac{\omega}{\sigma+d} = \pi + \tan^{-1}\frac{\omega}{\sigma+c}$$

or

$$\tan^{-1}\frac{\dfrac{\omega}{\sigma}+\dfrac{\omega}{\sigma+d}}{1-\dfrac{\omega}{\sigma}\cdot\dfrac{\omega}{\sigma+d}}=\pi+\tan^{-1}\frac{\omega}{\sigma+c}$$

or

$$\tan^{-1}\frac{\dfrac{\omega(2\sigma+d)}{\sigma(\sigma+d)}}{\dfrac{\sigma^2+d\sigma-\omega^2}{\sigma(\sigma+d)}}=\pi+\tan^{-1}\frac{\omega}{\sigma+c}$$

Applying 'tan' on both the sides, we get

$$\frac{\omega(2\sigma+d)}{\sigma^2+d\sigma-\omega^2}=\frac{\omega}{\sigma+c}$$

or

$$\frac{2\sigma+d}{\sigma^2+d\sigma-\omega^2}=\frac{1}{\sigma+c}$$

or

$$2\sigma^2+2\sigma c+d\sigma+cd=\sigma^2+d\sigma-\omega^2$$

or

$$\sigma^2+2\sigma c+cd+\omega^2=0$$

or

$$\sigma^2+2\sigma c+c^2-c^2+cd+\omega^2=0$$

or

$$(\sigma+c)^2+\omega^2=c^2-cd$$

The above is the equation of a circle (Fig. 6.4) with centre $(-c, 0)$ and radius $\sqrt{c^2-cd}$ when the x-axis is σ and the y-axis is $j\omega$.

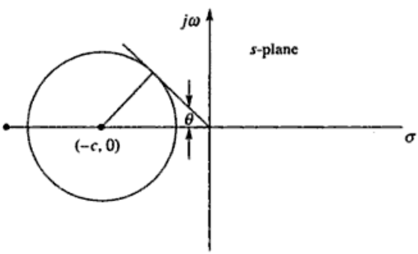


Fig. 6.4

Again, we know that the standard form of the characteristic equation for a second-order system is

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

where $s_1, s_2 = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$ with $\cos\theta = \zeta$(damping coefficient).

Hence the minimum value of the damping coefficient in Fig 6.4 will be

$$\zeta_{\min} = \cos\theta = \frac{\sqrt{c^2-(c^2-cd)}}{c} = \frac{\sqrt{cd}}{c} = \sqrt{\frac{d}{c}}$$

From the above example, it is understood that the roots of the characteristic equation of any system will certainly describe a locus.

In general, if we take

$$G(s)H(s) = \frac{K\prod_{i=1}^{m}(s+z_i)}{\prod_{j=1}^{n}(s+p_j)}$$

where $\prod_{i=1}^{m}(s+z_i)$ means $(s+z_1)(s+z_2)\ldots(s+z_m)$ and $\prod_{j=1}^{n}(s+p_j)$ means $(s+p_1)(s+p_2)\ldots(s+p_n)$, then $z_1, z_2, \ldots, z_m$ will be the zeros and $p_1, p_2, \ldots, p_n$ will be the poles of the loop transfer function. Therefore,

$$\frac{K\prod_{i=1}^{m}|s+z_i|}{\prod_{j=1}^{n}|s+p_j|} = 1$$

and

$$\sum_{i=1}^{m}\angle(s+z_i) - \sum_{j=1}^{n}\angle(s+p_j) = \pm(2q+1)\pi$$

where $q = 0, 1, 2, \ldots$.

From the above equation, it is clear that $K = \dfrac{\prod_{j=1}^{n}|s_0+p_j|}{\prod_{i=1}^{m}|s_0+z_i|}$

where s_0 is the solution point when $\sum_{i=1}^{m}\angle(s_0+z_i) - \sum_{j=1}^{n}\angle(s_0+p_j) = \pm(2q+1)\pi$ will be satisfied.

Obviously, this will throw us to a trial and error graphical method. But for coming to a quicker solution, some rules have been framed for determining an approximate sketch of the root locus.

Rules for the construction of root locus

(a) Since the roots of the characteristic equation are either real or complex conjugate or a combination of both, the locus must be symmetrical about the σ-axis of the s-plane.

(b) Since the characteristic equation can be written as

$$\prod_{j=1}^{n}(s+p_j)+K\prod_{i=1}^{m}(s+z_i) = 0$$

the root locus originates from the open-loop pole with $K = 0$ and terminates either on an open-loop zero or on infinity with $K = \infty$. Hence, the number of branches terminating on infinity is equal to the difference between the open-loop poles and zeros. When $K = 0$,

$$\prod_{j=1}^{n}(s+p_j) = 0 \qquad (j = 1, 2, \ldots, n)$$

and the root locus branches start at the open-loop poles. Again,

$$\frac{1}{K}\prod_{j=1}^{n}(s+p_j)+\prod_{i=1}^{m}(s+z_i) = 0$$

Therefore, when K tends to infinity,

$$\prod_{i=1}^{m}(s+z_i) = 0$$

Hence the roots are located at $-z_i (i = 1, 2, \ldots, m)$. Therefore, the m branches of the root locus terminate on the open-loop zeros.

If $n > m$, then it can be proved as follows that $(n - m)$ zeros will be at infinity. We know that

$$\frac{\prod_{i=1}^{m}(s+z_i)}{\prod_{j=1}^{n}(s+p_j)} = \frac{1}{K}$$

As $K \to \infty$, if we put $s \to \infty\, e^{j\phi}$, then the left-hand side and the right-hand side will be equal. Suppose

$$\frac{s+a}{(s+b)(s+c)} = \frac{1}{K} \qquad \text{or} \qquad \frac{1+\dfrac{a}{s}}{\left(1+\dfrac{b}{s}\right)(s+c)} = \frac{1}{K}$$

As $K \to \infty$, RHS $\to 0$. As $s \to \infty\, e^{j\phi}$, LHS $\to \dfrac{1}{1\cdot\infty} \to 0$. Hence, LHS = RHS.

(c) The locus will describe any point on the real axis when the sum of the numbers of the open-loop poles and zeros on the real axis to that point is odd.

This can be proved very easily in the following manner:

$$\angle G(s)H(s) = \angle \frac{\prod_{i=1}^{m}(s+z_i)}{\prod_{j=1}^{n}(s+p_j)} = (m_r - n_r)\ 180^\circ = \pm\ (2q+1)\pi, \qquad q = 0, 1, 2$$

since the poles and zeros m_r and n_r on the real axis will make an angle of 180° to the right of the point.

In Fig. 6.5, P is a pole on the real axis and A is a point. Here P is on the right of A. Hence P will describe an angle 180° at A.

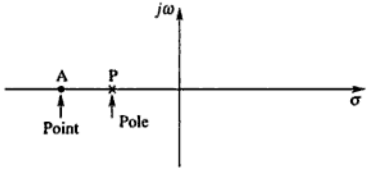


Fig. 6.5

Since $(m_r - n_r)\pi = \pm\ (2q + 1)\pi$ (for $q = 0, 1, 2, \ldots$), $(m_r - n_r)$ is an odd number. When $(m_r - n_r)$ is an odd number, obviously $(m_r + n_r)$ is also odd.

Suppose the loop transfer function of a system is

$$G(s)H(s) = \frac{K_1(s+1)(s+2)}{s(s+3)(s+4)}$$

The poles are at O, C and D positions (Fig. 6.6). The zeros are at A and B positions. On the right-side of A, the number of poles and zeros is only one, i.e. one pole at O. This being an odd

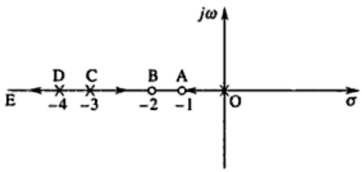


Fig. 6.6

number, the root locus will lie on the real axis OA. Hence the path of the root locus will be from the pole at O to zero at A. Again, to the right-side of pole C, the sum of the number of poles and zeros is 3, therefore, the root locus will lie on CB. That means from pole C the root locus will reach to the zero B.

To the right-side of pole D, the number of poles and zeros is four; it is an even number; therefore, the root locus will not lie on DC, rather the root locus will be on the left-side of D. The pole at D will move towards infinity through the real axis line DE as shown by the arrow. Thus, here it is very much clear that the number of branches of the root locus which will move towards infinity is equal to $(n - m)$ where n is the number of poles and m is the number of zeros.

(d) Usually the number of branches of the root locus which tend to infinity are described along straight line asymptotes whose angles are expressed as

$$\phi_A = \frac{(2q+1)\pi}{n-m} \qquad q = 0, 1, 2, \ldots, (n - m - 1)$$

The centroids of the asymptotes will be given by

$$\frac{\Sigma \text{ real parts of poles } - \Sigma \text{ real parts of zeros}}{\text{number of poles } - \text{ number of zeros}}$$

For example, in the case of $G(s)H(s) = \dfrac{K_1(s+1)(s+2)}{s(s+3)(s+4)}$, the centroid of the asymptote will be

$$\frac{-3-4-(-1-2)}{3-2} = -4$$

that is, the point D in Fig. 6.6.

The angles of the asymptotes will be

$$\frac{(2q+1)\,180°}{3-2} = 180°, \text{ with } q = 0 \qquad (\because n - m - 1 = 3 - 2 - 1)$$

(e) The points at which the multiple roots of the characteristic equation exist are termed breakaway points. The solution of $\dfrac{dK}{ds} = 0$ are the breakaway points. [K is the parameter of the loop transfer function $G(s)H(s)$.]

For example, if the characteristic equation $1 + G(s)\,H(s) = 0$ has a multiple root at $s = -a$ of multiplicity r, then $1 + G(s)\,H(s) = (s + a)^r A(s)$ where $A(s)$ does not contain the factor $(s + a)$. Now,

$$\frac{d}{ds}[1 + G(s)H(s)] = r(s + a)^{r-1} A(s) + (s + a)^r \frac{dA(s)}{ds}$$

$$= (s + a)^{r-1} \left[rA(s) + (s+a)\frac{dA(s)}{ds} \right]$$

At $s = -a$, the right-hand side is zero. Therefore,

$$\frac{d}{ds}[G(s)H(s)] = 0 \quad \text{at} \quad s = -a$$

Again, when we put $s = -a$ in the equation

$$1 + G(s)H(s) = (s + a)^r A(s)$$

then $1 + G(s)H(s)$ becomes equal to zero, and we get the characteristic equation. Suppose

$$1 + G(s)H(s) = 1 + K\frac{B(s)}{C(s)} = 0$$

then,

$$G(s)H(s) = K\frac{B(s)}{C(s)}$$

Therefore,

$$\frac{d}{ds}[G(s)H(s)] = \frac{K[B'(s)\cdot C(s) - C'(s)B(s)]}{[C(s)]^2}$$

or

$$B'(s)C(s) - C'(s)B(s) = 0$$

Again

$$1 + K\frac{B(s)}{C(s)} = 0$$

or

$$K = -\frac{C(s)}{B(s)}$$

From the above equation, it is also observed that the value of K changes with the change in the value of s and this change can be found by evaluating $\frac{dK}{ds}$ as

$$\frac{dK}{ds} = -\left[\frac{C'(s)B(s) - B'(s)C(s)}{[B(s)]^2}\right]$$

Now in the case of multiple roots, we have already proved that

$$B'(s)C(s) - C'(s)B(s) = 0$$

Therefore, $\frac{dK}{ds} = 0$ is the condition for the breakaway point. Hence a breakaway point on the real axis is defined as that point on the real axis where two or more branches of the root locus depart from or arrive at the real axis. Figures 6.7 and 6.8 explain this by showing σ_b as the breakaway point.

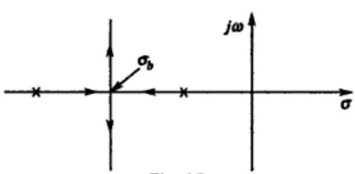


Fig. 6.7

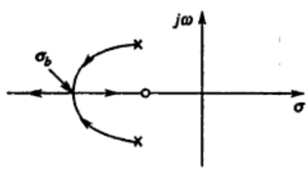


Fig. 6.8

EXAMPLE 6.1 Show that the breakaway point σ_b on the real axis satisfies

$$\sum_{i=1}^{n} \frac{1}{\sigma_b + p_i} = \sum_{i=1}^{n} \frac{1}{\sigma_b + z_i}$$

Solution Suppose the open-loop gain of a system is expressed as

$$K\frac{(s+z_1)(s+z_2)\dots(s+z_m)}{(s+p_1)(s+p_2)\dots(s+p_n)} = K\frac{\mathrm{N}}{\mathrm{D}}$$

Therefore the characteristic equation will be

$$1 + K\frac{\mathrm{N}}{\mathrm{D}} = 0 \quad \text{or} \quad K\frac{\mathrm{N}}{\mathrm{D}} = -1 \quad \text{or} \quad K = -\frac{\mathrm{D}}{\mathrm{N}}$$

At the breakaway point,

$$\frac{dK}{ds} = 0 = \frac{d}{ds}\left[\frac{(s+p_1)(s+p_2)\dots(s+p_n)}{(s+z_1)(s+z_2)\dots(s+z_m)}\right]$$

or

$$\frac{\left[\frac{d}{ds}(s+p_1)(s+p_2)\dots(s+p_n)\right]\left[(s+z_1)(s+z_2)\dots(s+z_m)\right] - \left[\frac{d}{ds}(s+z_1)(s+z_2)\dots(s+z_m)\right]\left[(s+p_1)(s+p_2)\dots(s+p_n)\right]}{[(s+z_1)(s+z_2)\dots(s+z_m)]^2} = 0$$

We know from calculus,

$$\frac{d}{dx}[u_1\ u_2\ u_3\ \dots\ u_n] = [u_1\ u_2\ u_3\ \dots\ u_n]\left[\frac{\frac{\partial u_1}{\partial x}}{u_1} + \frac{\frac{\partial u_2}{\partial x}}{u_2} + \frac{\frac{\partial u_3}{\partial x}}{u_3} + \dots + \frac{\frac{\partial u_n}{\partial x}}{u_n}\right]$$

Therefore, $\frac{dK}{ds} = 0$ reduces to

$$\frac{\mathrm{N}\cdot\mathrm{D}\left[\frac{\frac{\partial(s+p_1)}{\partial s}}{s+p_1} + \frac{\frac{\partial(s+p_2)}{\partial s}}{s+p_2} + \dots + \frac{\frac{\partial(s+p_n)}{\partial s}}{s+p_n}\right] - \mathrm{D}\cdot\mathrm{N}\left[\frac{\frac{\partial(s+z_1)}{\partial s}}{s+z_1} + \frac{\frac{\partial(s+z_2)}{\partial s}}{s+z_2} + \dots + \frac{\frac{\partial(s+z_m)}{\partial s}}{s+z_m}\right]}{N^2} = 0$$

or

$$\mathrm{N}\cdot\mathrm{D}\left[\frac{1}{s+p_1} + \frac{1}{s+p_2} + \dots + \frac{1}{s+p_n}\right] - \mathrm{D}\cdot\mathrm{N}\left[\frac{1}{s+z_1} + \frac{1}{s+z_2} + \dots + \frac{1}{s+z_m}\right] = 0$$

Since the breakaway point is on the real axis at σ_b, then the above can be written as follows.

$$N \cdot D\left[\frac{1}{\sigma_b + p_1} + \frac{1}{\sigma_b + p_2} + \ldots + \frac{1}{\sigma_b + p_n}\right] - D \cdot N\left[\frac{1}{\sigma_b + z_1} + \frac{1}{\sigma_b + z_2} + \ldots + \frac{1}{\sigma_b + z_m}\right] = 0$$

or

$$\sum_{i=1}^{n} \frac{1}{\sigma_b + p_i} = \sum_{i=1}^{m} \frac{1}{\sigma_b + z_i}. \text{ Proved.}$$

EXAMPLE 6.2 Determine the breakaway point of the system whose open-loop transfer function is $G(s)H(s) = \dfrac{K}{s(s+3)^2}$.

Solution Since there is no zero in the open-loop transfer function,

$$\sum_{i=1}^{n} \frac{1}{\sigma_b + p_i} = 0$$

where σ_b is the breakaway point on the real axis. All the poles of the open-loop transfer function, that is, 0, –3, –3 are on the real axis and, therefore, the breakaway point will also be on the real axis. Now,

$$\frac{1}{\sigma_b} + \frac{1}{\sigma_b + 3} + \frac{1}{\sigma_b + 3} = 0$$

or

$$\frac{1}{\sigma_b} = -\frac{2}{\sigma_b + 3} \quad \text{or} \quad \sigma_b = -1$$

EXAMPLE 6.3 Determine the root locus of the system having open-loop gain

$$\frac{K(s+2)}{(s+1+j\sqrt{3})(s+1-j\sqrt{3})}.$$

Solution Poles of the open-loop gain are $-1 + j\sqrt{3}$ and $-1 - j\sqrt{3}$.

The zero of the open-loop gain is = –2.

The centroid of the asymptote $= \dfrac{-1-1-(-2)}{2-1} = 0$

The angle of the asymptotic line $= \dfrac{(2q+1)\pi}{2-1} = \pi \qquad (\because q = n - m - 1 = 2 - 1 - 1 = 0)$

The characteristic equation is

$$1 + \frac{K(s+2)}{(s+1+j\sqrt{3})(s+1-j\sqrt{3})} = 0$$

or

$$K(s + 2) = -[(s + 1)^2 + 3] \quad \text{or} \quad K = -\frac{s^2 + 2s + 4}{s + 2}$$

or

$$\frac{dK}{ds} = 0 = \frac{(2s + 2)(s + 2) - s^2 - 2s - 4}{(s + 2)^2} \quad \text{or} \quad s = 0 \quad \text{or} \quad -4.$$

The root locus will exist on the right-side of the zero (i.e. -2) on account of the existence of an odd number of poles and zeros on the real axis, i.e. one. Therefore, the breakaway point will be -4. Of course we are always considering K as a positive number, starting from zero and approaching towards ∞. Hence the root locus will be as shown in the Fig. 6.9.

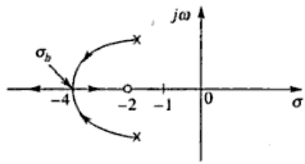


Fig. 6.9 *Example 6.3.*

EXAMPLE 6.4 Show whether the root locus for the Example 6.3 will be changed when the parameter K is negative.

Solution If the value of K is negative, then the root locus will be changed. The reason is as follows.

The characteristic equation of the system as shown in Example 6.3 is

$$1 + \frac{K(s + 2)}{(s + 1 + j\sqrt{3})(s + 1 - j\sqrt{3})} = 0$$

or

$$1 + \frac{K(s + 2)}{s^2 + 2s + 1 + 3} = 0$$

or

$$s^2 + s(K + 2) + 4 + 2K = 0$$

Applying Routh's criterion, we get

$$\begin{array}{lll} s^2 & 1 & 4 + 2K \\ s & K + 2 & 0 \\ s^0 & 4 + 2K & \end{array}$$

When the value of K is positive, $K + 2$ and $4 + 2K$ will be positive. Hence the root locus will not intersect the $j\omega$-axis. That is why, the breakaway point which was taken -4 is absolutely correct. But, if K is negative, then at the value of $K = -2$, the root locus will touch or intersect the $j\omega$-axis.

In other words at $K = -2$,

$$s^2 + (4 + 2K) = 0 \quad \text{or} \quad s^2 = 0 \quad \text{or} \quad s = 0, 0.$$

Hence the root locus will touch the $j\omega$-axis at the origin. The breakaway point will be at zero instead of -4. Therefore, the root locus will be as shown in Fig. 6.10.

At $K = -3$, obviously from the Routh's criterion, it is clear that the system will be unstable. That means the root locus will move to the right-side of $j\omega$-axis.

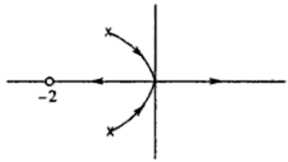


Fig. 6.10 Example 6.4.

EXAMPLE 6.5 Determine the root locus of the system whose open-loop gain is

$$G(s)H(s) = \frac{K}{s(s+4)(s^2+4s+20)}$$

Solution The poles of the open-loop gain are $0, -4, -2 + j4, -2 - j4$.

$$\text{The centroid of the asymptotes} = \frac{0-4-2-2}{4} = -\frac{8}{4} = -2$$

The angles of the asymptotes are:

$$= \frac{(2q+1)\pi}{4} \quad \text{with } q = 0, 1, 2, 3 \qquad (\because n - m - 1 = 4 - 0 - 1 = 3)$$

$$= 45°, 135°, 225°, 315°$$

Again

$$K = -s(s+4)(s^2+4s+20)$$

$$= -(s^4 + 8s^3 + 36s^2 + 80s)$$

and

$$\frac{dK}{ds} = -(4s^3 + 24s^2 + 72s + 80) = 0$$

or

$$s^3 + 6s^2 + 18s + 20 = 0$$

or

$$(s+2)(s^2+4s+10) = 0$$

Therefore, the breakaway points will be -2 or $-2 \pm j\sqrt{6}$

Figure 6.11 shows the root locus. The intersection points of the root locus on the $j\omega$-axis can also be found from the Routh's criterion.

The characteristic equation is

$$s(s+4)(s^2+4s+20) + K = 0 \quad \text{or} \quad s^4 + 8s^3 + 36s^2 + 80s + K = 0$$

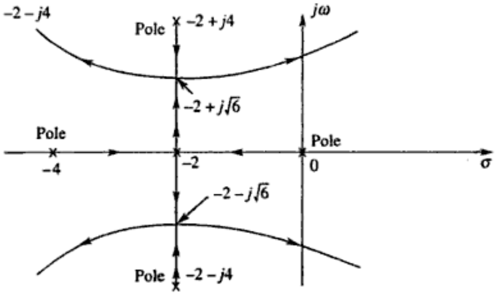


Fig. 6.11 *Example 6.5.*

The Routh's criterion is

s^4	1	36	K
s^3	1	10	
s^2	26	K	
s	$\frac{260-K}{26}$	0	
s^0	K		

When $\frac{260-K}{26} = 0$, the locus will intersect at $K = 260$.

The value of s at that time will be

$$26s^2 + 260 = 0 \qquad \text{or} \qquad s = \pm j\sqrt{10}$$

Hence the intersecting points on the $j\omega$-axis will be $j\sqrt{10}$ and $-j\sqrt{10}$.

EXAMPLE 6.6 Determine the root locus of the system whose open-loop transfer function is

$$G(s)H(s) = \frac{K(s+2)}{(s+1)^2}$$

Solution Poles and zeros of the open-loop gain are:

$$\text{Pole} = -1, -1$$

$$\text{Zeros} = -2$$

For the breakaway point

$$\frac{dK}{ds} = \frac{-d}{ds}\left[\frac{(s+1)^2}{s+2}\right] = 0$$

or

$$\frac{d}{ds}\left[\frac{s^2+2s+1}{s+2}\right]=0$$

or

$$s^2 + 4s + 3 = 0$$

or

$$s = -3 \quad \text{or} \quad -1$$

For the asymptote:

$$\text{Centroid} = \frac{-1-1-(-2)}{2-1} = 0$$

$$\text{Angle} = \frac{(2q+1)\pi}{2-1} \quad \text{with } q = 0$$

If we apply the formula, $\sum_{i=1}^{n} \frac{1}{\sigma_b + p_i} = \sum_{i=1}^{m} \frac{1}{\sigma_b + z_i}$, we get

$$\frac{1}{\sigma_b+1}+\frac{1}{\sigma_b+1}=\frac{1}{\sigma_b+2} \qquad \text{or} \qquad \sigma_b = -3$$

Hence $\sigma_b = -3$ must be the breakaway point.

The root locus will be as shown in Fig. 6.12. The locus will start from the poles (– 1, – 1) and reach the breakaway point at – 3. Then one path will go towards the zero at – 2 and the other path to infinity.

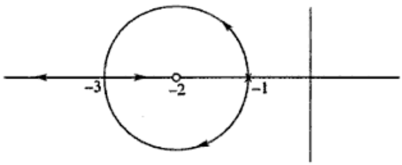


Fig. 6.12 Example 6.6.

6.3 CONSTRUCTION OF ROOT CONTOUR

So far, we have studied the root locus of the closed-loop feedback system when the open-loop gain parameter K is varying. Suppose that both the gain parameter K and the poles vary, then the root locus plots obtained are termed *root contours*. For example, say, the characteristic equation is $s^2 + as + K = 0$, where both a and K are varying. Therefore,

$$1 + \frac{K}{s^2+as} = 0 \qquad \text{or} \qquad 1+\frac{K}{s(s+a)} = 0$$

Here in the open-loop, the transfer function is $\frac{K}{s(s+a)}$ where both K and the pole $-a$ are varying. From the characteristic equation

$$s^2 + as + K = 0 \qquad \text{or} \qquad 1 + \frac{as}{s^2 + K} = 0$$

it can be easily assumed, as if the a is the gain parameter, where

$$a = -\frac{(s^2 + K)}{s}$$

Now for determining the breakaway point,

$$\frac{da}{ds} = 0 = \frac{d}{ds}\left[-\frac{(s^2 + K)}{s}\right] = \frac{2s \cdot s - (s^2 + k)}{s^2}$$

or

$$2s^2 - s^2 - K = 0 \qquad \text{or} \qquad s = \pm\sqrt{K}$$

Applying the Routh's criterion, we can also calculate the intersection point of the locus with the $j\omega$-axis.

s^2	1	K
s	a	0
s^0	K	

When $a = 0$, the intersection point will be found by the following equation:

$$s^2 + K = 0 \qquad \text{or} \qquad s = \pm j\sqrt{K}$$

Since in $\frac{as}{s^2 + K}$, the poles are $s = \pm j\sqrt{K}$, and the zero is $s = 0$, for different values of K, one path of the root locus will approach zero, i.e. the origin in this example and the other path will be towards ∞ and the breakaway point will be $\sqrt{K}$ on the real axis. Figure 6.13 shows the root contour.

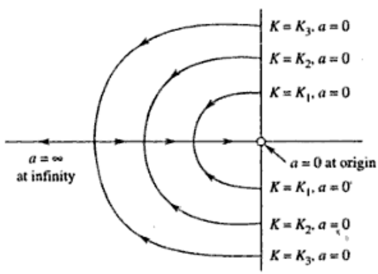


Fig. 6.13 *Root contour.*

EXAMPLE 6.7 If the open-loop transfer function is $\frac{K}{s(s+1)(s+\beta)}$ where both K and β are varying, find the root contour of the feedback control system.

Solution The characteristic equation of the system is

$$1+\frac{K}{s(s+1)(s+\beta)}=0 \qquad \text{or} \qquad s(s+1)(s+\beta)+K=0$$

or

$$s^2(s+1)+\beta(s+1)s+K=0$$

or

$$1+\frac{\beta s(s+1)}{s^2(s+1)+K}=0$$

In this problem, first of all β is to be assumed 0, then the characteristic equation will be

$$s^2(s+1)+K=0$$

or

$$1+\frac{K}{s^2(s+1)}=0$$

The poles of the open-loop transfer function are 0, 0 and –1.

$$\text{Centroid} = -\frac{1}{3}$$

$$\text{Angle of asymptote} = \frac{(2q+1)}{3}\pi \quad \text{with } q = 0, 1, 2$$

$$= 60°, 180°, 300°$$

For the breakaway point

$$\frac{dK}{ds} = -\frac{d}{ds}[s^2(s+1)] = 0$$

or

$$s(3s+2)=0$$

i.e.

$$s=0 \quad \text{or} \quad s=-\frac{2}{3}$$

It is very clear that the breakaway point will be zero since between 0 and –1 no root locus will exist on the real axis as the number of poles and zeros on the right-side of the pole –1 are even, i.e. 2 poles.

Hence the breakaway point will be at 0, not at $-\frac{2}{3}$. Figure 6.14 shows the root locus for $\beta = 0$.

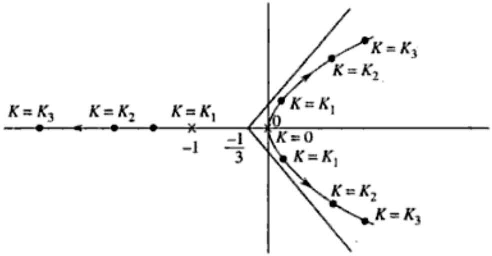


***Fig. 6.14** Example 6.7: Root locus for $\beta = 0$.*

Now, the root contours are drawn by increasing the value of β from 0 to ∞.

If the Routh's criterion is applied to the characteristic equation,

$$s^2(s + 1) + \beta s(s + 1) + K = 0$$

the point of intersection on the $j\omega$-axis for each value of K can be determined.

From the characteristic equation $s^3 + s^2(1 + \beta) + \beta s + K = 0$, the Routh's criterion is

$$\begin{array}{lcc} s^3 & 1 & \beta \\ s^2 & 1+\beta & K \\ s & \dfrac{(1+\beta)\beta - K}{1+\beta} & 0 \\ s^0 & K & \end{array}$$

Thus,

$$\frac{(1+\beta)\beta - K}{1+\beta} = 0 \quad \text{or} \quad \beta^2 + \beta - K = 0$$

or

$$\beta = \frac{-1 \pm \sqrt{1+4K}}{2}$$

Taking β as positive, i.e. $\beta = \dfrac{-1+\sqrt{1+4K}}{2}$, we get from $s^2(1 + \beta) + K = 0$, $s = \pm j\sqrt{\dfrac{K}{1+\beta}}$, which will be the intersection point on the $j\omega$-axis at the value $\beta = \dfrac{-1+\sqrt{1+4K}}{2}$.

The breakaway point on the real axis will be determined as follows:

$$\beta = \frac{-\left[s^2(s+1)+K\right]}{s(s+1)}$$

or

$$\frac{d\beta}{ds} = \frac{-d}{ds}\left[\frac{s^2(s+1)+K}{s(s+1)}\right] = 0$$

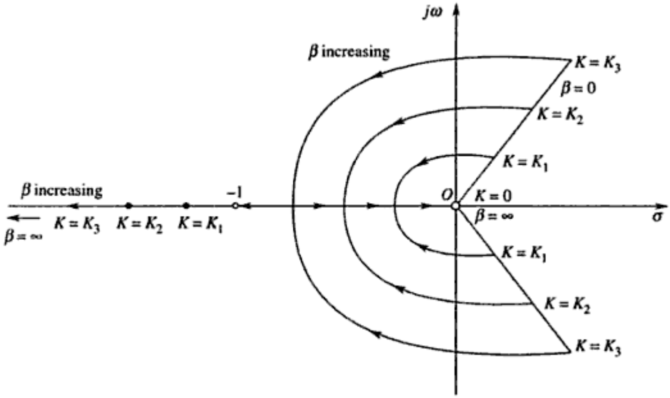


Fig. 6.15 Example 6.7: Root contour.

or

$$s^4 + 2s^3 + s^2 - 2sK - K = 0$$

For different values of K, the factorization is made to get the value of s on the real axis. Suppose for $K = 1$, it is σ. Then,

$$\sigma^4 + 2\sigma^3 + \sigma^2 - 2\sigma - 1 = 0$$

Say

$$y = \sigma^4 + 2\sigma^3 + \sigma^2 - 2\sigma - 1$$

Draw a curve y vs. σ and find out from the curve at what value of σ, y is zero. This will be the breakaway point. The graphical method is better when normal factorization is cubersome.

Finally, the shape of the root contour will be as shown in Fig. 6.15.

SUMMARY

The importance of the root locus method for the control system theory is emphasized. Its correlation with the Mason's gain formula is shown. The method of drawing the root locus is described. The minimum value of the damping coefficient is determined. The rules for the construction of the root locus are given. The construction process of root contour is also described.

QUESTIONS

1. Explain the usefulness of root locus diagram in the design of feedback control systems.
2. Sketch the root locus of the system whose open-loop transfer function is

$$G(s)H(s) = \frac{K(s+1)}{s(s-1)(s^2+4s+16)}$$

Find the range of K for which the closed-loop system is stable.

3. The open-loop transfer function of a unity feedback control system is given by

$$G(s) = \frac{K(s+4)}{s(s+1)}$$

Sketch the root locus plot of the system and determine the value of the gain K so that the system is critically damped.

4. The open-loop transfer function of a control system with positive feedback is given by

$$G(s) = \frac{K}{s(s^2+4s+4)}$$

Sketch the root locus diagram of the system as a function of K.

5. Sketch the root locus plots and briefly explain qualitatively the improvements in system performance (stability and steady-state error) that are obtainable by introducing a compensating zero $(s + 3)$ to the unity feedback system whose open-loop transfer function is

$$G(s) = \frac{K}{s(s+2)(s+6)}$$

6. Construct the root locus diagram of the system of which the poles and zeros of $G(s)H(s)$ are: poles at 0, 0, -2, -2 and a zero at -4.

7. The open-loop transfer function of a unity feedback system is given by

$$G(s) = \frac{K}{(s+3)(s+5)(s^2+2s+2)}$$

In the root locus diagram for this system:

(i) Determine the number of branches of root loci.
(ii) Determine the locations at which these branches start.
(iii) Determine the locations at which these branches terminate.
(iv) Find the angles of the asymptotes and the point at which the asymptotes intersect the real axis.
(v) Find the breakaway point.
(vi) Find the points where the root loci cross the imaginary axis.

Also, plot the root loci.

8. Sketch the root locus diagram for the closed-loop system having a loop transfer function given by

$$G(s)H(s) = \frac{K(s+2)}{s(s+1)}$$

as K varies from zero to infinity.

9. Define breakaway point in the root locus diagram.

10. What is root contour? When is the construction of root contour required? Explain clearly.

11. Sketch the root locus of a unity feedback system having

$$G(s) = \frac{2(s+\alpha)}{s(s+2)(s+10)}$$

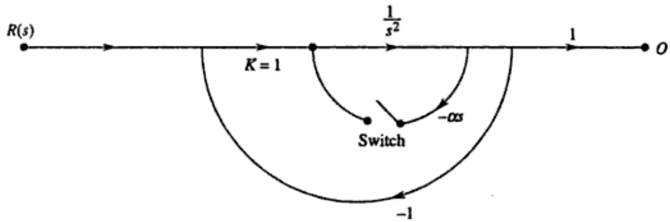
12. Prove that the following closed-loop unity feedback system has a damping factor greater than 0.5 when $K = 1$

$$G(s) = \frac{K(s+13)}{s(s+3)(s+7)}$$

13. Determine the real root of the following characteristic equation

$$s^3 - 3s^2 + 4s - 5 = 0$$

14. The signal flow graph of a control system is shown in the figure below. With the switch closed, draw the root locus plot of the system with α as a varying parameter. Show that the complex root branches are part of a circle. From the root locus plot, determine the value of α such that the resulting system has a damping ratio of 0.5. For the value of α, find the overall transfer function in factored form.



15. Sketch the root locus diagram for the system having open-loop transfer function $G(s)$ with unity feedback. The value of $G(s)$ is given by

$$G(s) = \frac{K}{s(1+s\tau_1)(1+s\tau_2)}$$

where K, τ_1, and τ_2 have positive values. Show how the root locus gets modified with the addition of (i) a pole (ii) a zero. Discuss the effect of each on the performance of the system, if the system is already stable before the addition.

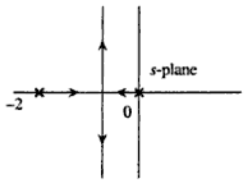
16. Consider the open-loop transfer function of a unity feedback system

$$G(s) = \frac{K(s+3)}{s(s^2+2s+2)(s+5)(s+6)}$$

In the root locus diagram for this system:

(i) Determine the number of branches of root loci.
(ii) Determine the locations at which these branches start.
(iii) Determine the locations at which these branches terminate.
(iv) Is $s = -10 + j0$ a point on the root locus (give valid reasons)?
(v) At what points is the imaginary axis crossed by the root loci and what is the corresponding value of gain K?

17. What are the effects of adding a first-order factor, namely $(s + 6)$ in the denominator of a certain $G(s)$ for which the root locus is given in the following figure. Sketch roughly the ensuing root-loci.



Chapter 7 Analysis of Frequency Response

7.1 INTRODUCTION

The frequency response is defined as the magnitude and phase relationship between the sinusoidal input and the steady-state output of the system.

Suppose the input is $A_1 \sin \omega t$ and the output is $A_2 \sin (\omega t + \theta)$. The frequency response test on a system or component is generally made by maintaining the amplitude A_1 fixed and finding out A_2 and θ for a suitable range of frequencies. The frequency response is calculated from the sinusoidal transfer function that is obtained by substituting $j\omega$ in place of s in the system transfer function.

Suppose the second-order system is

$$T(s) = \frac{O(s)}{R(s)} = \frac{\omega_n^2}{(s)^2 + 2\zeta\omega_n(s) + \omega_n^2}$$

where ω_n is the undamped natural frequency of oscillations and ζ is the damping factor.

The frequency response transfer function will be

$$T(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2}$$

$$= \frac{\omega_n^2}{-\omega^2 + 2j\zeta\omega\omega_n + \omega_n^2}$$

$$= \frac{1}{-\dfrac{\omega^2}{\omega_n^2} + 2j\zeta\dfrac{\omega}{\omega_n} + 1}$$

$$= \frac{1}{-v^2 + 2j\zeta v + 1}$$

where $v = \omega/\omega_n$ and is termed the *normalized* driving signal frequency. Therefore,

$$|T(j\omega)| = \frac{1}{\sqrt{(1 - v^2)^2 + 4\zeta^2 v^2}}$$

and

$$\angle |T(j\omega)| = -\tan^{-1}\frac{2\zeta v}{1-v^2}$$

The steady-state output of the system is

$$o(t) = \frac{1}{\sqrt{(1-v^2)^2 + 4\zeta^2 v^2}} \sin\left(\omega t - \tan^{-1}\frac{2\zeta v}{1-v^2}\right)$$

When $v = 0$, the magnitude $|T(j\omega)| = 1$ and the phase angle $\angle T(j\omega) = 0°$.

When $v = 1$, $|T(j\omega)| = \frac{1}{2\zeta}$ and the phase angle $\angle T(j\omega) = -\frac{\pi}{2}$.

When $v \to \infty$, $|T(j\omega)| = 0$ and $\angle T(j\omega) = -\pi$.

For the maximum value of $|T(j\omega)|$, $\frac{d}{dv}[|T(j\omega|] = 0$. That is,

$$\frac{d}{dv}\left[\frac{1}{\sqrt{(1-v^2)^2 + 4\zeta^2 v^2}}\right] = 0$$

or

$$-\frac{1}{2}\frac{[2(1-v^2)\cdot(-2v) + 8\zeta^2 v]}{2\sqrt{(1-v^2)^2 + 4\zeta^2 v^2}\,[(1-v^2)^2 + 4\zeta^2 v^2]} = 0$$

or

$$-4v + 4v^3 + 8\zeta^2 v = 0 \qquad \text{or} \qquad v^2 = 1 - 2\zeta^2$$

or

$$\left(\frac{\omega}{\omega_n}\right)^2 = 1 - 2\zeta^2 \qquad \text{or} \qquad \frac{\omega}{\omega_n} = \sqrt{1-2\zeta^2} \qquad \text{or} \qquad \omega = \omega_n\sqrt{1-2\zeta^2}$$

Hence the maximum value of $|T(j\omega)|$ is

$$= \frac{1}{\sqrt{(1-v^2)^2 + 4\zeta^2 v^2}}$$

$$= \frac{1}{\sqrt{(1-1+2\zeta^2)^2 + 4\zeta^2(1-2\zeta^2)}} = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

The frequency where $|T(j\omega)|$ is maximum is termed the *resonant frequency*.

The phase angle at the resonant frequency will be

$$\angle T(j\omega) = -\tan^{-1}\frac{2\zeta v}{1-v^2}$$

$$= -\tan^{-1}\frac{2\zeta\sqrt{1-2\zeta^2}}{1-1+2\zeta^2} = -\tan^{-1}\frac{\sqrt{1-2\zeta^2}}{\zeta}$$

The nature of the frequency response curves for magnitude and phase angle of the second-order system against the normalized frequency v is shown in Figs. 7.1 and 7.2 respectively.

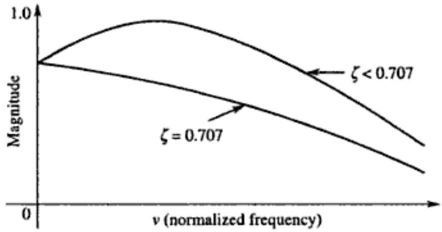


Fig. 7.1 Frequency response characteristics of magnitude of the second-order system.

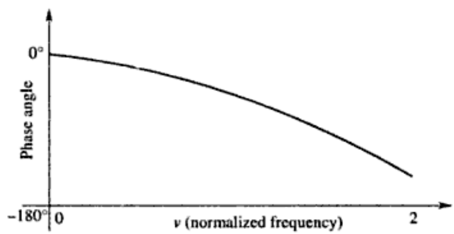


Fig. 7.2 Frequency response characteristic of phase angle of the second-order system.

Bandwidth and cut-off frequency

The range of frequencies over which the $|T(j\omega)|$ is equal to or greater than $\frac{1}{\sqrt{2}}$ is defined as the bandwidth. The frequency at which $|T(j\omega)|$ is $\frac{1}{\sqrt{2}}$ is termed the *cut-off frequency* (Fig. 7.3).

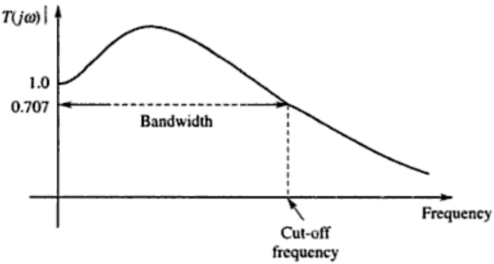


***Fig. 7.3** Frequency response of a feedback control system.*

In Fig. 7.3, the bandwidth for the control system can be obtained from the equality, $|T(j\omega)| = \frac{1}{\sqrt{2}}$ due to similarity with low pass filters. Hence, in this case the bandwidth is equal to the cut-off frequency satisfying the relation

$$|T(j\omega)| = \frac{1}{\sqrt{(1-v^2)^2 + (2\zeta v)^2}} = \frac{1}{\sqrt{2}}$$

or

$$(1 - v^2)^2 + (2\zeta v)^2 = 2 \qquad \text{or} \qquad v^4 - 2(1 - 2\zeta^2)v^2 - 1 = 0$$

Therefore,

$$v^2 = \frac{2(1-2\zeta^2) \pm \sqrt{4(1-2\zeta^2)^2 + 4}}{2}$$

$$= 1 - 2\zeta^2 \pm \sqrt{2 - 4\zeta^2 + 4\zeta^4}$$

or

$$v = \sqrt{1 - 2\zeta^2 + \sqrt{2 - 4\zeta^2 + 4\zeta^4}} \qquad (\because v > 0)$$

Therefore, the bandwidth will be equal to $\sqrt{1 - 2\zeta^2 + \sqrt{2 - 4\zeta^2 + 4\zeta^4}}$.

7.2 METHODS OF DESCRIBING FREQUENCY RESPONSE

The polar plot is one of the methods of describing the frequency response. For the *RC* filter circuit shown in Fig. 7.4, we have

$$E_0(s) = \frac{E_i(s)}{R + \frac{1}{Cs}} \cdot \frac{1}{Cs} = \frac{\frac{E_i(s)}{Cs}}{\frac{RCs + 1}{Cs}}$$

$$= \frac{E_i(s)}{1 + RCs} = \frac{E_i(s)}{1 + \tau s} \quad (\tau = RC = \text{time constant})$$

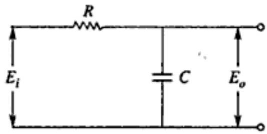


Fig. 7.4 RC filter.

Therefore,

$$\frac{E_0(s)}{E_i(s)} = \frac{1}{1 + \tau s}$$

or

$$T(s) = \frac{1}{1 + \tau s}$$

For frequency response, $s = j\omega$. Therefore,

$$T(j\omega) = \frac{1}{1 + j\omega\tau}$$

That is,

$$|T(j\omega)| = \frac{1}{\sqrt{1 + \omega^2\tau^2}} \quad \text{and} \quad \angle T(j\omega) = \angle -\tan^{-1}\omega\tau$$

When

$$\omega = 0, \quad |T(j\omega)| = 1 \quad \text{and} \quad \angle T(j\omega) = -\tan^{-1} 0 = 0°$$

When

$$\omega = \frac{1}{\tau}, \quad |T(j\omega)| = \frac{1}{\sqrt{2}} \quad \text{and} \quad \angle T(j\omega) = -\tan^{-1} 1 = -\frac{\pi}{4}$$

When

$$\omega = \infty, \quad |T(j\omega)| = 0 \quad \text{and} \quad \angle |T(j\omega)| = -\tan^{-1} \infty = -\frac{\pi}{2}$$

If we draw the polar plot with the above data, we get, for $\omega = 0$, $\omega = \frac{1}{\tau}$ and $\omega = \infty$, the points as plotted in Fig. 7.5, at B, C, and A respectively. The triangle ABC is developed from trigonometry. Now,

$$\cos A = \frac{AB^2 + AC^2 - BC^2}{2AB \cdot AC} = \frac{1 + \frac{1}{2} - BC^2}{2 \cdot 1 \cdot \frac{1}{\sqrt{2}}}$$

or

$$\frac{1}{\sqrt{2}} = \frac{\frac{3}{2} - BC^2}{\sqrt{2}} \quad \text{or} \quad BC = \frac{1}{\sqrt{2}}$$

Since $AC = BC = \frac{1}{\sqrt{2}}$, $\angle CAB = \angle CBA = 45°$, hence $\angle ACB = 90°$, and the polar plot is a circle.

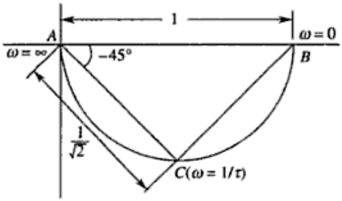


Fig. 7.5 Polar plot of the transfer function of RC filter.

If the transfer function is $T(j\omega) = \dfrac{1}{j\omega(1 + j\omega\tau)}$, it may be rearranged as follows:

$$T(j\omega) = \frac{j\omega(1 - j\omega\tau)}{j\omega(1 + j\omega\tau)\, j\omega(1 - j\omega\tau)} = \frac{j\omega(1 - j\omega\tau)}{-\omega^2(1 + \omega^2\tau^2)}$$

$$= \frac{j\omega + \omega^2\tau}{-\omega^2(1 + \omega^2\tau^2)} = \frac{+\omega^2\tau}{-\omega^2(1 + \omega^2\tau^2)} + j\frac{\omega}{-\omega^2(1 + \omega^2\tau^2)}$$

or

$$T(j\omega) = -\frac{\tau}{(1 + \omega^2\tau^2)} - \frac{j}{\omega(1 + \omega^2\tau^2)}$$

When $\omega = 0$, $T(j\omega) = -\tau - j\infty = \infty \angle -90°$

When $\omega = \infty$, $T(j\omega) = -0 - j0 = 0 \angle -180°$

The polar plot with the above data will be as shown in Fig. 7.6.

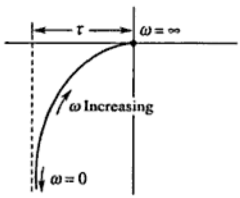


Fig. 7.6 Polar plot of $\dfrac{1}{j\omega(1 + j\omega\tau)}$.

Similarly, the polar plot of $\dfrac{1}{(1 + j\omega\tau_1)(1 + j\omega\tau_2)}$ will be as shown in Fig. 7.7.

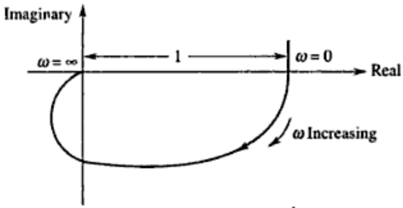


Fig. 7.7 Polar plot of $\dfrac{1}{(1 + j\omega\tau_1)(1 + j\omega\tau_2)}$.

The polar plot of $\dfrac{1}{j\omega(1 + j\omega\tau_1)(1 + j\omega\tau_2)}$ will be as shown in Fig. 7.8.

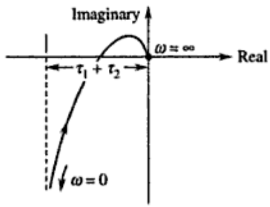


Fig. 7.8 Polar plot of $\dfrac{1}{j\omega(1 + j\omega\tau_1)(1 + j\omega\tau_2)}$.

The polar plot of $\dfrac{1}{(1 + j\omega\tau_1)(1 + j\omega\tau_2)(1 + j\omega\tau_3)}$ will be as shown in Fig. 7.9.

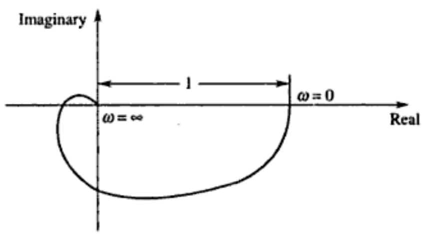


Fig. 7.9 Polar plot of $\dfrac{1}{(1 + j\omega\tau_1)(1 + j\omega\tau_1)(1 + j\omega\tau_3)}$.

The polar plot of $\frac{1}{(j\omega)^2(1+j\omega\tau_1)(1+j\omega\tau_2)}$ will be as shown in Fig. 7.10.

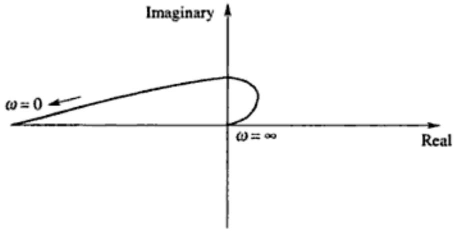


Fig. 7.10 Polar plot of $\frac{1}{(j\omega)^2(1+j\omega\tau_1)(1+j\omega\tau_2)}$

The polar plot of $\frac{1}{(j\omega)^2(1+j\omega\tau_1)(1+j\omega\tau_2)(1+j\omega\tau_3)}$ will be as shown in Fig. 7.11.

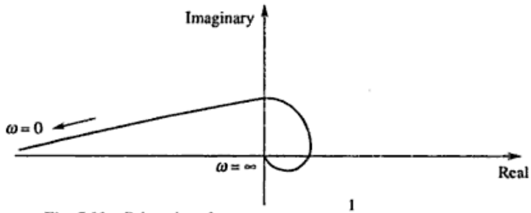


Fig. 7.11 Polar plot of $\frac{1}{(j\omega)^2(1+j\omega\tau_1)(1+j\omega\tau_2)(1+j\omega\tau_3)}$.

Inverse polar plot

The inverse polar plot is the curve of $\frac{1}{T(j\omega)}$ vs. ω. For example:

$$\text{If } T(j\omega) = \frac{1}{1+j\omega\tau}, \text{ then } \frac{1}{T(j\omega)} = 1 + j\omega\tau = \sqrt{1+\omega^2\tau^2}\ \angle\tan^{-1}\omega\tau$$

The curve is shown in Fig. 7.12.

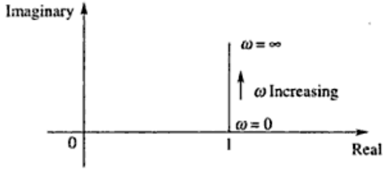


Fig. 7.12 Inverse polar plot of $\frac{1}{1+j\omega\tau}$.

7.3 BODE PLOT

The Bode plots, developed by H.W. Bode, consist of two graphs—the magnitude of the frequency response and the phase angle of the frequency response of the transfer function plotted against the frequency in logarithmic scale. Logarithmic scales are used for Bode plots since they simplify construction, manipulation, interpretation and help to develop graphs over a wide range of frequencies. In many cases, the graphs come as straight lines. The magnitude of the transfer function for any value of ω is plotted in decibel units, that is, as 20 log $|T(j\omega)|$, where the base of the logorithm is 10. For example, if $|T(j\omega)| = 10$, then the magnitude is 20 log 10 = 20 dB.

The plot of decibel magnitude versus log ω is termed the Bode *magnitude plot* and the plot of phase angle versus log ω is termed the Bode *phase angle* plot. The curves are usually drawn on a semilog paper using the log scale for frequency and the linear scale for the magnitude in decibels and the phase angle in degrees.

Suppose, $T(j\omega) = \frac{1}{(1+\omega^2\tau^2)^{1/2}} \angle -\tan^{-1}\omega\tau$, then the log magnitude is

$$20 \log |T(j\omega)| = 20 \log(1 + \omega^2\tau^2)^{-1/2} = -10 \log(1 + \omega^2\tau^2)$$

If the frequency $\omega << \frac{1}{\tau}$, then the log magnitude will be:

$$20 \log |T(j\omega)| = -10 \log 1 = 0 \text{ dB}$$

If $\omega >> \frac{1}{\tau}$, then the log magnitude will be

$$20 \log |T(j\omega)| = -20 \log \omega\tau = -20 \log \omega - 20 \log \tau$$

Thus, it is observed that 20 log $|T(j\omega)|$ versus log ω will provide a straight line graph having a slope –20 dB per unit change in log ω. A unit change in log ω indicates

$$\log\left(\frac{\omega_2}{\omega_1}\right) = 1 \quad \text{or} \quad \omega_2 = 10\omega_1$$

This change is called a decade. If $\omega_2 = 2\omega_1$, then the change is called the octave. Therefore, – 20 log 2 = – 6 dB.

Now, the log magnitude versus the log frequency curve of $\frac{1}{1+j\omega\tau}$ may be approximated by two straight line asymptotes. One straight line will be at 0 dB for the frequency range $0 < \omega \le \frac{1}{\tau}$ and the other will be a straight line with a slope '– 20 dB/decade' for the frequency range $\frac{1}{\tau} \le \omega < \infty$. From the above analysis, another definition is important. This is termed the *corner frequency* or *break frequency*. The corner frequency is the meeting point of two asymptotes. Hence after drawing the asymptotic curves, we can evaluate the error and apply corrections over the approximate curve.

Now the error in log magnitude for $0 < \omega \le \frac{1}{\tau}$ is provided by

$$-10 \log (1 + \omega^2\tau^2) + 10 \log 1$$

Hence at the corner frequency $\omega = \frac{1}{\tau}$, the error is

$$-10 \log (1 + 1) + 10 \log 1 = -3 \text{ dB}$$

Similarly the error at $\omega = \frac{1}{2\tau}$ is

$$-10 \log\left(1+\frac{1}{4}\right) + 10 \log 1 = -10 \log\left(\frac{5}{4}\right) + 10 \log 1$$

$$= -10 \log 5 + 10 \log 4 + 10 \log 1$$

$$= -1 \text{ dB}$$

In this way, the errors can be calculated at any point between $\omega = 0$ to $\omega = \frac{1}{\tau}$

When $\frac{1}{\tau} \le \omega < \infty$, the error magnitude can be determined from the following expression

$$-10 \log (1 + \omega^2\tau^2) + 20 \log \omega\tau$$

since the log magnitude was approximated as $-20 \log \omega\tau$.

When $\omega = \frac{1}{\tau}$, the error is

$$-10 \log (1 + 1) + 20 \log 1 = -3 \text{ dB}$$

This matches with our previous calculated value at the lower frequency range end point. Similarly when $\omega = \frac{2}{\tau}$, the error will be

$$-10 \log (1 + 4) + 20 \log 2 = -1 \text{ dB}$$

For the second-order system, the Bode plot can be drawn as follows. Suppose

$$T(j\omega) = \frac{1}{1 + j2v\zeta - v^2}$$

Therefore,

$$|T(j\omega)| = 20 \log \left| \frac{1}{1 + j2\zeta v - v^2} \right|$$

$$= 20 \log \frac{1}{\sqrt{(1 - v^2)^2 + (2\zeta v)^2}}$$

$$= 20 \log 1 - 20 \log \sqrt{(1 - v^2)^2 + (2\zeta v)^2} = -20 \log [(1 - v^2)^2 + (2\zeta v)^2]^{1/2}$$

$$= -10 \log [(1 - v^2)^2 + 4\zeta^2 v^2]$$

If $v << 1$, the $|T(j\omega)|$ is given by: $10 \log 1 = 0$

If $v >> 1$, the $|T(j\omega)|$ is given by: $-10 \log v^4 = -40 \log v$

Hence the log magnitude curve will be two straight line asymptotes—one will be a horizontal line at 0 dB for $v << 1$ and the other will be a slope of -40 dB/decade for $v >> 1$. The two asymptotes will meet on the 0 dB line at $v = 1$, i.e. at the corner frequency of the plot. The error between the actual magnitude and this asymptotic approximation is as follows.

For $0 < v \leq 1$, the error will be found as

$$-10 \log [(1 - v^2)^2 + 4\zeta^2 v^2] + 10 \log 1$$

For $1 < v < \infty$, the error will stand as

$$-10[(1 - v^2)^2 + 4\zeta^2 v^2] + 40 \log v$$

7.3.1 Method of Plotting the Bode Plot

Let us take an example and explain the procedure of plotting the Bode plot. Suppose

$$T(j\omega) = \frac{10(1 + j\omega)}{(j\omega)^2 \left[1 + j\frac{\omega}{4} - \left(\frac{\omega}{4}\right)^2\right]}$$

The log magnitude is

$$20 \log |T(j\omega)| = 20 \log 10 + 20 \log |(1 + j\omega)| + 20 \log \left| \frac{1}{(j\omega)^2} \right| + 20 \log \left| \frac{1}{1 + j\frac{\omega}{4} - \left(\frac{\omega}{4}\right)^2} \right|$$

Now,

$$20 \log |(1 + j\omega)| = 20 \log \sqrt{1 + \omega^2} = 10 \log (1 + \omega^2)$$

When $\omega << 1$, then

$$20 \log |(1 + j\omega)| = 0$$

When $\omega >> 1$, then

$$20 \log |(1 + j\omega)| = 10 \log \omega^2 = 20 \log \omega$$

It means that the slope is 20. Hence the approximate value of the above expression will be 0 up to $\omega = 1$ and then at $\omega = 1$, it will be a straight line with a slope of 20 dB per decade (see Graph 1, Fig. 7.13).

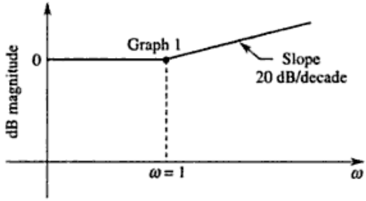


Fig. 7.13

Now,

$$20 \log \left|\left(\frac{1}{j\omega}\right)^2\right| = 20 \log \frac{1}{\omega^2} = 20 \log 1 - 40 \log \omega = -40 \log \omega$$

When $\omega = 0.1$, the dB value is $-40 \log \frac{1}{10} = 40$. When $\omega = 1$, the dB value is 0. Here the slope is -40 dB/decade and the curve is a straight line (see Graph 2, Fig. 7.14).

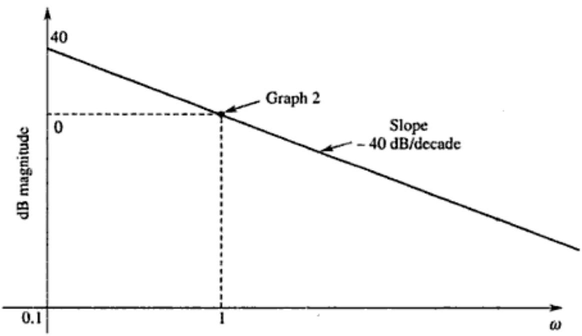


Fig. 7.14

Now,

$$20 \log \left| \frac{1}{1 + j\frac{\omega}{4} - \left(\frac{\omega}{4}\right)^2} \right| = 20 \log \left| \frac{1}{\sqrt{\left(1 - \left(\frac{\omega}{4}\right)^2\right)^2 + \left(\frac{\omega}{4}\right)^2}} \right|$$

Here $v = \frac{\omega}{4}$, the corner frequency will be at $v = 1 = \frac{\omega}{4}$ or $\omega = 4$.

Hence the curve will be as per Graph 3 (Fig. 7.15).

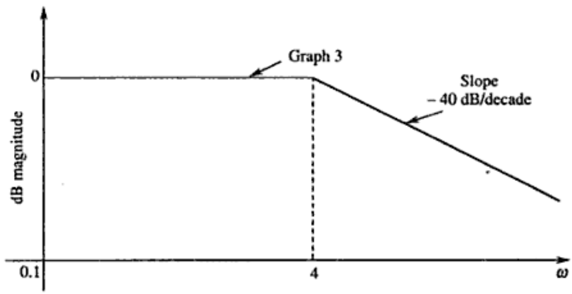


Fig. 7.15

Finally,

$$20 \log_{10} 10 = 20$$

It will have a constant 20 dB magnitude and the curve will be as shown in Graph 4 (Fig. 7.16).

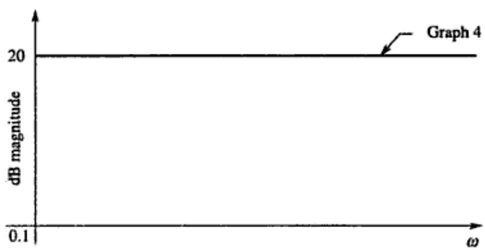


Fig. 7.16

The combination of all the Graphs 1–4 (Figs. 7.13 – 7.16) will be the Bode plot of the complete transfer function with asymptotic approximation (see Fig. 7.17).

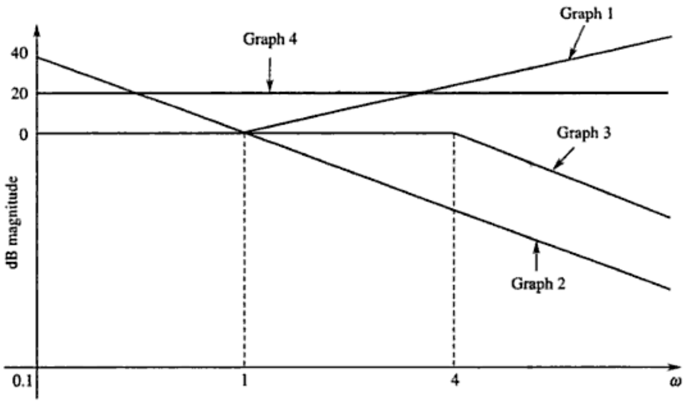


Fig. 7.17 *Bode plot.*

Instead of adding point by point all the graphs, there is another way of solving the problem, which is explained below.

Let us take an example,

$$T(j\omega) = \frac{8\left(1+\dfrac{j\omega}{2}\right)}{j\omega(1+2j\omega)\left[1+j0.4\left(\dfrac{\omega}{8}\right)-\left(\dfrac{\omega}{8}\right)^2\right]}$$

Now the log magnitude is

$$20 \log |T(j\omega)| = 20 \log \left|\frac{8}{j\omega}\right| + 20 \log \left|\left(1+\frac{j\omega}{2}\right)\right| - 20 \log |(1 + 2j\omega)| -$$

$$20 \log \left|\left(1+j0.4\frac{\omega}{8}-\left(\frac{\omega}{8}\right)^2\right)\right|$$

Now,

$$20 \log \left|\frac{8}{j\omega}\right| = 20 \log \frac{8}{\omega} = 20 \log 8 - 20 \log \omega$$

It is a straight line in Bode plot with magnitude of $20 \log_{10} 8$ (= $60 \log_{10} 2 = 60 \times 0.3 = 18$) at $\omega = 1$. Also,

$$20 \log \left| \frac{1}{1 + j2\omega} \right| = 20 \log \frac{1}{\sqrt{1 + 4\omega^2}}$$

$$= 20 \log 1 - 20 \log \sqrt{1 + 4\omega^2} = -10 \log (1 + 4\omega^2)$$

When $4\omega^2 << 1$ or $\omega^2 << \frac{1}{4}$, the dB value is 0. When $4\omega^2 >> 1$ or $\omega^2 >> \frac{1}{4}$, the dB value is

$$-10 \log 4\omega^2 = -10 \log 4 - 20 \log \omega \text{ with the slope as } -20 \text{ dB/decade.}$$

At $\omega = \frac{1}{2}$, the dB value is

$$-10 \log 4 - 20 \log \frac{1}{2} = -20 \log 2 - 20 \log 1 + 20 \log 2 = -20 \log 1 = 0$$

Therefore, the corner frequency is at $\omega = \frac{1}{2}$.

Now,

$$20 \left| \log \left(1 + j\frac{\omega}{2} \right) \right| = 20 \log \sqrt{1 + \frac{\omega^2}{4}} = 10 \log \left(1 + \frac{\omega^2}{4} \right)$$

For $\frac{\omega^2}{4} << 1$, the dB value is 0. For $\frac{\omega^2}{4} >> 1$, the dB value is

$$10 \log \frac{\omega^2}{4} = 10 \log \omega^2 - 10 \log 4 = 20 \log 2 - 10 \log 4 \qquad \text{(where } \omega = 2\text{)}$$

$$= 20 \log 2 - 20 \log 2 = 0$$

Hence the slope is 20 dB/decade and $\omega = 2$ is the corner frequency.

Now,

$$20 \log \left| \frac{1}{1 + j0.4\frac{\omega}{8} - \left(\frac{\omega}{8}\right)^2} \right| = 20 \log \left| \frac{1}{1 + j2(0.2)\frac{\omega}{8} - \left(\frac{\omega}{8}\right)^2} \right|$$

Here $v = \frac{\omega}{8}$, since $v = 1$ is the corner frequency condition. Therefore, $\omega = 8$ is the corner frequency and the slope is, -40 dB/decade. Here, the corner frequency is $\omega = 8$ and the slope is -40 dB/decade.

Thus, the complete data for drawing the final curve is as follows:

Item	*Corner frequency*	*Details of the curve*
$\frac{8}{j\omega}$	None	Slope – 20 dB/decade. Passing through 18 dB at $\omega = 1$. At $\omega = 0.8$, the log magnitude is 20 dB.
$\frac{1}{1+j2\omega}$	0.5	Slope is – 20 dB/decade.
$1+j\frac{\omega}{2}$	2	Slope is +20 dB/decade
$\frac{1}{1+j.4\frac{\omega}{8}-\left(\frac{\omega}{8}\right)^2}$	8	Slope is – 40 dB/decade

The Bode plot of the above data drawn on the semi-log paper is shown in Fig. 7.18.

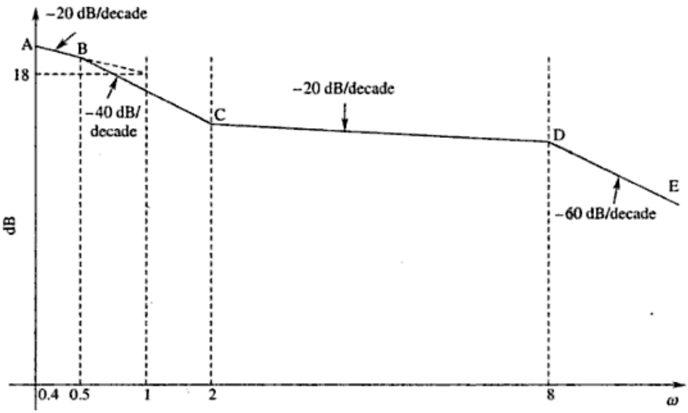


***Fig. 7.18** Bode plot.*

In the curve shown in Fig. 7.18, the slope of the curve at point A is – 20 db/decade and this is for $\frac{8}{j\omega}$. As soon as point B, i.e. $\omega = 0.5$ (the next corner frequency) is reached, the slope is changed to – 40 dB/decade = (= (–20 – 20)). This slope will continue up to point C until the next corner frequency $\omega = 2$ is reached. The portion BC is on account of $\frac{1}{1+j2\omega}$. The slope CD becomes -20 dB/decade $(=(-40+20))$ on account of $1+j\frac{\omega}{2}$. The slope DE becomes -60 dB/decade $(=(-20-40))$ on account of $\dfrac{1}{1+j.4\frac{\omega}{8}-\left(\frac{\omega}{8}\right)^2}$.

This is the method of drawing all the asymptotic curves.

Now we have to apply the error to derive the actual curve.

For $20\log\left|\frac{1}{1+j2\omega}\right| = -10\log(1+4\omega^2)$, the error in log magnitude for $0<\omega\le\frac{1}{2}$ is

$$-10\log(1+4\omega^2)+10\log 1$$

Hence, the error at the corner frequency $\omega=\frac{1}{2}$ is

$$-10\log(1+1)+0=-10\log 2=-3\text{ dB}.$$

The error at $\omega=\frac{1}{4}$ is

$$-10\log\left(1+4\cdot\frac{1}{16}\right)+10\log 1 = -10\log\left(\frac{5}{4}\right)+0 = -10\log 5+10\log 4=-1\text{ dB}$$

The error in log magnitude for $\frac{1}{2}\le\omega<\infty$ is

$$-10\log(1+4\omega^2)+10\log 4\omega^2$$

At $\omega=\frac{1}{2}$

$$\text{Error}=-10\log 2+10\log 1=-3\text{ dB}$$

At $\omega=1$

$$\text{Error}=-10\log 5+10\log 4=-1\text{ dB}$$

Hence on the asymptotic curves the above values need to be incorporated to get the actual curve.

Now, for $1+j\frac{\omega}{2}$, when $0<\omega\le 2$

$$\text{Error}=+10\log\left(1+\frac{\omega^2}{4}\right)-10\log 1$$

At the corner frequency, $\omega = 2$

$$\text{Error} = 10 \log 2 = 3 \text{ dB}$$

When $2 \le \omega < \infty$

$$\text{Error} = 10 \log\left(1 + \frac{\omega^2}{4}\right) - 10 \log \frac{\omega^2}{4}$$

At the corner frequency $\omega = 2$

$$\text{Error} = 10 \log 2 = 3 \text{ dB}$$

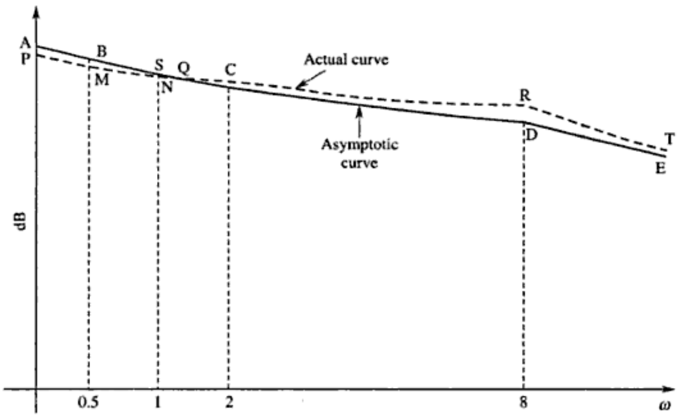


Fig. 7.19 Actual and asymptotic curves.

Figure 7.19 shows both the actual and the asymptotic curves. The error at $\omega = 0.5$ is found to be BM which is equal to – 3 dB. The error at $\omega = 1$ is found to be SN which is equal to – 1 dB. Similarly, the errors at different points are calculated and the actual curve is obtained.

The Bode plot is not complete until the phase angle plot is also made on the same semi-log paper because the transfer function can be fully identified only by both its magnitude and phase angle.

For example, if the transfer function is $\frac{1}{1+j\omega\tau}$, the magnitude is $\frac{1}{\sqrt{(1+\omega^2\tau^2)}}$ and the phase angle is $-\tan^{-1}\omega\tau$. When $\omega = \frac{1}{\tau}$, the phase angle is $-\tan^{-1} 1 = -45°$. When $\omega = \frac{1}{2\tau}$, the phase angle is $-\tan^{-1}\frac{1}{2}$. When $\omega = \frac{2}{\tau}$, the phase angle is $-\tan^{-1}\frac{2}{\tau}\cdot\tau = -\tan^{-1} 2$.

For different values of frequency ω on the horizontal axis, the phase angle is determined and plotted on the same semi-log paper.

SUMMARY

The frequency response of the system is defined. Methods of describing the frequency response are enumerated. The polar plot of frequency response is explained. The Bode plot of frequency response is also explained. Examples of how to plot the Bode plot are described. Asymptotic approximation of the Bode plot is first illustrated through examples. Then the error is incorporated and the actual magnitude curve of the Bode plot is drawn. The method of drawing the frequency plot of phase of the transfer function is also explained.

QUESTIONS

1. Write a short note on polar plots.
2. Construct the Bode plot for the following transfer functions:

   (a) $G(s) = \dfrac{100}{s^2(1+0.05s)(1+0.08s)(1+0.125s)}$

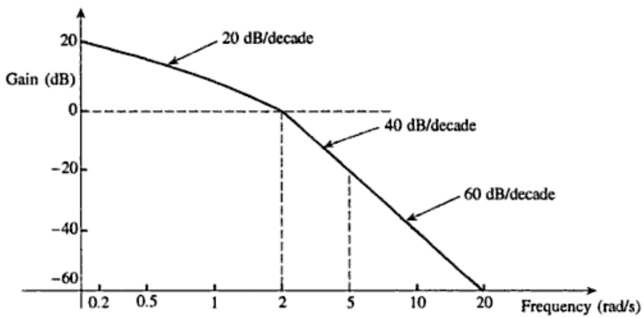
   (b) $G(s) = \dfrac{10(s+10)}{s(s+5)(s+2)}$.

3. Draw the Bode diagram for the system whose open-loop transfer function is

$$G(s)H(s) = \frac{10(s+50)}{s(s+5)}$$

   Apply the corrections as well.

4. The open-loop gain of a servomechanism is plotted in the figure below against frequency using the Bode plot. Assuming that all the elements are of first-order type, write down the open-loop transfer function of the system and plot its open-loop phase against frequency on the same scale. State any assumption made.



Chapter

8 Stability in Frequency Response Systems

8.1 INTRODUCTION

The first and foremost requirement of stability is that the roots of the characteristic equation must not lie in the right-half of the s-plane or on the imaginary axis because that will produce instability or cause sustained oscillations of the system. In frequency domain analysis, the Nyquist stability criterion is one of the very important tools used to determine the stability of the system.

Suppose a function $f(s)$ is expressed as follows.

$$f(s) = \frac{(s-a_1)(s-a_2)\cdots(s-a_m)}{(s-b_1)(s-b_2)\cdots(s-b_n)}$$

We know that $s = \sigma + j\omega$. Let

$$f(s) = A + jB$$

For every point s in the s-plane (as shown in Fig. 8.1) at which $f(s)$ is analytic, a corresponding point in the $f(s)$ plane (as shown in Fig. 8.2) will be observed. On the other hand, it can be said that for a contour in the s-plane, which does not go through any singular point, there will be a contour in the $f(s)$ plane. Now,

$$\angle f(s) = \angle (s-a_1) + \angle (s-a_2) + \cdots - \angle (s-b_1) - \angle (s-b_2) - \cdots$$

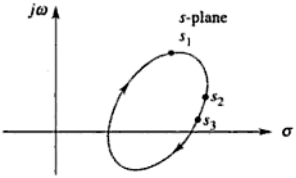


Fig. 8.1 *s-plane contour.*

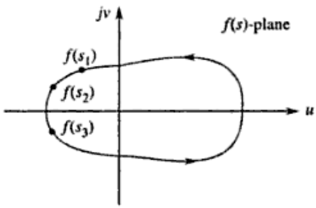


Fig. 8.2 *f(s)-plane contour.*

In Fig. 8.3, if the point s describes the contour in the clockwise direction, then the phasor $s - a_1$ will develop a net angle -2π since the root a_1 is inside the contour. But the other roots $a_2, a_3, \ldots$

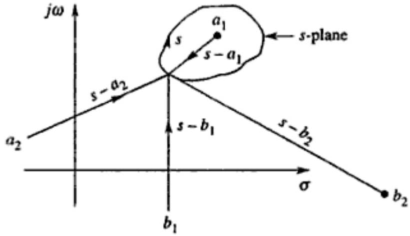


Fig. 8.3 s-plane contour enclosing a zero of f(s).

and b_2, b_3, ... will develop zero net angle. Hence the $f(s)$ phasor develops a net phase angle change of -2π (Fig. 8.4). Thus it can be said that the tip of the $f(s)$ vector will describe a closed contour about the origin of the $f(s)$ in the clockwise direction. To speak the truth, the exact shape of the closed contour in the $f(s)$-plane is not the main important point, rather it is essential to know that this contour encircles the origin once. Thus it can be said that for each zero of $f(s)$ enclosed by the s-plane contour, the $f(s)$-plane contour encircles the origin once in the clockwise direction. Again, in the case of poles, i.e. b_1, b_2, ... if any pole is encircled by the s-plane in the clockwise direction, the $f(s)$-plane contour will encircle the origin in the counterclockwise direction since the pole is in the denominator of $f(s)$. The angle described will be $-(-2\pi) = +2\pi$.

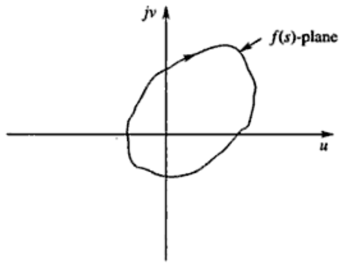


Fig. 8.4 f(s)-plane contour corresponding to Fig. 8.3.

8.2 NYQUIST STABILITY CRITERION

We know that the closed-loop control system provides the characteristic equation equal to $1 + G(s)H(s) = 0$, where

$$G(s)H(s) = \frac{K(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)} \qquad \text{with } m \le n$$

Therefore,

$$f(s) = 1 + G(s)H(s) = 1 + \frac{K(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$$

$$= \frac{(s+p_1)(s+p_2)\cdots(s+p_n) + K(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$$

$$= \frac{(s+z_1')(s+z_2')\cdots(s+z_n')}{(s+p_1)(s+p_2)\cdots(s+p_n)} = 0$$

or

$$(s + z_1')(s + z_2') \ldots (s + z_n') = 0$$

We know that the roots of the characteristic equation should lie in the left-half of the s-plane. Here the roots of the characteristic equation are nothing but the zeros of the $f(s)$. From this it can be concluded that even if the open-loop system is unstable, the closed-loop system may be stable.

Now, for determining the presence of any zeros of $f(s) = 1 + G(s)H(s)$ in the right-half s-plane, a contour is chosen that completely encloses the right-half of the the s-plane. This contour is termed the *Nyquist contour.*

Figure 8.5 describes the Nyquist contour. The points A and B are at infinity on the $+j\omega$ and $-j\omega$ axes, respectively, and the arc is of radius r tending towards infinity. Therefore, the whole contour drawn in the clockwise direction describes the right-half of the s-plane. Now let us consider that M zeros and N poles of $f(s)$ are in the right-hand side of s-plane. If s moves along the Nyquist contour in the s-plane, a closed contour can be developed in the $f(s)$-plane which will enclose the origin $N - M$ times in the counterclockwise direction. We have already seen that, to make a closed-loop system stable, there must not be any zero in the right-half s-plane. Hence $M = 0$.

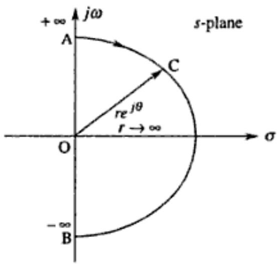


Fig. 8.5 Nyquist contour.

Hence, for a stable system, the following will be the criterion. If s moves along the Nyquist contour in the s-plane, a closed contour can be developed in the $f(s)$-plane which will enclose the origin N times in the counterclockwise direction. This 'N times' is nothing but the right-half s-plane poles of the open-loop transfer function $G(s)H(s)$. Again it can be written

$$G(s)H(s) = [1 + G(s)H(s)] - 1$$

Now the contour of $G(s)H(s)$ corresponding to the Nyquist contour is the same as the contour of $1 + G(s)H(s)$ drawn from the point $(-1 + j0)$. Therefore, it can be said that the encirclement of the origin by the contour of $1 + G(s)H(s)$ is equivalent to the encirclement of the point $(-1 + j0)$ by the contour $G(s)H(s)$. This is shown in Fig. 8.6.

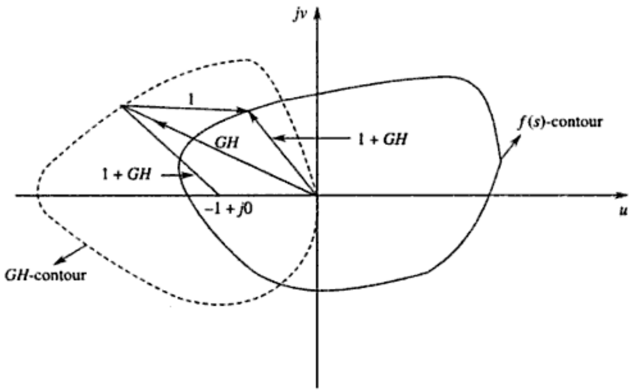


Fig. 8.6 Nyquist criterion.

Therefore the Nyquist criterion can be stated as follows:

If the contour of the open-loop transfer function $G(s)H(s)$ corresponding to the Nyquist contour in the s-plane encircles the point $(-1 + j0)$ in the counterclockwise direction as many times as the number of right-half s-plane poles of $G(s)H(s)$, then the closed-loop system is stable. It means that if both the open-loop and the closed-loop systems are to be stable, the net encirclement would have to be zero.

EXAMPLE 8.1 Apply the Nyquist criterion to find out the stability of $G(s)H(s) = \dfrac{1}{s(s+1)}$.

Solution Since in $G(s)H(s) = \dfrac{1}{s(s+1)}$, one pole is at the origin, there will be a slight modification to the Nyquist contour. Figure 8.7 describes the Nyquist contour.

Since there is one pole at the origin, a circle $re^{j\phi}$ is drawn taking origin as the centre assuming $r \to 0$.

$$s = \lim_{r \to 0} re^{j\phi} \qquad (\phi \text{ varies from } -90^\circ \text{ to } +90^\circ \text{ via } 0^\circ)$$

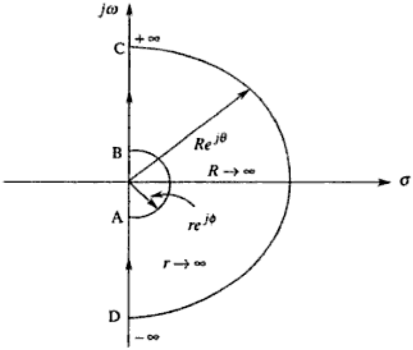


Fig. 8.7 Example 8.1: Nyquist contour.

Now,

$$\lim_{r \to 0} \left(\frac{1}{s(s+1)} \right)$$

$$= \lim_{r \to 0} \left[\frac{1}{re^{j\phi}(re^{j\phi}+1)} \right]$$

$$= \lim_{r \to 0} \left(\frac{1}{re^{j\phi}} \right)$$

The value of $G(s)H(s)$ will approach ∞ as r tends to zero and $-\phi$ will vary from $+90°$ to $-90°$ via 0° as s moves from A to B in Fig. 8.7.

Figure 8.8 shows the locus along the curve PQ in the clockwise direction, where both P and Q are at infinity.

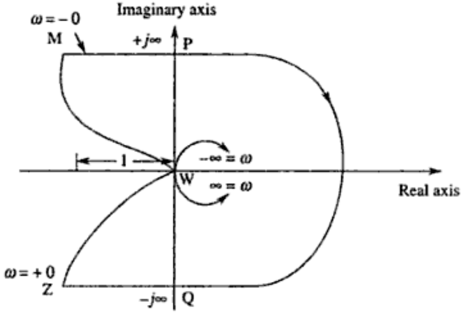


Fig. 8.8 Example 8.1.

When s moves from B to C (see Fig. 8.7), the $G(s)H(s) = \dfrac{1}{j\omega(1 + j\omega)}$. This is nothing but the polar plot ZW, where Z will meet Q at infinity when the value of ω is 0 and the polar plot will be at W, when the value of ω is infinity.

When s will move from C to D along the curve in Fig. 8.7, the value of $s = Re^{j\theta}$, and when $R \to \infty$ and θ varies from $+90°$ to $-90°$, we get

$$G(s)H(s) = \lim_{R\to\infty} \frac{1}{Re^{j\theta}(Re^{j\theta} + 1)}$$

$$= \lim_{R\to\infty} \frac{1}{R^2e^{2j\theta} + Re^{j\theta}}$$

$$= \lim_{R\to\infty} \frac{\dfrac{1}{R^2 e^{2j\theta}}}{1 + \dfrac{e^{j\theta}}{Re^{2j\theta}}}$$

$$= \lim_{R\to\infty} \frac{1}{R^2 e^{2j\theta}} = 0 \cdot e^{-2j\theta}$$

That is, the value of $G(s)H(s)$ is zero and θ varies from $-180°$ to $+180°$ via 0°.

When s will move from D to A in Fig. 8.7, the $G(s)H(s) = \dfrac{1}{-j\omega(1 - j\omega)}$, the polar plot WM will occur. M will meet P at infinity at $\omega = -0$, and at W the value of ω will be $-\infty$.

EXAMPLE 8.2 Develop the Nyquist plot of the feedback system whose open-loop transfer function is

$$G(s)H(s) = \frac{K}{(\tau_1 s + 1)(\tau_2 s + 1)}$$

and find out whether the system is stable or not.

Solution We have already studied the polar plot of $\dfrac{1}{(1 + j\omega\tau_1)(1 + j\omega\tau_2)}$. (See Fig. 8.9.)

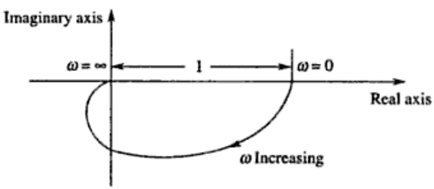


Fig. 8.9 Example 8.2: Polar plot of open-loop transfer function.

Now if we have to develop the closed-loop contour, then we have to draw the plots, i.e. from 0 to ∞ and – ∞ to – 0. Hence the complete polar plot of $\frac{K}{(1+\tau_1 j\omega)(1+\tau_2 j\omega)}$ is shown in Fig. 8.10.

The plot of Fig. 8.10 clearly indicates that it has not encircled the point $-1 + j0$. Hence both the open-loop system and the closed-loop system are stable.

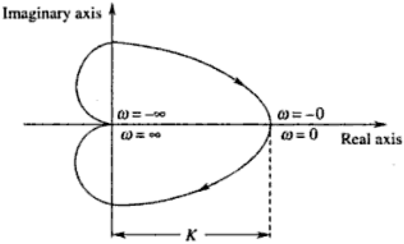


Fig. 8.10 Example 8.2: Polar plot of closed-loop transfer function.

EXAMPLE 8.3 Find the stability of both the open-loop and the closed-loop systems, when the open-loop transfer function is $G(s)H(s) = \frac{s+3}{(s+1)(s-1)}$.

Solution Since $G(s)H(s) = \frac{s+3}{(s+1)(s-1)}$ (see Fig. 8.11), one of the poles of the open-loop transfer function is in the right-half s-plane. Hence the open-loop system is unstable. Now let us draw the

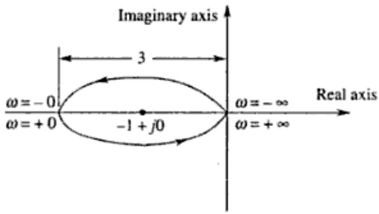


Fig. 8.11 Example 8.3.

locus of $G(j\omega)H(j\omega)$ and find out with the help of the Nyquist criterion whether or not the closed loop system is stable. Thus,

$$G(j\omega)H(j\omega) = \frac{j\omega+3}{(j\omega+1)(j\omega-1)}$$

When $\omega = 0$, $$G(j\omega)H(j\omega) = \frac{3}{1(-1)} = -3$$

When $\omega = \infty$, $$|G(j\omega)H(j\omega)| = \frac{\sqrt{\omega^2+9}}{\sqrt{\omega^2+1}\,\sqrt{\omega^2-1}}$$

$$= \frac{\omega\sqrt{1+\frac{9}{\omega^2}}}{\omega\sqrt{1+\frac{1}{\omega^2}}\,\omega\sqrt{1-\frac{1}{\omega^2}}} = 0$$

$$\angle G(j\omega)H(j\omega) = \tan^{-1}\frac{\omega}{3} - \tan^{-1}\frac{\omega}{1} - \tan^{-1}\frac{\omega}{-1}$$

$$= -\frac{\pi}{2}$$

When $\omega = -\infty$

$$\angle G(j\omega)H(j\omega) = \tan^{-1}\frac{\omega}{3} - \tan^{-1}\frac{\omega}{1} - \tan^{-1}\frac{\omega}{-1}$$

$$= \frac{\pi}{2}$$

When $\omega = 2$, $$|G(j\omega)H(j\omega)| = \frac{\sqrt{13}}{\sqrt{5}\sqrt{3}}$$

Therefore,

$$\angle G(j\omega)H(j\omega) = \tan^{-1}\frac{2}{3} - \tan^{-1}2 - \tan^{-1}(-2)$$

$$= 33.7^\circ - 63.5^\circ - 116.5^\circ$$

$$= -146.3^\circ$$

Hence this point will be in the third quadrant.

As ω increases from 0 to '+ ∞', the locus will move in the anticlockwise direction and enclose $-1 + j0$ once in the anticlockwise direction. Now, in the open-loop transfer function,

$$G(j\omega)H(j\omega) = \frac{j\omega+3}{(j\omega+1)(j\omega-1)}$$

one of the poles is in the right-hand side of the s-plane. But it satisfies the Nyquist criterion by enclosing once in its contour the point $(-1 + j0)$ in the anticlockwise direction. Therefore, the closed-loop system is stable.

8.2.1 Nyquist Contour with Open-loop Poles on the $j\omega$-Axis

Suppose there are three poles on the $j\omega$-axis, one at the origin and the others at $+j\omega_1$ and $-j\omega_1$. In this case, the Nyquist contour needs to be drawn as shown in Fig. 8.12. At the origin, the circle will be expressed mathematically as $re^{j\theta}$ where $r \to 0$. At $\pm j\omega_1$, the circles will be represented by $j\omega_1 + r_1e^{j\theta}$ and $-j\omega_1 + r_1e^{+j\theta}$, respectively, where $r_1 \to 0$.

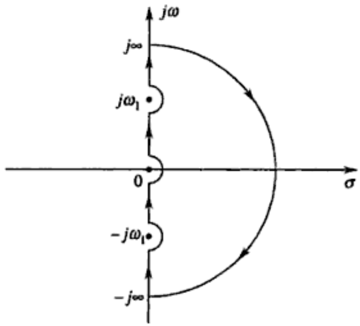


Fig. 8.12 Nyquist contour with open-loop poles on the jω-axis.

8.3 STUDY OF RELATIVE STABILITY

From the Nyquist criterion, it is very much clear that the stability criterion totally depends on the encirclement of the point $(-1 + j0)$ by the open-loop transfer function contour. Hence, it can be concluded in another way that if the polar plot of open-loop transfer function approaches closer to the point $(-1 + j0)$, the system tends towards instability.

From Figs. 8.13 and 8.14, it is quite clear that the system in Fig. 8.13 is more stable than the system in Fig. 8.14. For Fig. 8.13, the polar plot is drawn in Fig. 8.15. For Fig. 8.14, the polar plot is drawn in Fig. 8.16. In Fig. 8.16, it may be observed that the polar plot is much nearer to the point $(-1 + j0)$.

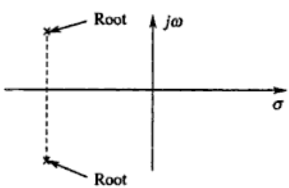


Fig. 8.13

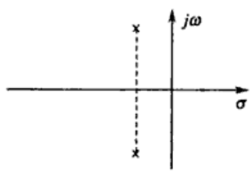


Fig. 8.14

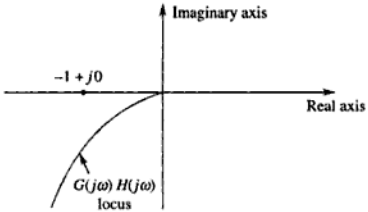


Fig. 8.15

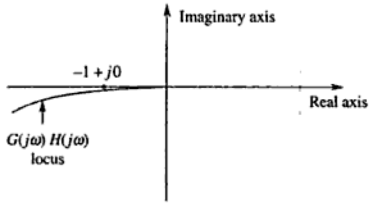


Fig. 8.16

8.4 GAIN MARGIN AND PHASE MARGIN

Suppose A is the locus of the open-loop transfer function $G(j\omega)H(j\omega)$ of a system (Fig. 8.17). Let us draw a unit circle taking its centre at the origin. The locus of $G(j\omega)H(j\omega)$ crosses the real axis at a frequency ω_2 making an intercept x on the real axis. The locus intersects the unit circle at a frequency ω_1 and the phase angle at this intersecting point is θ. If the value of x approaches unity and the phase angle θ tends to zero, the relative stability of the system is reduced. Hence, the relative stability can be measured in terms of the intercept x and the phase angle θ. The above concepts are used to define gain margin and phase margin for practical measurement of relative stability.

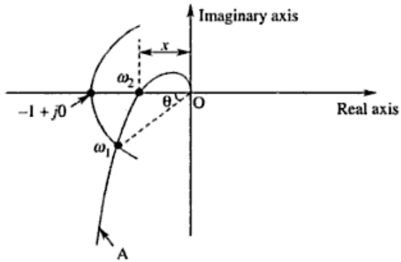


Fig. 8.17 Locus of the open-loop transfer function $G(j\omega)H(j\omega)$.

We know that the polar plot of

$$G(s)H(s) = \frac{K}{s(\tau_1 s + 1)(\tau_2 s + 1)}$$

will be as shown in Fig. 8.18 for different values of K.

The point of intersection of the polar plot with the negative real axis can be found out by making the imaginary part of $G(j\omega)H(j\omega)$ equal to zero. Now,

$$\frac{K}{j\omega(j\omega\tau_1 + 1)(j\omega\tau_2 + 1)} = -\frac{K(1 - j\omega\tau_1)(1 - j\omega\tau_2)j}{\omega(1 + \omega^2\tau_1^2)(1 + \omega^2\tau_2^2)}$$

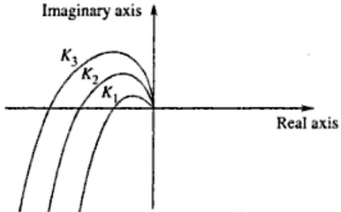


Fig. 8.18 Polar plot of $\dfrac{K}{s(\tau_1 s+1)(\tau_2 s+1)}$.

$$= \frac{-K(j+\omega\tau_1)(1-j\omega\tau_2)}{\omega(1+\omega^2\tau_1^2)(1+\omega^2\tau_2^2)} = \frac{-K(j+\omega\tau_2+\omega\tau_1-j\omega^2\tau_1\tau_2)}{\omega(1+\omega^2\tau_1^2)(1+\omega^2\tau_2^2)}$$

$$= \frac{-K\omega(\tau_1+\tau_2)}{\omega(1+\omega^2\tau_1^2)(1+\omega^2\tau_2^2)} + j\frac{K(\omega^2\tau_1\tau_2-1)}{\omega(1+\omega^2\tau_1^2)(1+\omega^2\tau_2^2)}$$

Putting the imaginary part of $G(j\omega)H(j\omega)$ equal to zero, we have

$$\frac{K(\omega^2\tau_1\tau_2-1)}{\omega(1+\omega^2\tau_1^2)(1+\omega^2\tau_2^2)} = 0$$

or

$$\omega = \frac{1}{\sqrt{\tau_1\tau_2}}$$

The magnitude of the real part will be

$$\frac{-K\omega(\tau_1+\tau_2)}{\omega(1+\omega^2\tau_1^2)(1+\omega^2\tau_2^2)} = \frac{-K(\tau_1+\tau_2)}{\left(1+\dfrac{1}{\tau_1\tau_2}\tau_1^2\right)\left(1+\dfrac{1}{\tau_1\tau_2}\tau_2^2\right)}$$

$$= \frac{-K(\tau_1+\tau_2)}{\left(1+\dfrac{\tau_1}{\tau_2}\right)\left(1+\dfrac{\tau_2}{\tau_1}\right)} = \frac{-K(\tau_1+\tau_2)}{\left(\dfrac{\tau_2+\tau_1}{\tau_2}\right)\left(\dfrac{\tau_1+\tau_2}{\tau_1}\right)}$$

$$= \frac{-K\tau_1\tau_2}{\tau_1+\tau_2}$$

Now, for the system to be stable,

$$\frac{K\tau_1\tau_2}{\tau_1+\tau_2} < 1$$

or

$$K < \frac{\tau_1+\tau_2}{\tau_1\tau_2}$$

8.4.1 Gain Margin

Gain margin (GM) is the factor by which the system gain can be increased to drive it to the verge of stability. In Fig. 8.17, it is observed that at $\omega = \omega_2$ the phase angle of $G(j\omega)H(j\omega)$ is 180° and the magnitude $|G(j\omega)H(j\omega)|$ is x. If the gain of the system is increased by the factor $\frac{1}{x}$, then $|G(j\omega)H(j\omega)|$ will become $x \cdot \frac{1}{x} = 1$ at $\omega = \omega_2$. Then the polar plot of $G(j\omega)H(j\omega)$ will pass through the point $(-1 + j0)$ making the system to move towards instability. Hence the gain margin can be defined as the reciprocal of the gain at the frequency at which the phase angle becomes 180°. The frequency at which the phase angle becomes 180° is termed the *phase cross-over frequency*. Therefore, GM = $\frac{1}{x}$, where $x = |G(j\omega)\ H(j\omega)|$ at $\omega = \omega_2$. In decibel, the gain margin will be expressed as

$$\text{GM} = -20 \log x \quad \text{dB.}$$

8.4.2 Phase Margin

The frequency at which $|G(j\omega)H(j\omega)| = 1$ is termed the *gain cross-over frequency*. It is found by the intersection of $G(j\omega)H(j\omega)$ and the unit circle drawn with centre as the origin. At this frequency (ω_1) the phase angle of $G(j\omega_1)H(j\omega_1)$ is $(-\pi + \theta)$. If an additional phase lag equal to θ is applied at the gain cross-over frequency, the phase angle of $G(j\omega_1)H(j\omega_1)$ will change to $-180°$, and the magnitude will remain unity.

The system will approach the door of instability. That is why, this phase lag θ is defined as the phase margin (PM). In other words, the phase margin is termed the amount of additional phase lag at the gain cross-over frequency needed to take the system towards instability.

Hence, the phase margin can be expressed as

$$\text{PM} = [\angle G(j\omega)H(j\omega) \text{ when } \omega = \omega_1] + 180°$$

where ω_1 is the gain cross-over frequency.

EXAMPLE 8.4 Determine the relationship between the phase margin and the damping factor.

Solution Let us consider the open-loop transfer function

$$G(s)H(s) = \frac{K}{s(\tau s + 1)} = \frac{\omega_n^2}{s(s + 2\zeta\omega_n)}$$

Now,

$$\frac{K}{s(\tau s + 1)} = \frac{K}{\tau s^2 + s} = \frac{K}{\tau s\left(s + \frac{1}{\tau}\right)} = \frac{\frac{K}{\tau}}{s\left(s + \frac{1}{\tau}\right)}$$

Since,

$$\frac{K}{s(\tau s + 1)} = \frac{\omega_n^2}{s(s + 2\zeta\omega_n)}$$

we get

$$\frac{K}{\tau} = \omega_n^2 \quad \text{or} \quad \omega_n = \sqrt{\frac{K}{\tau}} \quad \text{and} \quad \frac{1}{\tau} = 2\zeta\omega_n$$

or

$$2\zeta\sqrt{\frac{K}{\tau}} = \frac{1}{\tau} \quad \text{or} \quad 2\,\zeta\sqrt{K} = \frac{1}{\sqrt{\tau}}$$

or

$$\zeta = \frac{1}{2\sqrt{K\tau}}$$

At the gain cross-over frequency $\omega = \omega_1$, the magnitude $|G(j\omega)H(j\omega)| = 1$. That is,

$$\frac{\omega_n^2}{\omega_1\sqrt{\omega_1^2 + 4\zeta^2\omega_n^2}} = 1$$

or

$$\omega_n^2 = \omega_1\sqrt{(\omega_1^2 + 4\zeta^2\omega_n^2)}$$

or

$$\omega_n^4 = \omega_1^2(\omega_1^2 + 4\zeta^2\omega_n^2)$$

or

$$\omega_1^4 + 4\zeta^2\omega_n^2\omega_1^2 - \omega_n^4 = 0$$

Therefore,

$$\omega_1^2 = \frac{-4\,\zeta^2\omega_n^2 \pm \sqrt{16\zeta^4\omega_n^4 + 4\omega_n^4}}{2}$$

$$= -2\,\zeta^2\omega_n^2 \pm \omega_n^2\sqrt{4\zeta^4 + 1}$$

or

$$\frac{\omega_1^2}{\omega_n^2} = -\,2\zeta^2 \pm \sqrt{4\zeta^4 + 1}$$

Since $\dfrac{\omega_1^2}{\omega_n^2}$ is a positive number,

$$\frac{\omega_1^2}{\omega_n^2} = \sqrt{4\zeta^4 + 1} - 2\zeta^2$$

or

$$\omega_1 = \left(\sqrt{4\zeta^4 + 1} - 2\zeta^2\right)^{1/2}\omega_n$$

Again, phase margin (PM)

$$\theta = [\angle G(j\omega)\, H(j\omega) \text{ for } \omega = \omega_1] + 180^\circ$$

$$= -90^\circ - \tan^{-1} \frac{\omega_1}{2\zeta\omega_n} + 180^\circ$$

$$= 90^\circ - \tan^{-1} \frac{\omega_1}{2\zeta\omega_n}$$

or

$$= 90 - \tan^{-1}\left[\frac{1}{2\zeta}\left(\sqrt{4\zeta^4 + 1} - 2\zeta^2\right)^{1/2}\right]$$

or

$$\tan(90^\circ - \theta) = \tan \tan^{-1}\left[\frac{1}{2\zeta}\left(\sqrt{4\zeta^4 + 1} - 2\zeta^2\right)^{1/2}\right]$$

or

$$\cot \theta = \frac{1}{2\zeta}\left(\sqrt{4\zeta^4 + 1} - 2\zeta^2\right)^{1/2}$$

or

$$\tan \theta = 2\zeta \left(\frac{1}{\sqrt{4\zeta^4 + 1} - 2\zeta^2}\right)^{1/2}$$

Therefore,

$$\theta = \tan^{-1}\left[2\zeta\left(\frac{1}{(4\zeta^4 + 1)^{1/2} - 2\zeta^2}\right)^{1/2}\right].$$

Method of calculating the open-loop gain if the gain margin is provided and the open-loop transfer function is given as

$$G(s) = \frac{K}{s(\tau_1 s + 1)(\tau_2 s + 1)}$$

There are two methods of solving this problem: (a) graphical method and (b) analytical method.

Graphical method. Let us assume $K = 1$. Then,

$$G(j\omega) = \frac{1}{j\omega(\tau_1 j\omega + 1)(\tau_2 j\omega + 1)}$$

or

$$|G(j\omega)| = \frac{1}{\omega\sqrt{1 + \tau_1^2\omega^2}\,\sqrt{1 + \tau_2^2\omega_2}}$$

$$\angle G(j\omega) = -90^\circ - \tan^{-1} \tau_1\omega - \tan^{-1} \tau_2\omega$$

We have to draw the polar plot of $G(j\omega)$. (See Fig. 8.19.)

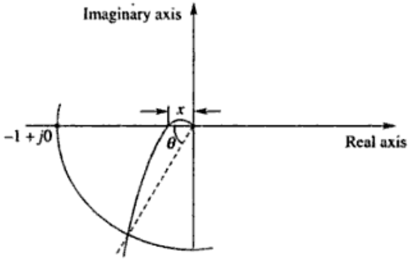


Fig. 8.19 Polar plot of G(jω).

Suppose the gain margin obtained is

$$\gamma = 20 \log \frac{1}{x} \qquad \text{and} \qquad \text{phase margin} = \theta$$

Now to obtain the given gain margin Z, the Nyquist plot should intersect the negative real axis, say, at b. Therefore,

$$20 \log \frac{1}{b} = Z \qquad \text{or} \qquad \log \frac{1}{b} = \frac{Z}{20}$$

or

$$10^{Z/20} = \frac{1}{b} \qquad \text{or} \qquad b = \frac{1}{10^{Z/20}}$$

Now to achieve b, the system gain is to be increased by the factor $\frac{b}{x}$. Hence the value of $K = \frac{b}{x}$. Therefore,

$$K = \frac{1}{10^{Z/20} \cdot x}$$

Analytical method. We have already calculated, $b = \frac{1}{10^{Z/20}}$.

The plot of $G(j\omega)$ intersects the real axis where the imaginary part is equal to zero. That means the imaginary part of $G(j\omega) = 0$. Therefore,

$$\text{Imaginary part of } \frac{K}{j\omega(\tau_1 j\omega + 1)(\tau_2\, j\omega + 1)} = 0$$

From the above, ω can be determined. Say, the value is $\omega = \omega_2$.

Therefore,

$$|G(j\omega)| \text{ at } \omega = \omega_2 = b = \frac{1}{10^{Z/20}}$$

From the above, the value of K can be found out.

Method of calculating the open-loop gain if the phase margin is provided and the open-loop transfer function is given as

$$G(s) = \frac{K}{s(\tau_1 s + 1)(\tau_2 s + 1)}$$

Graphical method. Suppose the phase margin is α. The polar plot of

$$G(j\omega) = \frac{1}{j\omega(\tau_1 j\omega + 1)(\tau_2 j\omega + 1)}$$

is redrawn as shown in Fig. 8.20.

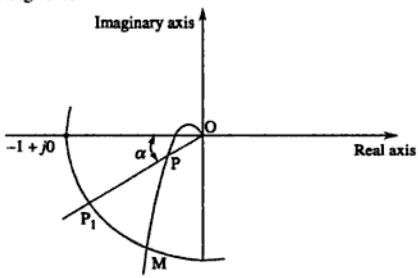


Fig. 8.20 Polar plot of G(jω).

A circle drawn with the centre O intersects the polar plot at the point M. The angle α is drawn and that intersects the polar plot and the circle at P and P_1, respectively. Now, to make α as the phase margin, the point P is to be shifted to P_1 Hence the system gain needs to be increased by a factor $\frac{OP_1}{OP}$. Therefore, the value of K will be $\frac{OP_1}{OP}$.

Analytical method. Suppose $\omega = \omega_1$ is the gain cross-over frequency. Then from

$$G(j\omega) = \frac{K}{j\omega(\tau_1 j\omega + 1)(\tau_2 j\omega + 1)}$$

we get

$$\angle G(j\omega) = -90° - \tan^{-1} \tau_1\omega - \tan^{-1} \tau_2\omega$$

Therefore,

$$\angle G(j\omega) + 180° = \alpha$$

or

$$+90^\circ - \tan^{-1} \tau_1\omega_1 - \tan^{-1} \tau_2\omega_1 = \alpha$$

From the above ω_1 can be calculated. Therefore,

$$|G(j\omega)|_{\omega=\omega_1} = \frac{K}{\omega_1\sqrt{1+(\tau_1\omega_1)^2}\sqrt{1+(\tau_2\omega_1)^2}} = 1$$

Putting the value of ω_1, we can find the value of K.

Method of determining the gain margin and phase margin with the help of Bode plot

Suppose the open-loop transfer function is $\frac{K}{s(\tau_1 s+1)(\tau_2 s+1)}$. The polar plot of this function is drawn again in Fig. 8.21. The gain margin is 20 log $\frac{1}{x}$ = – 20 log x.

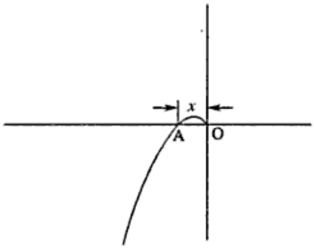


Fig. 8.21 Polar plot of $\frac{K}{s(\tau_1 s+1)(\tau_2 s+1)}$.

Since x is less than one, – 20 log x is a positive number. At the point A, the phase angle is – 180°. Corresponding to this – 180°, the decibel observed in the Bode plot will be 20 log x and applying a negative sign, we will get the gain margin with the help of the Bode plot.

Figure 8.22 shows how the gain margin is calculated. Corresponding to the phase angle curve, the phase angle – 180° is to be located, say it is at N. Now, corresponding to the point N, OM is the value of frequency. Corresponding to the point M, Q will be the point on the magnitude (decibel) curve. The y-axis value of the point Q will be 20 log x and, therefore, – 20 log x will represent the gain margin.

For phase margin, we know that $|G(j\omega)| = 1$. Hence 20 log $|G(j\omega)| = 0$.

Suppose, in the Bode plot, the value of 20 log $|G(j\omega)| = 0$, is at S. Corresponding to the point S, the point Z will be the phase angle of $\angle G(j\omega)$. The angle WZ in degrees plus 180° will be the phase margin.

It is also possible to adjust the system gain for a specified GM or PM with the help of the Bode plot. Suppose the gain margin found in the Bode plot is x_1 and it is to be increased to x_2, then we have to determine how the open-loop gain is to be changed. That means $x_2 - x_1$ will be

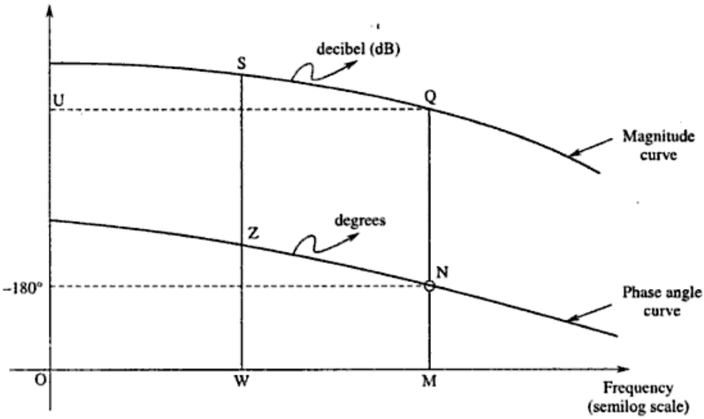


Fig. 8.22 Bode plot to calculate gain margin and phase margin.

a positive number. Therefore, the system gain is to be reduced by $-(x_2 - x_1)$ dB or by the factor y, where

$$20 \log y = x_2 - x_1 \quad \text{or} \quad \log y = \frac{x_2 - x_1}{20}$$

or

$$y = 10^{(x_2 - x_1)/20}$$

That means if the gain factor was K, it will now be $\frac{K}{y}$.

Suppose we have to reduce the phase margin. Say, the phase margin found in the Bode plot is θ_1, now we have to make it θ_2. For the θ_2 phase margin, the phase angle in the Bode plot will be $(\theta_2 - 180°)$. Corresponding to $(\theta_2 - 180°)$ phase angle in the Bode plot, the frequency is read from the horizontal axis of the Bode plot. Corresponding to that frequency, the magnitude of $G(j\omega)$ is found out from the Bode plot. If the angle θ_2 is to be made the phase margin, the above magnitude is made zero because 20 log 1 = 0 and at the phase margin, $|G(j\omega)| = 1.0$. If this value is found less than zero by some amount p dB, then the system gain is to be raised upwards by p dB. It means that the system gain is to be increased K_1 times, where

$$20 \log K_1 = p$$

If the gain of the system was earlier K, it would now be KK_1.

8.5 CONSTANT-M CIRCLES

The closed-loop frequency response of a system can be expressed as

$$T(j\omega) = \frac{O(j\omega)}{R(j\omega)}$$

where $O(j\omega)$ and $R(j\omega)$ are the output and the input, respectively, in terms of the frequency response. If $G(j\omega)$ is the open-loop frequency response, then in a closed-loop unity-feedback system the transfer function will be

$$T(j\omega) = \frac{O(j\omega)}{R(j\omega)} = \frac{G(j\omega)}{1+G(j\omega)} = \frac{x+jy}{1+x+jy} = Me^{j\beta}$$

Therefore, the magnitude M of $T(j\omega)$ is given by

$$M = \frac{|x+jy|}{|1+x+jy|} = \frac{\sqrt{x^2+y^2}}{\sqrt{(1+x)^2+y^2}}$$

or

$$M^2 = \frac{x^2+y^2}{(1+x)^2+y^2}$$

Rearranging this equation yields

$$x^2(M^2-1) + 2xM^2 + y^2(M^2-1) + M^2 = 0$$

or

$$x^2 + \frac{2xM^2}{M^2-1} + y^2 + \frac{M^2}{M^2-1} = 0 \quad \text{(dividing throughout by } M^2-1\text{)}$$

or

$$x^2 + \frac{2xM^2}{M^2-1} + \left(\frac{M^2}{M^2-1}\right)^2 - \left(\frac{M^2}{M^2-1}\right)^2 + y^2 + \frac{M^2}{M^2-1} = 0 \quad \left(\text{adding and subtracting the term } \frac{M^2}{M^2-1}\right)$$

or

$$\left(x+\frac{M^2}{M^2-1}\right)^2 + y^2 + \frac{M^4-M^2-M^4}{(M^2-1)^2} = 0$$

or

$$\left(x+\frac{M^2}{M^2-1}\right)^2 + y^2 = \left(\frac{M}{M^2-1}\right)^2$$

This is the equation of a circle with the centre at $\left(-\frac{M^2}{M^2-1}, 0\right)$ and with radius $\frac{M}{M^2-1}$.

Thus for various values of M, a large number of constant-M circles can be drawn. Figure 8.23 shows the different constant-M circles drawn for some values of M.

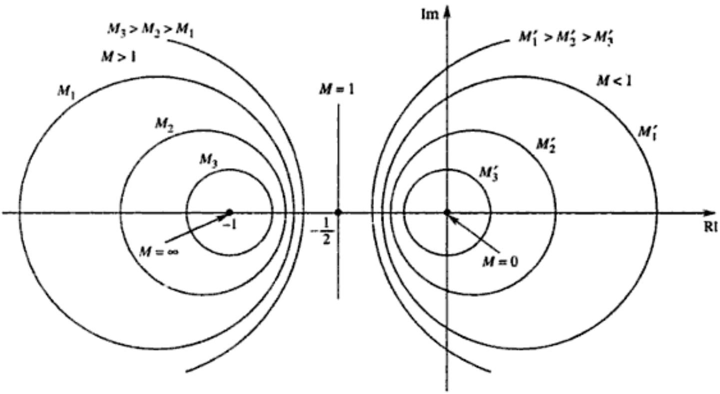


Fig. 8.23 Constant-M circles.

Now:

(i) For $M > 1$, as M increases, the radius of the M-circle reduces and the centre located on the negative real axis proceeds towards $(-1 + j0)$. For, $M = \infty$, radius $= \dfrac{M}{M^2 - 1} = \dfrac{\frac{1}{M}}{1 - \frac{1}{M^2}} = 0$ and the centre $\left(= -\dfrac{M^2}{M^2 - 1} = -\dfrac{1}{1 - \frac{1}{M^2}} = -1 \right)$ is at $(-1, 0)$.

(ii) For $M = 1$, radius $= \dfrac{M}{M^2 - 1} = \infty$, centre $\left[\dfrac{-M^2}{M^2 - 1} = \infty \right]$ is at $(-\infty, 0)$. Therefore, for $M = 1$, the centre is at infinity and the radius is infinity. This indicates that it is a straight line parallel to the y-axis.

Now we know that,

$$y^2 + \left[x + \frac{M^2}{M^2 - 1} \right]^2 = \frac{M^2}{(M^2 - 1)^2}$$

A line parallel to the y-axis means that $x = K$. The intercept on the x-axis by this line is $(K, 0)$. Substituting $x = K$, $y = 0$ in the preceding equation, we have

$$\left(K + \frac{M^2}{M^2 - 1}\right)^2 = \frac{M^2}{(M^2 - 1)^2}$$

or

$$K + \frac{M^2}{M^2 - 1} = \frac{M}{M^2 - 1}$$

or

$$K = \frac{M}{M^2 - 1} - \frac{M^2}{M^2 - 1} = \frac{-M}{M + 1}$$

Now, when $M = 1$, we get $K = -\frac{1}{2}$. Hence the intercept on the x-axis will be at $\left(-\frac{1}{2}, 0\right)$ for the straight line $(M = 1)$.

(iii) For $M < 1$, as M decreases, the radius of the M-circle reduces and the centre located on the positive real axis approaches towards the origin. For $M = 0$, radius $= \frac{M}{M^2 - 1} = 0$, and the centre $\left(-\frac{M^2}{M^2 - 1} = 0\right)$ will be at (0, 0).

8.6 CONSTANT-N CIRCLES

The phase angle of the closed-loop transfer function $T(j\omega)$ is given by

$$\beta = \angle\left(\frac{x + iy}{1 + x + iy}\right)$$

$$= \tan^{-1}\frac{y}{x} - \tan^{-1}\frac{y}{1 + x} = \tan^{-1}\frac{\frac{y}{x} - \frac{y}{1 + x}}{1 + \frac{y}{x}\frac{y}{1 + x}}$$

$$= \tan^{-1}\frac{y}{x + x^2 + y^2}$$

or

$$\tan\beta = \frac{y}{x + x^2 + y^2}$$

or

$$N = \frac{y}{x + x^2 + y^2} \qquad \text{where } N \text{ is the phase angle in degrees.}$$

or

$$x^2 + x + y^2 - \frac{y}{N} = 0$$

or

$$x^2 + 2\frac{x}{2} + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + y^2 - 2\frac{y}{2N} + \left(\frac{1}{2N}\right)^2 - \left(\frac{1}{2N}\right)^2 = 0$$

or

$$\left(x + \frac{1}{2}\right)^2 + \left(y - \frac{1}{2N}\right)^2 = \frac{1}{4} + \frac{1}{4N^2} = \frac{N^2 + 1}{4N^2}$$

or

$$\left(x + \frac{1}{2}\right)^2 + \left(y - \frac{1}{2N}\right)^2 = \frac{1}{2^2}\frac{N^2 + 1}{N^2}$$

The centres of the circles described by the above equation will be at $\left(-\frac{1}{2}, \frac{1}{2N}\right)$ with radius of value $\frac{1}{2N}\sqrt{N^2 + 1}$.

Then for different values of β, a large number of N-circles can be drawn. These circles pass through (0, 0) since the circle equation $\left(x + \frac{1}{2}\right)^2 + \left(y - \frac{1}{2N}\right)^2 = \frac{1}{4} + \frac{1}{4N^2}$ is satisfied.

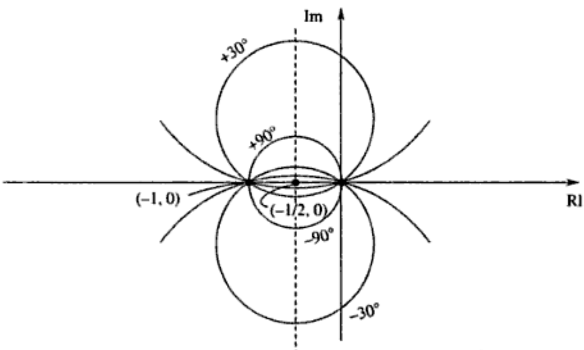


Fig. 8.24 Constant-N circles.

Figure 8.24 shows a large number of *N* circles. The circles

$$\left(x+\frac{1}{2}\right)^2 + \left(y-\frac{1}{2N}\right)^2 = \frac{1}{2^2}\frac{N^2+1}{N^2}$$

also pass through (–1, 0) since

$$\text{L.H.S.} = \left(-1+\frac{1}{2}\right)^2 + \left(\frac{1}{2N}\right)^2 = \frac{1}{4} + \frac{1}{4N^2} = \text{R.H.S.}$$

8.7 METHOD OF DETERMINING CLOSED-LOOP FREQUENCY RESPONSE FOR NONUNITY FEEDBACK SYSTEMS

We know that the closed-loop transfer function of a nonunity feedback system is expressed as

$$T(j\omega) = \frac{G(j\omega)}{1+G(j\omega)\,H(j\omega)}$$

where $H(j\omega)$ is the nonunity feedback path transfer function. Now, by modification

$$T(j\omega) = \frac{1}{H(j\omega)}\left[\frac{G(j\omega)\,H(j\omega)}{1+G(j\omega)\,H(j\omega)}\right]$$

$$= \frac{1}{H(j\omega)}\left[\frac{G_1(j\omega)}{1+G_1(j\omega)}\right] = \frac{1}{H(j\omega)}T_1(j\omega)$$

where $G_1(j\omega) = G(j\omega)H(j\omega)$ and $T_1(j\omega) = \dfrac{G_1(j\omega)}{1+G_1(j\omega)}$.

Hence constant-*M* and constant-*N* circles can be found out for the unity feedback transfer function $T_1(j\omega)$, and the nonunity feedback transfer function $T(j\omega)$ can then be determined by multiplying $T_1(j\omega)$ by $\dfrac{1}{H(j\omega)}$.

This procedure can be easily applied by Bode plot. With the help of the Bode plot, $T_1(j\omega)$ and $H(j\omega)$ are drawn and the log magnitude and phase angle of $H(j\omega)$ can be subtracted from those of $T_1(j\omega)$ to get the log magnitude and phase angle of $T(j\omega)$. The result will therefore be the Bode plot for the closed-loop frequency response $T(j\omega)$.

8.8 NICHOLS CHART

For ease in design work, N.B. Nichols transformed the constant-*M* and constant-*N* circles in polar coordinates to log magnitude and phase angle coordinates. The resulting chart is known as the

Nichols chart. The Nichols chart is therefore a decibel magnitude–phase angle plot of the loci of constant dB magnitude and phase angle of $\frac{O}{R}(j\omega)$, graphed as $|G(j\omega)|$ versus $\angle G(j\omega)$. Now,

$$T(j\omega) = \frac{G(j\omega)}{1+G(j\omega)} = \frac{|G(j\omega)|\angle\phi_G}{1+|G(j\omega)|\angle\phi_G}$$

$$= \frac{|G(j\omega)|\angle\phi_G}{1+|G(j\omega)|\cos\phi_G + j|G(j\omega)|\sin\phi_G}$$

Therefore,

$$\angle T(j\omega) = \angle\phi_G - \tan^{-1}\left[\frac{|G(j\omega)|\sin\phi_G}{1+|G(j\omega)|\cos\phi_G}\right]$$

or

$$\tan[\angle T(j\omega)] = \tan\left[\phi_G - \tan^{-1}\frac{G\sin\phi_G}{1+G\cos\phi_G}\right] = N$$

or

$$N = \frac{\tan\phi_G - \dfrac{G\sin\phi_G}{1+G\cos\phi_G}}{1+\tan\phi_G \cdot \dfrac{G\sin\phi_G}{1+G\cos\phi_G}}$$

$$= \frac{\dfrac{\sin\phi_G}{\cos\phi_G} - \dfrac{G\sin\phi_G}{1+G\cos\phi_G}}{1+\dfrac{\sin\phi_G}{\cos\phi_G}\cdot\dfrac{G\sin\phi_G}{1+G\cos\phi_G}}$$

$$= \frac{\sin\phi_G(1+G\cos\phi_G) - G\sin\phi_G\cos\phi_G}{\cos\phi_G(1+G\cos\phi_G)+G\sin^2\phi_G} = \frac{\sin\phi_G}{\cos\phi_G + G}$$

or

$$G + \cos\phi_G - \frac{\sin\phi_G}{N} = 0$$

That means,

$$|G(j\omega)| + \cos\phi_G - \frac{1}{N}\sin\phi_G = 0 \tag{8.1}$$

For a fixed value of N, the locus can therefore be drawn in the following manner:

(a) Take the value of ϕ_G.
(b) Calculate $|G(j\omega)|$ from Eq. (8.1).
(c) Plot the value of $20\log_{10}|G(j\omega)|$ versus phase angle ϕ_G.

Figure 8.25 shows the dB magnitude vs. the phase angle curve for a constant, $N = -\tan 60° = -\sqrt{3}$.

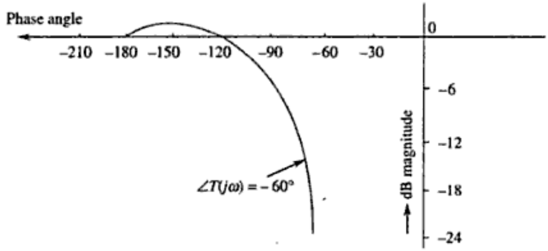


Fig. 8.25 *dB magnitude vs. phase angle plot for N = –tan 60°.*

Again

$$\frac{O(j\omega)}{R(j\omega)} = \frac{|G(j\omega)|\angle\phi_G}{1+|G(j\omega)|\angle\phi_G}$$

$$= \frac{|G(j\omega)|[\cos\phi_G + j\sin\phi_G]}{1+|G(j\omega)|\cos\phi_G + j|G(j\omega)|\sin\phi_G}$$

$$= \frac{|G(j\omega)|}{\sqrt{(1+|G(j\omega)|\cos\phi_G)^2 + (|G(j\omega)|\sin\phi_G)^2}}$$

or

$$M = \frac{|G(j\omega)|}{\sqrt{1+2|G(j\omega)|\cos\phi_G + |G(j\omega)|^2}}$$

or

$$M^2 = \frac{|G(j\omega)|^2}{1+2|G(j\omega)|\cos\phi_G + |G(j\omega)|^2}$$

or

$$|G(j\omega)|^2(M^2-1) + 2M^2|G(j\omega)|\cos\phi_G + M^2 = 0$$

or

$$|G(j\omega)|^2 + \frac{2M^2}{M^2-1}|G(j\omega)|\cos\phi_G + \frac{M^2}{M^2-1} = 0 \tag{8.2}$$

For a fixed value of M, the locus can therefore be drawn in the following manner:

(a) Find out numerical values of $|G(j\omega)|$.

(b) Solve the resultant equation for ϕ_G, excluding values $|G(j\omega)|$ for which $|\cos\phi_G| > 1$, from Eq. (8.2).

(c) Plot the dB magnitude versus phase angle, that is, 20 log $|G(j\omega)|$ versus phase angle.

Figure 8.26 is the dB magnitude versus the phase angle plot for constant $M = \sqrt{2}$, that means, 20 log $|T(j\omega)|$ = 20 log $\sqrt{2}$ = 3 dB.

Now Fig. 8.26 and Fig. 8.25 are shown in the combined form in Nichols chart for different constant values of M and N (see Fig. 8.27).

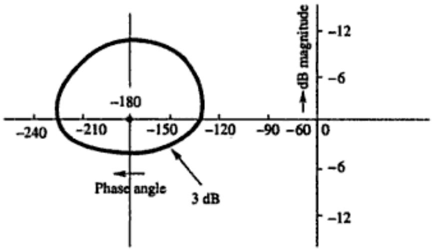


Fig. 8.26 dB magnitude vs. phase angle plot for $M = \sqrt{2}$

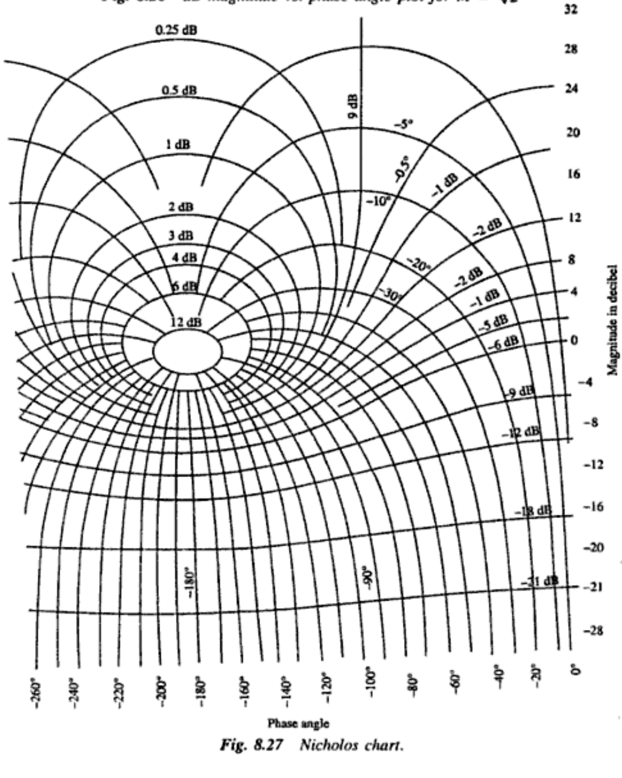


Fig. 8.27 Nicholos chart.

SUMMARY

The fundamental concept of stability in frequency response systems is discussed. The Nyquist contour is defined. The Nyquist stability criterion is explained. Examples to apply the Nyquist criterion are taken and solved. Study of relative stability is made. Gain margin and phase margin are defined. The method of determining open-loop gain, if the gain margin is given as data, is explained. The method of calculating the open-loop gain, if the phase margin is provided, is shown. The method of determining the gain margin and the phase margin with the help of the Bode plots is described. Procedures for determining constant-M circles and constant-N circles in the case of the closed-loop frequency response of a system are explained. The procedure of drawing the Nichols chart is shown.

QUESTIONS

1. Draw the complete Nyquist plot for the system with the open-loop transfer function $G(s)H(s) = \dfrac{K}{s(s-1)}$. Comment on the closed-loop stability of the system.

2. Comment, using the Nyquist criterion, on the closed-loop stability of the system whose open-loop transfer function is

$$G(s)H(s) = \frac{(0.5)(1+5s)}{s^2(1+0.5s)}$$

Find also the gain margin and the phase margin.

3. State the Nyquist stability criterion and investigate the stability of the closed-loop system with the following open-loop transfer function

$$G(s)H(s) = \frac{2(s+3)}{s(s-1)}$$

by drawing the Nyquist plot. Wherever the Nyquist plot crosses the real or imaginary axis, determine the frequency and the intercept value.

4. Sketch the Nyquist plot for the control system whose loop transfer function is given by

$$G(s)\,H(s) = \frac{1}{s(1+0.2s)(1+0.5s)}$$

Determine the gain margin and comment on the stability of the system.

5. Illustrate with suitable Nyquist diagrams the distinguishing features of a control system which has absolute stability and a system having conditional stability.

6. What do you mean by "Constant-M circles" and "Constant-N circles"?

7. How do you calculate the closed-loop frequency response for nonunity feedback systems? Explain clearly.

8. Describe Nichols Chart. Discuss its importance in finding the frequency response of systems?

9. State and explain the Nyquist stability criterion. How do you study relative stability from the Nyquist criterion?

10. Define gain margin, phase margin, phase cross-over frequency, and gain cross-over frequency.

11. Consider a feedback system having the characteristic equation

$$1+\frac{K}{(s+1)(s+1.5)(s+2)}=0$$

It is desired that all the roots of the characteristic equation should have real parts less than −1. Extend the Nyquist stability criterion to find the largest value of K, satisfying this condition.

12. (a) Make a rough sketch of the Nyquist's plot for a system whose open-loop transfer function is $G(s)\,H(s)=\dfrac{5}{s(1+0.2s)\,(1+s)}$.
 (b) Is the above system stable?
 (c) Define the "gain margin" (G.M.) of a system and determine the G.M. of the system specified in (a).
 (d) Define the "phase margin" (ϕ) of a system and indicate how can this be determined from the Nyquist plot?

13. (a) State and explain the frequency domain specifications for a control system.
 (b) Sketch the Nyquist plot for

$$GH(s)=\frac{1}{s^4(s+p)}; \qquad p>0$$

14. Give the relative advantages and disadvantages of the two graphical methods used in the control systems, namely the Nyquist plot and the Bode plot. State with reasons which method would you choose for design purposes.

15. Sketch the Bode-plot, on a plain paper, for the following function and give the approximate values of the gain margin and phase margin

$$G(s)=\frac{10(s+2)}{s(s+6)\,(s+10)}$$

Chapter 9 Compensators and Controllers

9.1 INTRODUCTION

Let us first of all study what compensation is and why is it required in control systems. To explain this we take an example of an open-loop transfer function

$$G(s) = \frac{60K}{s(s+2)(s+6)}$$

We know that the velocity error constant, K_v, for unit-ramp input is

$$\lim_{s\to 0} sG(s) = \lim_{s\to 0} \frac{s.60K}{s(s+2)(s+6)} = 5K$$

Therefore, the steady-state error $= \frac{1}{5K}$. If K is 1, then the steady-state error is 0.2. Suppose $60K = 35$ or $K = \frac{35}{60}$. Then,

$$G(s) = \frac{35}{s(s+2)(s+6)}$$

The characteristic equation is

$$s(s+2)(s+6) + 35 = 0$$

or

$$s^3 + 8s^2 + 12s + 35 = 0$$

or

$$(s+7)(s^2 + s + 5) = 0$$

That is,

$$s = -7 \qquad \text{or} \qquad s = \frac{-1 \pm j\sqrt{19}}{2}$$

The centroid of the asymptotes is

$$\frac{-2-6}{3} = -2.66$$

The angles of the asymptotes are

$$\frac{(2q+1)180°}{3} = 60°,\ 180°,\ 300° \qquad \text{with } q = 0,\ 1,\ 2.$$

For breakaway point

$$\frac{dK}{ds} = -\frac{d}{ds}[s(s+2)(s+6)] = 0$$

or

$$\frac{d}{ds}(s^3 + 8s^2 + 12s) = 0$$

or

$$3s^2 + 16s + 12 = 0$$

That is,

$$s = \frac{-8 \pm 2\sqrt{7}}{3} = -0.906,\ -4.426$$

Therefore, the breakaway point will be at −0.906 since the value − 4.426 will have an even number of poles, i.e. 2.

Now, when $K = \frac{35}{60}$, we got three roots, that is: −7, $\frac{-1 + j\sqrt{19}}{2}$, $\frac{-1 - j\sqrt{19}}{2}$.

Similarly, when $K = 1$, we will get two conjugate roots: $-0.3 + j2.8$ and $-0.3 - j2.8$. These roots are usually called the dominant pair of roots.

The $60K$ is nothing but the system gain. From Fig. 9.1, it is clear that when the system gain is 60, the undamped natural frequency

$$\text{OA} = \omega_n = \sqrt{(0.3)^2 + (2.8)^2} = 2.85$$

$$\cos\theta = \zeta = \text{damping coefficient}$$
$$= \cos 84° = 0.105$$

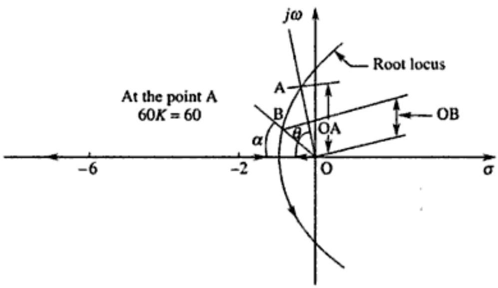


Fig. 9.1

$$\text{Settling time} = \frac{4}{\zeta\omega_n} = \frac{4}{0.105 \times 2.85} = 13.36 \text{ s}$$

Usually the value of ζ is 0.6 and the settling time is less than 4 s. So, $\cos \alpha = 0.6$ or $\alpha = \cos^{-1} 0.6$. We now draw the angle α. If we draw ($\alpha = \cos^{-1} 0.6$), we will find OB = ω_n = 1.26. Therefore,

$$\text{Settling time} = \frac{4}{0.6 \times 1.26} = 5.3 \text{ s}$$

$$60K = 10.5$$

The value of $60K$ will be found from $|s(s + 2)(s + 6)| = 60K$. The value of s will be found from the coordinates of the point B. Now,

$$K_v = \lim_{s \to 0} s \frac{10.5}{s(s+2)(s+6)} = \frac{10.5}{12} = 0.875$$

$$\text{Steady-state error} = \frac{1}{0.875} = 1.14$$

Thus, it is observed that to get the required value of the damping coefficient, the steady-state error increases and the settling time improves. Hence by changing the gain $60K$, all the requirements are not fulfilled.

Now from the point of view of mathematical experimentation, we insert one zero in the expression for the open-loop transfer function. That is, we make the open-loop transfer function

$$G(s) = \frac{60K(s+3)}{s(s+2)(s+6)}$$

The centroid of asymptotes is

$$\frac{-2-6+3}{3-1} = -\frac{5}{2} = -2.5$$

The angles of asymptotes are

$$\frac{(2q+1)\pi}{3-1} = \frac{\pi}{2}, \frac{3\pi}{2} \quad \text{with } q = 0, 1$$

For breakaway point:

$$s(s + 2)(s + 6) + 60K(s + 3) = 0$$

or

$$-\frac{s(s+2)(s+6)}{60(s+3)} = K$$

Therefore,

$$\frac{dK}{ds} = \frac{d}{ds}\left[\frac{(s^2+2s)(s+6)}{60(s+3)}\right] = 0$$

or

$$\frac{d}{ds}\left[\frac{(s^2+2s)(s+6)}{(s+3)}\right]=0$$

or

$$\frac{d}{ds}\left[\frac{s^3+8s^2+12s}{s+3}\right]=0$$

or

$$\frac{(3s^2+16s+12)(s+3)-s^3-8s^2-12s}{(s+3)^2}=0$$

or

$$2s^3 + 17s^2 + 48s + 36 = 0$$

The breakaway point will have a real value and that can be determined very easily by the graphical method. Say, the value is σ. Then,

$$y = 2\sigma^3 + 17\sigma^2 + 48\sigma + 36$$

We draw the curve graphically and find the value σ when $y = 0$. The root locus is shown in Fig. 9.2.

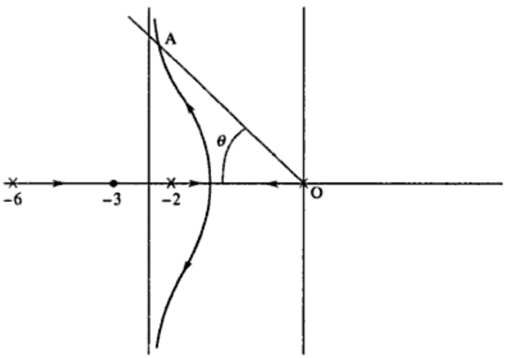


Fig. 9.2

Again we draw θ where $\cos\theta = 0.6$ (considering damping coefficient $\zeta = 0.6$). Calculate OA, that is, ω_n, and it will be found to be 3.4. The value $60K = |s(s+2)(s+6)|$, where s is equal to the coordinate of point A and it will be found that:

$$60K = 16$$

Now,

$$K_v = \lim_{s \to 0} s \frac{60K(s+3)}{s(s+2)(s+6)} = \frac{60K \times 3}{2 \times 6} = 4$$

$$\text{Settling time} = \frac{4}{\zeta \omega_n} = \frac{4}{0.6 \times 3.4} = 1.96 \text{ s}$$

The settling time is well below 4 s. The steady state error = $\frac{1}{K_v} = \frac{1}{4} = 0.25$.

Thus it is observed that incorporation of one zero improves the system. *This is the fundamental meaning of compensation.* We have now understood *compensation* from the mathematical viewpoint. Let us see how to make it happen practically. Naturally, we have to take the help of some electrical circuits.

The compensation is usually classified as follows:

(a) Series compensation or cascade compensation
(b) Feedback compensation or parallel compensation
(c) State feedback compensation

9.2 TYPES OF COMPENSATION

9.2.1 Series Compensation or Cascade Compensation

This is the most commonly used system where the controller is placed in series with the controlled process. Figure 9.3 shows the series compensation.

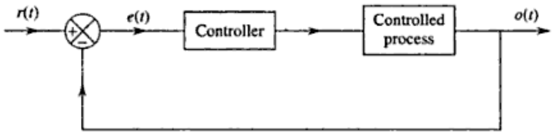


Fig. 9.3 Series compensation.

9.2.2 Feedback Compensation or Parallel Compensation

This is the system where the controller is placed in the minor feedback path as shown in Fig. 9.4.

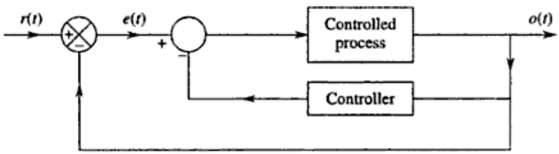


Fig. 9.4 Feedback compensation or parallel compensation.

9.2.3 State Feedback Compensation

This is a system which generates the control signal by feeding back the state variables through constant real gains. The scheme is termed *state feedback*. It is shown in Fig. 9.5.

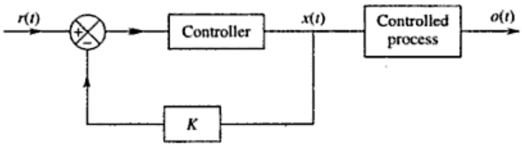


Fig. 9.5 *State feedback compensation.*

The compensation schemes shown in Figs. 9.3, 9.4, and 9.5 have one degree of freedom, since there is only one controller in each system. The demerit with one degree of freedom controllers is that the performance criteria that can be realized are limited. That is why there are compensation schemes which have two degree freedoms, such as:

(a) Series-feedback compensation
(b) Feedforward compensation

9.2.4 Series-Feedback Compensation

Series-feedback compensation is the scheme for which a series controller and a feedback controller are used. Figure 9.6 shows the series-feedback compensation scheme.

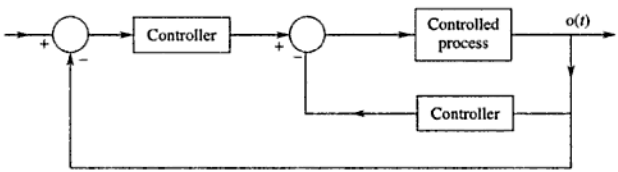


Fig. 9.6 *Series-feedback compensation.*

9.2.5 Feedforward Compensation

The feedforward controller is placed in series with the closed-loop system which has a controller in the forward path (Fig. 9.7). In Fig. 9.8, the feedforward-controller is placed in parallel with the controller in the forward path.

The commonly used controllers in the above-mentioned compensation schemes are now described in the section below.

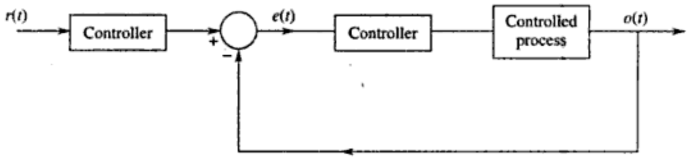


Fig. 9.7 Feedforward controller in series with the closed-loop system.

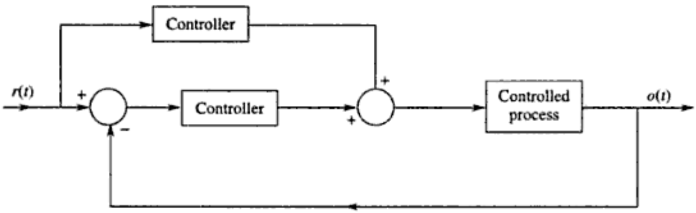


Fig. 9.8 Feedforward controller in parallel with the controller in the forward path.

9.3 LEAD COMPENSATOR

It has a zero and a pole with zero closer to the origin. The general form of the transfer function of the lead compensator is

$$G(s) = \frac{s + \frac{1}{\tau}}{s + \frac{1}{\beta\tau}}$$

where $\beta < 1$ and $\tau > 0$. Thus,

$$G(s) = \frac{\beta.(\tau s + 1)}{\beta\tau s + 1}$$

When $s = j\omega$

$$G(j\omega) = \beta\,\frac{(\tau j\omega + 1)}{\beta\tau j\omega + 1}$$

Since $\beta < 1$, $\angle G(j\omega)$ is leading in nature.

Figure 9.9 is the example of a lead compensator. Let us find out, why it is so?

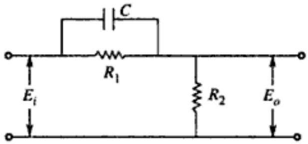


Fig. 9.9 Lead compensator.

Here,

$$E_o(s) = \frac{E_1(s)R_2}{\dfrac{R_1 \times \dfrac{1}{Cs}}{R_1 + \dfrac{1}{Cs}} + R_2}$$

$$\frac{E_o(s)}{E_i(s)} = \frac{R_2}{\dfrac{R_1 \times \dfrac{1}{Cs} + R_2\left(R_1 + \dfrac{1}{Cs}\right)}{R_1 + \dfrac{1}{Cs}}} = \frac{R_2R_1 + \dfrac{R_2}{Cs}}{R_1R_2 + \dfrac{1}{Cs}(R_1 + R_2)}$$

$$= \frac{Cs\,R_1R_2 + R_2}{Cs\,R_1R_2 + R_1 + R_2}$$

$$= \frac{R_2(Cs\,R_1 + 1)}{(R_1 + R_2)\left(\dfrac{CsR_1R_2}{R_1 + R_2} + 1\right)}$$

$$= \left(\frac{R_2}{R_1 + R_2}\right)\frac{CR_1s + 1}{\left(\dfrac{CR_1R_2s}{R_1 + R_2} + 1\right)}$$

Substituting

$$\tau = CR_1; \qquad \beta\tau = \frac{CR_1R_2}{R_1 + R_2} \quad (\because\ \tau = CR_1)$$

we can see that the above transfer function tallies with

$$G(s) = \beta\,\frac{\tau s + 1}{\beta\tau s + 1}$$

Moreover, β, being $\dfrac{R_2}{R_1 + R_2}$, is less than one; and τ, being CR_1, is greater than zero. Thus, Fig 9.9 is an example of lead compensator.

9.4 LAG COMPENSATOR

It has a zero and a pole with the zero situated on the left of the pole on the negative real axis. The general form of the transfer function of the lag compensator is

$$G(s) = \frac{s + \frac{1}{\tau}}{s + \frac{1}{\alpha\tau}} = \frac{\alpha(\tau s + 1)}{\alpha\tau s + 1}$$

where $\alpha > 1$, $\tau > 0$. Therefore, the frequency response of the above transfer function will be

$$G(j\omega) = \frac{\alpha(\tau j\omega + 1)}{\alpha\tau j\omega + 1}$$

Since $\alpha > 1$, the $\angle G(j\omega)$ will be lagging in nature.

Figure 9.10 shows an example of a lag compensator. Here,

$$E_o(s) = \frac{E_i(s)}{R_1 + R_2 + \frac{1}{Cs}}\left(R_2 + \frac{1}{Cs}\right)$$

or

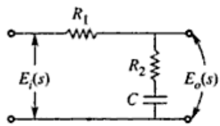


Fig. 9.10 Lag compensator.

$$\frac{E_o(s)}{E_i(s)} = \frac{R_2 + \frac{1}{Cs}}{R_1 + R_2 + \frac{1}{Cs}}$$

$$= \frac{R_2Cs + 1}{(R_1 + R_2)Cs + 1}$$

$$= \frac{R_2C\left(s + \frac{1}{R_2C}\right)}{(R_1 + R_2)C\left(S + \frac{1}{(R_1 + R_2)C}\right)}$$

$$= \frac{R_2}{(R_1 + R_2)} \frac{s + \frac{1}{R_2C}}{\left(s + \frac{1}{(R_1 + R_2)C}\right)} = \frac{R_2}{(R_1 + R_2)} \frac{\left(s + \frac{1}{R_2C}\right)}{\left(s + \frac{R_2}{(R_1 + R_2)R_2C}\right)}$$

Now, comparing with

$$G(s) = \frac{s + \frac{1}{\tau}}{s + \frac{1}{\alpha\tau}}$$

we get

$$\frac{1}{\tau} = \frac{1}{R_2C}; \qquad \frac{1}{\alpha\tau} = \frac{R_2}{(R_1 + R_2)R_2C}$$

or

$$\frac{1}{\alpha\tau} = \frac{R_2}{(R_1 + R_2)}\frac{1}{\tau} \qquad \left(\because \frac{1}{\tau} = \frac{1}{R_2C}\right)$$

or

$$\alpha = \frac{R_1 + R_2}{R_2}$$

Therefore,

$$\frac{E_o(s)}{E_i(s)} = \frac{1}{\alpha}\frac{s + \frac{1}{\tau}}{s + \frac{1}{\alpha\tau}}$$

Since, $\alpha = \dfrac{R_1 + R_2}{R_2}$, $\alpha > 1$ and since $\tau = R_2C$, $\tau > 0$.

Thus, Fig. 9.10 is the lag compensator. Only one multiplying factor $\frac{1}{\alpha}$ appears, but that does not interact with the phase relationship.

9.5 LAG-LEAD COMPENSATOR

The lag-lead compensator is the combination of a lag compensator and a lead compensator. The lag-section is provided with one real pole and one real zero, the pole being to the right of zero, whereas the lead section has one real pole and one real zero with the zero being to the right of the pole.

The transfer function of the lag-lead compensator will be

$$G(s) = \left(\frac{s + \frac{1}{\tau_1}}{s + \frac{1}{\alpha\tau_1}}\right)\left(\frac{s + \frac{1}{\tau_2}}{s + \frac{1}{\beta\tau_2}}\right)$$

where $\alpha > 1$, $\beta < 1$.

Figure 9.11 shows the lag-lead network, where

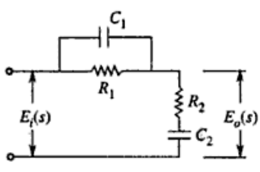


Fig. 9.11 Lag-lead compensator.

$$E_o(s) = \frac{E_i(s)}{\dfrac{R_1 \times \frac{1}{sC_1}}{R_1 + \frac{1}{sC_1}} + R_2 + \dfrac{1}{sC_2}}\left(R_2 + \frac{1}{sC_2}\right)$$

or

$$\frac{E_o(s)}{E_i(s)} = \frac{\left(R_1 + \frac{1}{sC_1}\right)\left(R_2 + \frac{1}{sC_2}\right)}{R_1 \frac{1}{sC_1} + \left(R_2 + \frac{1}{sC_2}\right)\left(R_1 + \frac{1}{sC_1}\right)}$$

$$= \frac{\frac{(sC_1R_1 + 1)}{sC_1}\frac{(sC_2R_2 + 1)}{sC_2}}{\frac{R_1}{sC_1} + \frac{(R_2sC_2 + 1)}{sC_2}\frac{(R_1sC_1 + 1)}{sC_1}}$$

$$= \frac{\frac{(1 + sC_1R_1)(1 + sC_2R_2)}{s^2C_1C_2}}{\frac{R_1sC_2 + R_2sC_2 + 1 + R_1R_2s^2C_1C_2 + R_1sC_1}{s^2C_1C_2}}$$

$$= \frac{(1 + sC_1R_1)(1 + sC_2R_2)}{s^2R_1R_2C_1C_2 + s(R_1C_1 + R_2C_2) + 1 + R_1sC_2}$$

$$= \frac{C_1R_1\,C_2R_2\left(s + \frac{1}{C_1R_1}\right)\left(s + \frac{1}{C_2R_2}\right)}{R_1R_2C_1C_2\left[s^2 + \left\{\frac{1}{R_2C_2} + \frac{1}{R_1C_1} + \frac{1}{R_2C_1}\right\}s + \frac{1}{R_1R_2C_1C_2}\right]}$$

$$= \frac{\left(s + \frac{1}{C_1R_1}\right)\left(s + \frac{1}{C_2R_2}\right)}{s^2 + \left(\frac{1}{R_1C_1} + \frac{1}{R_2C_1} + \frac{1}{R_2C_2}\right)s + \frac{1}{R_1R_2C_1C_2}}$$

If the above transfer function is compared with

$$G(s) = \frac{\left(s + \frac{1}{\tau_1}\right)\left(s + \frac{1}{\tau_2}\right)}{\left(s + \frac{1}{\alpha\tau_1}\right)\left(s + \frac{1}{\beta\tau_2}\right)}$$

then,

$$\frac{1}{\tau_1} = \frac{1}{C_1R_1}, \qquad \frac{1}{\tau_2} = \frac{1}{C_2R_2}$$

$$\frac{1}{\alpha\tau_1} + \frac{1}{\beta\tau_2} = \frac{1}{R_1C_1} + \frac{1}{R_2C_1} + \frac{1}{R_2C_2}$$

$$\frac{1}{\alpha\beta\tau_1\tau_2} = \frac{1}{R_1R_2C_1C_2}$$

$$\tau_1 = C_1R_1$$

$$\tau_2 = C_2R_2$$

$$\alpha\beta\tau_1\tau_2 = R_1R_2C_1C_2$$

or

$$\alpha\beta C_1R_1C_2R_2 = R_1R_2C_1C_2$$

or

$$\alpha\beta = 1 \quad \text{or} \quad \beta = \frac{1}{\alpha}$$

Therefore,

$$G(s) = \frac{\left(s + \frac{1}{\tau_1}\right)\left(s + \frac{1}{\tau_2}\right)}{\left(s + \frac{1}{\alpha\tau_1}\right)\left(s + \frac{\alpha}{\tau_2}\right)} \quad \text{where } \alpha > 1$$

and

$$\frac{1}{R_1C_1} + \frac{1}{R_2C_1} + \frac{1}{R_2C_2} = \frac{1}{\alpha\tau_1} + \frac{\alpha}{\tau_2}$$

9.6 CONTROLLERS

9.6.1 PI Controller

At ideal condition, the circuit shown in Figure 9.12 will have a magnitude $\left|\frac{E_o}{E_i}\right| = \frac{R_2}{R_1}$. This is termed P-controller (Proportional controller). Here, the output voltage is proportional to the input voltage. Since there is no time constant in the circuit, the response of the circuit is very fast.

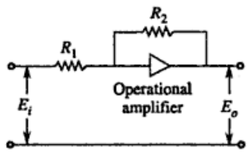


Fig. 9.12 P-controller.

The P-controller has the disadvantage of a permanent error between the desired and the actual output voltage. Figure 9.13 shows the example of an integ.al controller. The current in the capacitor will be equal to the current in the resistor in the case of an ideal condition. If the current in the resistor is i_1 and

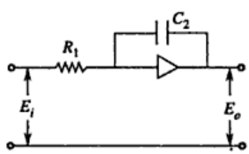


Fig. 9.13 Integral controller.

that in the capacitor is i_2, then $|i_1| = |i_2| = \frac{E_i}{R_1}$. Now,

$$|E_o| = \frac{\int |i_2| dt}{C_2} = \frac{1}{R_1 C_2} \int |E_i| dt$$

Hence the output voltage is equal to $\frac{1}{R_1 C_2}$ (integration of the input voltage).

If Laplace transform is taken,

$$|E_o(s)| = \frac{|E_i(s)|}{s R_1 C_2}$$

or

$$\text{Transfer function, } \left|\frac{E_o(s)}{E_i(s)}\right| = \frac{1}{s R_1 C_2}$$

Figure 9.14 shows the combination of P and I controllers. That is why, it is called proportional plus integral controller. This is a controller which produces an output signal consisting of two terms, one proportional to the actuating signal and the other proportional to its integral.

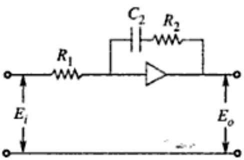


Fig. 9.14 P and I controller.

Let the current i_1 in the resistance R_1 be equal to the current i_2 in the resistance R_2 and capacitance C_2. Therefore, under the ideal condition

$$|i_1| = |i_2| = \frac{|E_i|}{R_1}$$

Now,

$$|E_o| = |i_2| R_2 + \frac{\int |i_2| dt}{C_2}$$

$$= \frac{|E_i|}{R_1} R_2 + \frac{\int |E_i| dt}{R_1 C_2}$$

Taking Laplace transforms,

$$|E_o(s)| = |E_i(s)| \left(\frac{R_2}{R_1}\right) + \frac{|E_i(s)|}{s R_1 C_2}$$

or

$$\frac{|E_o(s)|}{|E_i(s)|} = \left(\frac{R_2}{R_1}\right)\left[1 + \frac{1}{sR_2C_2}\right] = \frac{R_2}{R_1}\frac{sR_2C_2 + 1}{sR_2C_2}$$

or

$$|E_o(s)| = \frac{R_2}{R_1}\frac{(1 + sR_2C_2)}{sR_2C_2}|E_i(s)|$$

Figure 9.15 shows the combination of a P controller and an error detector. The $-E_i$ and E_i' are the reference and feedback signals, respectively. Since the point A is the virtual ground,

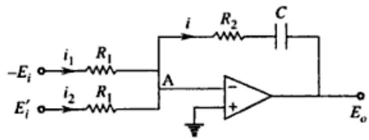


***Fig. 9.15** PI controller and an error detector.*

$$i_1 = -\frac{E_i}{R_1}, \qquad i_2 = \frac{E'i}{R_1}$$

or

$$E_o = -\left[\frac{1}{C}\int i\,dt + iR_2\right]$$

$$= -\left[\frac{1}{C}\int\left(-\frac{E_i}{R_1} + \frac{E_i'}{R_1}\right)dt + \left(-\frac{E_i}{R_1} + \frac{E_i'}{R_1}\right)R_2\right]$$

or

$$E_o = \frac{1}{R_1C}\int(E_i - E_i')dt + \frac{R_2}{R_1}(E_i - E_i')$$

Taking Laplace transforms

$$E_o(s) = \frac{1}{R_1C}\frac{(E_i(s) - E_i'(s)}{s} + \frac{R_2}{R_1}[E_i(s) - E_i'(s)]$$

$$= \left(\frac{1}{R_1Cs} + \frac{R_2}{R_1}\right)[E_i(s) - E_i'(s)]$$

$$= \frac{R_2}{R_1}\left(1 + \frac{1}{R_2Cs}\right)[E_i(s) - E_i'(s)]$$

or

$$E_i(s) - E'_i(s) = \frac{R_1}{R_2}\left(\frac{R_2Cs}{R_2Cs+1}\right)E_o(s)$$

Figure 9.16 shows the combination of a PI controller, a limiter, and an error detector.

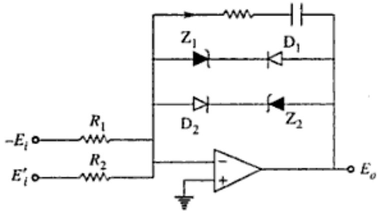


Fig. 9.16 PI controller, limiter, and error detector.

Zener diode Z_1 and diode D_1 place the limitation on the maximum positive voltage. Zener diode Z_2 and diode D_2 place the limitation on the maximum negative voltage since it is known from the circuit of Fig. 9.17. Figure 9.18 is the waveform of the output voltage.

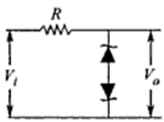


Fig. 9.17

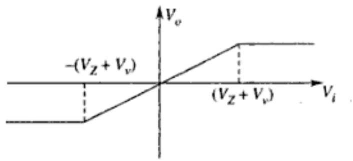


Fig. 9.18

In Fig. 9.18, V_Z is the breakdown voltage of the zener diode, and V_v is the barrier potential of the diode.

9.6.2 PD Controller

Figure 9.19 shows the circuit diagram of the PD controller.

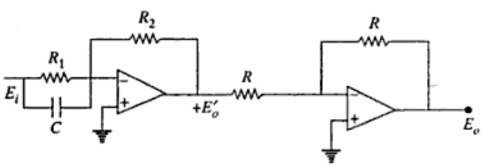


Fig. 9.19 PD controller.

Now,

$$\frac{-E'_o(s)}{R_2} = \frac{E_i(s)}{\dfrac{R_1 \times \dfrac{1}{sC}}{R_1 + \dfrac{1}{sC}}}$$

or

$$E'_o(s) = -R_2 \frac{E_i(s)}{\dfrac{\dfrac{R_1}{sC}}{\dfrac{sCR_1 + 1}{sC}}} = -R_2 \frac{E_i(s)}{\dfrac{R_1}{sCR_1 + 1}}$$

$$= -\frac{R_2(sCR_1 + 1)}{R_1} E_i(s)$$

Again

$$\frac{E_o(s)}{R} = -\frac{E'_o(s)}{R}$$

or

$$E_o(s) = -E'_o(s)$$

Therefore,

$$E_o(s) = +\frac{R_2}{R_1}(sCR_1 + 1)E_i(s)$$

or

$$\frac{E_o(s)}{E_i(s)} = \frac{R_2}{R_1} + sCR_2$$

Here R_2/R_1 is the constant portion and sCR_2 is the differential portion. Therefore, it is termed PD controller.

Figure 9.20 shows another circuit diagram of the PD (proportional and derivative) controller.

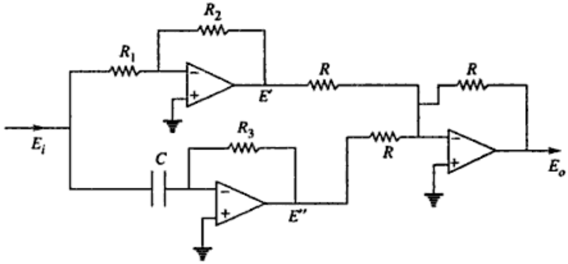


Fig. 9.20 PD controller.

Figure 9.21 being the same circuit shows the input voltage separately.

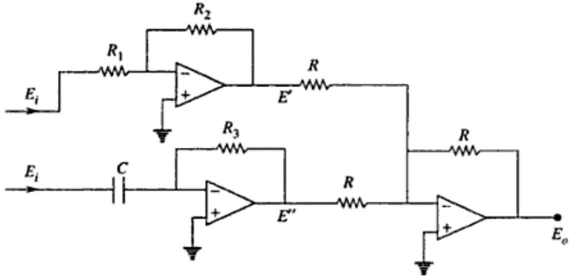


Fig. 9.21

Here,

$$\frac{E_i(s)}{R_1} = -\frac{E'(s)}{R_2}$$

and

$$\frac{E_i(s)}{\dfrac{1}{sC}} = -\frac{E''(s)}{R_3}$$

or

$$-E_i(s)\frac{R_2}{R_1} = E'(s)$$

and

$$-sCE_i(s)R_3 = E''(s)$$

From the above expressions, the circuit of Fig. 9.21 is modified to Fig. 9.22.

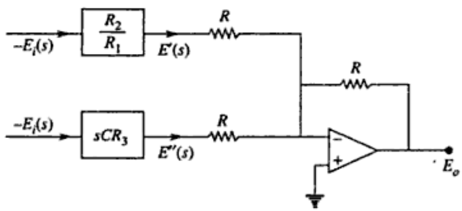


Fig. 9.22

Again,

$$\frac{E_o'(s)}{E'(s)} = -\frac{R}{R} = -1$$

and

$$\frac{E_o''(s)}{E''(s)} = -\frac{R}{R} = -1$$

Now,

$$E_o(s) = E'_o(s) + E''_o(s) = -E'(s) - E''(s)$$

$$= E_i(s)\left[\frac{R_2}{R_1} + sCR_3\right]$$

or

$$\frac{E_o(s)}{E_i(s)} = \frac{R_2}{R_1} + sCR_3$$

Thus it is proved that Fig. 9.20 is also a PD controller.

9.6.3 PID Controller

The PID controller is frequently used in industrial control systems. Figure 9.23 shows the circuit diagram of a PID contoller.

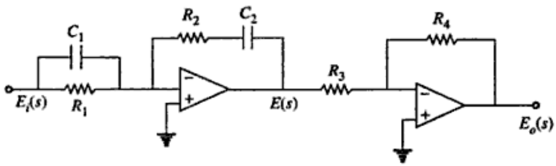


Fig. 9.23 PID Controller.

Here,

$$\frac{E_i(s)}{\dfrac{R_1\dfrac{1}{sC_1}}{R_1 + \dfrac{1}{sC_1}}} = -\frac{E(s)}{R_2 + \dfrac{1}{sC_2}}$$

or

$$\text{ss}\frac{E_i(s)}{E(s)} = -\frac{\dfrac{\dfrac{R_1}{sC_1}}{\dfrac{R_1 sC_1 + 1}{sC_1}}}{\dfrac{sC_2R_2 + 1}{sC_2}}$$

$$= -\frac{\dfrac{R_1}{R_1 sC_1 + 1}}{\dfrac{1 + sC_2R_2}{sC_2}} = -\frac{R_1 sC_2}{(1 + R_1C_1s)(1 + sC_2R_2)}$$

Also,

$$\frac{E(s)}{R_3} = -\frac{E_o(s)}{R_4}$$

or

$$E(s) = -\frac{R_3}{R_4}E_o(s)$$

Now,

$$\frac{E_i(s)}{E(s)} = -\frac{R_1 sC_2}{(1+sC_1R_1)(1+sC_2R_2)}$$

or

$$\frac{E_i(s)}{-\frac{R_3}{R_4}E_o(s)} = -\frac{R_1 sC_2}{(1+sC_1R_1)(1+sC_2R_2)}$$

or

$$\frac{E_i(s)}{E_o(s)} = \frac{R_3}{R_4}\frac{R_1 sC_2}{(1+sC_1R_1)(1+sC_2R_2)}$$

or

$$\frac{E_o(s)}{E_i(s)} = \frac{R_4(1+sC_1R_1)(1+sC_2R_2)}{R_3R_1sC_2}$$

$$= \frac{R_4R_2(1+sC_1R_1)(1+sC_2R_2)}{R_3R_1R_2sC_2}$$

$$= \frac{R_4R_2}{R_3R_1}\left[\frac{1+sC_1R_1+sC_2R_2+s^2C_1C_2R_1R_2}{sR_2C_2}\right]$$

$$= \frac{R_4R_2}{R_3R_1}\left[\frac{C_1R_1+C_2R_2}{R_2C_2}+\frac{1}{sR_2C_2}+sR_1C_1\right]$$

$$= \frac{R_4(C_1R_1+C_2R_2)}{R_3R_1C_2}\left[1+\frac{1}{(R_1C_1+R_2C_2)s}+\frac{R_1C_1R_2C_2}{R_1C_1+R_2C_2}s\right]$$

From the above expression, it is clear that in the transfer function, proportional, integral and derivative parts exist. Therefore, Fig. 9.23 is a PID controller.

SUMMARY

The need of compensation in control systems is explained. Comparison of the control systems is made by inserting one zero in the open-loop transfer function. The different schemes of compensation are described, that is, series compensation or cascade compensation, feedback compensation and state feedback compensation. Lead compensator, lag compensator, and lag-lead compensator are described with the help of circuits and their mathematical expressions. The controller circuits are explained in detail, for example, the PI controller, the PI controller with limiter and error detector, the PD controller, and the PID controller.

QUESTIONS

1. What is the need of compensators? Describe the different types of compensation schemes.
2. Write short notes on: (a) Lag compensator (b) Lead compensator (c) Lag-lead compensator.
3. Explain PI, PD, and PID controllers.
4. Explain the statement: "Lead compensator is visualized as a combination of a network and an amplifier".

Chapter 10

Nonlinear Control Systems

10.1 LIAPUNOV'S METHOD FOR STABILITY STUDY

Liapunov's method is a tool for determining the stability characteristics of nonlinear systems. This is the most general method for nonlinear systems. Of course, there are some limitations. The Liapunov's method is also applied to linear systems. Before we discuss the details of the Liapunov's method, we have to remember the following definitions which are very important for studying the Liapunov's method of stability.

If W is a real scalar function of the state variable x, i.e. $W = W(x) = V(x_1, x_2, x_3, \ldots)$ and the function W has always a positive or negative sign in a given region about the origin, except only at the origin, where the same is zero, then the function W is termed *positive* or *negative definite*.

Suppose $W = x_1^2 + x_2^2$, then it is a positive definite in a two-dimensional state according to the above definition.

Suppose $W = -m_1x_1^2 - m_2x_2^2 - m_3x_3^2$, then it is a negative definite in a three-dimensional state space, if m_1, m_2, and m_3 are greater than zero.

Similarly, if $W = m_1(x_1 + 4)^2 + m_2(x_2 - 4)^2$ with $m_1 > 0$, $m_2 > 0$, then it is not positive definite in a two-dimensional state space.

Again, if $W = x_1^2 + x_2^2$ is not positive in a three-dimensional state space, then at any value of x_3 it is zero if x_1 and x_2 are zero. Hence, it is not zero at the origin only in a three-dimensional state space.

The function $W = (x_1 + x_2)^2$ is not also positive definite in a two-dimensional space. Suppose $x_1 = 2$, and at that time $x_2 = -2$, then V becomes zero at ($x_1 = 2$, $x_2 = -2$), which is not obviously the origin.

A function is termed positive (or negative) semi-definite in a region when V has a positive (or negative) sign throughout that region except at certain points including the origin where it is zero. For example, $W = (x_1 + x_2)^2$ is positive semi-definite in a two-dimensional state space since it is zero at all points in W where $x_1 = -x_2$.

10.1.1 Application of the Liapunov's Method

Figure 10.1 shows an autonomous system with a mass-dashpot-spring arrangement, where the mass is M, the frictional coefficient is F, and the spring constant is K. Let us assume that the initial position x is 1 and the velocity $\dot{x}$ is 1. The differential equation of the system will then be

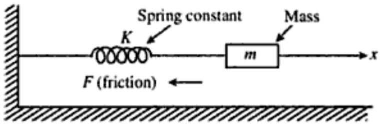


Fig. 10.1 Mass-dashpot-spring arrangement.

$$M\ddot{x} + F\dot{x} + Kx = 0$$

with

$$x(0) = 1 \quad \text{and} \quad \dot{x}(0) = 1$$

Let us assume that the state variables are, $x_1 = x$ and $x_2 = \dot{x}$. Therefore, we have

$$\dot{x}_1 = x_2$$

$$M\dot{x}_2 + Fx_2 + Kx_1 = 0$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{K}{M}x_1 - \frac{F}{M}x_2$$

$$x_1(0) = 1 \quad \text{and} \quad x_2(0) = 1$$

The above relations in matrix form can be expressed as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{K}{M} & -\frac{F}{M} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let us now consider $M = 1$, $F = 2$, and $K = 2$. Then,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

or

$$[\dot{x}] = A[x]$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}$$

Taking the Laplace transform of the above, we get

$$sX(s) - x(0) = AX(s)$$

or

$$sX(s) - AX(s) = x(0)$$

or

$$(sI - A)X(s) = x(0)$$

or

$$X(s) = [sI - A]^{-1}\, x(0)$$

Now,

$$sI - A = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} s & -1 \\ 2 & s+2 \end{bmatrix}$$

Now, let us find the inverse of $[sI - A]$.

First step: $$\begin{bmatrix} s+2 & -2 \\ 1 & s \end{bmatrix}$$

Second step: $$\begin{bmatrix} s+2 & 1 \\ -2 & s \end{bmatrix}$$

Third step: $$\frac{1}{s(s+2)+2}\begin{bmatrix} s+2 & 1 \\ -2 & s \end{bmatrix}$$

$$= \frac{1}{s^2+2s+2}\begin{bmatrix} s+2 & 1 \\ -2 & s \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{s+2}{s^2+2s+2} & \dfrac{1}{s^2+2s+2} \\ \dfrac{-2}{s^2+2s+2} & \dfrac{s}{s^2+2s+2} \end{bmatrix}$$

Thus,

$$X(s) = [sI - A]^{-1}\, x(0)$$

$$= \begin{bmatrix} \dfrac{s+2}{s^2+2s+2} & \dfrac{1}{s^2+2s+2} \\ \dfrac{-2}{s^2+2s+2} & \dfrac{s}{s^2+2s+2} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{s+3}{s^2+2s+2} \\ \dfrac{s-2}{s^2+2s+2} \end{bmatrix}$$

Now,

$$\frac{s+3}{s^2+2s+2} = \frac{s+3}{(s+1-j)(s+1+j)} = \frac{A}{s+1-j} + \frac{B}{s+1+j}$$

where

$$A = \lim_{s \to -1+j} \frac{s+3}{s+1+j} = \frac{-1+j+3}{-1+j+1+j} = \frac{2+j}{2j}$$

$$= 0.5 - j$$

and

$$B = \lim_{s \to -1-j} \frac{s+3}{s+1-j} = \frac{-1-j+3}{-1-j+1-j} = \frac{2-j}{-2j}$$

$$= 0.5 + j$$

Therefore,

$$\frac{s+3}{s^2+2s+2} = \frac{0.5-j}{s+1-j} + \frac{0.5+j}{s+1+j}$$

and

$$\mathcal{L}^{-1} \frac{s+3}{s^2+2s+2} = (0.5-j)e^{-(1-j)}t + (0.5+j)e^{-(1+j)}t$$

Again,

$$\frac{s-2}{s^2+2s+2} = \frac{A'}{s+1-j} + \frac{B'}{s+1+j}$$

where

$$A' = \lim_{s \to -1+j} \frac{s-2}{s+1+j} = \frac{-1+j-2}{-1+j+1+j} = \frac{j-3}{2j}$$

$$= 0.5 + 1.5j$$

and

$$B' = \lim_{s \to -1-j} \frac{s-2}{s+1-j} = \frac{-1-j-2}{-1-j+1-j} = \frac{-3-j}{-2j}$$

$$= 0.5 - 1.5j$$

Thus,

$$\mathcal{L}^{-1} \frac{s-2}{s^2+2s+2} = (0.5+1.5j)e^{-(1-j)}t + (0.5-1.5j)e^{-(1+j)}t$$

Therefore, $\mathscr{L}^{-1}X(s) = x(t)$

$$= \begin{bmatrix} (0.5-j)e^{-(1-j)}t + (0.5+j)e^{-(1+j)}t \\ (0.5+1.5j)e^{-(1-j)}t + (0.5-1.5j)e^{-(1+j)}t \end{bmatrix}$$

$$= \begin{bmatrix} (0.5-j)e^{-t}e^{jt} + (0.5+j)e^{-t}e^{-jt} \\ (0.5+1.5j)e^{-t}e^{jt} + (0.5-1.5j)e^{-t}e^{-jt} \end{bmatrix}$$

$$= \begin{bmatrix} (0.5-j)e^{-t}(\cos t + j\sin t) + (0.5+j)e^{-t}(\cos t - j\sin t) \\ (0.5+1.5j)e^{-t}(\cos t + j\sin t) + (0.5-1.5j)e^{-t}(\cos t - j\sin t) \end{bmatrix}$$

$$= \begin{bmatrix} (0.5\cos t + 0.5j\sin t - j\cos t + \sin t + 0.5\cos t - 0.5j\sin t + j\cos t + \sin t)e^{-t} \\ (0.5\cos t + 0.5j\sin t + 1.5j\cos t - 1.5\sin t + 0.5\cos t - 0.5j\sin t - 1.5j\cos t - 1.5\sin t)e^{-t} \end{bmatrix}$$

$$= \begin{bmatrix} (\cos t + 2\sin t)e^{-t} \\ (\cos t - 3\sin t)e^{-t} \end{bmatrix}$$

Now,

$$\cos t + 2\sin t = A\sin\alpha\cos t + A\cos\alpha\sin t$$

where

$$A\sin\alpha = 1, \quad A\cos\alpha = 2$$

$$A = \sqrt{5}, \quad \tan\alpha = \frac{1}{2} \qquad \text{or} \qquad \alpha = \tan^{-1}\frac{1}{2}$$

Thus,

$$\cos t + 2\sin t = \sqrt{5}\sin(t+\alpha)$$

Similarly,

$$\cos t - 3\sin t = A'\sin\alpha\cos t + A'\cos\alpha\sin t$$

where

$$A'\sin\alpha = 1, \quad A'\cos\alpha = -3$$

$$A' = \sqrt{10}, \quad \tan\alpha = -\frac{1}{3} \qquad \text{or} \qquad \alpha = \tan^{-1}\left(-\frac{1}{3}\right)$$

Therefore,

$$x(t) = \begin{bmatrix} \sqrt{5}\sin\left(t + \tan^{-1}\frac{1}{2}\right) \\ \sqrt{10}\sin\left(t + \tan^{-1}\left(-\frac{1}{3}\right)\right) \end{bmatrix}$$

From the above result, it is not difficult to understand that the system is asymptotically stable. Now, let us analyze the condition of the stored energy and how the same behaves with time.

The kinetic energy stored in the system will be $(1/2)m\dot{x}^2$.
Now,

$$x_1 = x \quad \text{and} \quad x_2 = \dot{x}$$

Thus, the kinetic energy will be $(1/2)Mx_2^2$. The potential energy stored in the spring will be $= (1/2)Kx^2 = (1/2)Kx_1^2$. Hence, the total stored energy will be

$$W = \frac{1}{2}(Kx_1^2 + Mx_2^2)$$

The rate of change of stored energy will be

$$\frac{dW}{dt} = Kx_1\dot{x}_1 + Mx_2\dot{x}_2$$

Now, we know

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{K}{M} & -\dfrac{F}{M} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{K}{M}x_1 - \frac{F}{M}x_2$$

Then,

$$\frac{dW}{dt} = Kx_1x_2 + Mx_2\left[-\frac{K}{M}x_1 - \frac{F}{M}x_2\right]$$

$$= Kx_1x_2 - Kx_1x_2 - Fx_2^2$$

$$= -Fx_2^2$$

Now, in the problem, we have considered: $M = 1$, $F = 2$, and $K = 2$.
Therefore,

$$W = \frac{1}{2}\left(2x_1^2 + x_2^2\right)$$

$$= x_1^2 + \frac{1}{2}x_2^2$$

Again,

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Therefore,

$$W(0) = x_1^2(0) + \frac{1}{2}x_2^2(0)$$

$$= 1 + \frac{1}{2} = \frac{3}{2}$$

and

$$\dot{W} = -Fx_2^2 = -2x_2^2$$

If the constant W curves are drawn in the phase space with x_1 and x_2 as the coordinate axes, the trajectory will remain always on constant W ellipses (Fig. 10.2).

If W reduces to zero, both x_1 and x_2 will be zero.

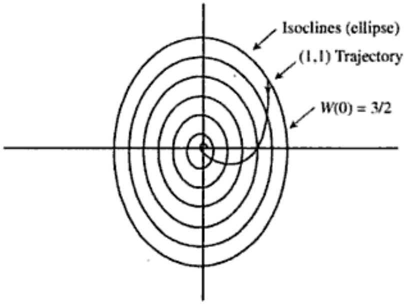


Fig. 10.2 *Constant W curves.*

When $x_1 = 0$ and $x_2 = 0$, we have

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

Since $\dot{x} = Ax$, we get $\dot{x} = 0$

This indicates that the mechanical system will have neither any velocity nor any acceleration after W attains the zero level. Hence the system is asymptotically stable. Since

$$W = x_1^2 + \frac{1}{2}x_2^2 \qquad \text{and} \qquad \dot{W} = -2x_2^2$$

it can be said that for the asymptotically stable condition, W is positive definite and $\dot{W}$ is negative definite. Even, it can be shown that for a system having positive definite W and a negative semi-definite $\dot{W}$, the system is stable. Of course, in that case asymptotic stability may not be certain.

Hence the Liapunov's first theorem can be described as follows:

If for an autonomous system with $\dot{x} = f(x)$, there exists a scalar function $W(x)$ which is real, continous with continuous first partial derivatives and with

(i) $W(0) = 0$

(ii) $W(x) > 0$ for all $x \neq 0$, and

(iii) $\dfrac{dW(x)}{dt} < 0$ for all $x \neq 0$

then the system is asymptotically stable in the neighbourhood of the origin. If the derivative $dW(x)/dt \le 0$, the system is still stable, but asymptotic stability cannot be assured. The function $W(x)$ is termed the Liapunov's function.

The success of the Liapunov's method totally depends on finding out or generating a suitable Liapunov's function.

In the case of linear systems, the direct methods for determining the suitable Liapunov's functions are available. But in the case of nonlinear methods, sometimes it is really difficult to determine the Liapunov's function and therefore the stability.

EXAMPLE 10.1 An autonomous system is expressed as follows:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -m_1x_2 - m_2x_1$$

Study the stability of the system using the Liapunov's method and considering the Liapunov's function as

$$W = x_1^2 + x_2^2$$

Solution Since $W = x_1^2 + x_2^2$, it is positive definite. Therefore,

$$W(0) = 0$$

$W(x) > 0$ for all except at $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Now,

$$\frac{dW}{dt} = \frac{\partial W}{\partial x_1}\cdot\frac{\partial x_1}{\partial t} + \frac{\partial W}{\partial x_2}\cdot\frac{\partial x_2}{\partial t}$$

$$= 2x_1\dot{x}_1 + 2x_2\dot{x}_2$$

$$= 2x_1\dot{x}_1 + 2x_2(-m_1x_2 - m_2x_1) \quad (\because\ \dot{x}_2 = -m_1x_2 - m_2x_1)$$

$$= 2x_1x_2 + 2x_2(-m_1x_2 - m_2x_1) \quad (\because\ \dot{x}_1 = x_2)$$

$$= 2x_1x_2 - 2x_2m_1x_2 - 2x_2m_2x_1$$

$$= (2x_1 - 2m_1x_2 - 2m_2x_1)x_2$$

$$= [2x_1(1 - m_2) - 2m_1x_2]x_2$$

Now, for $m_2 = 1$ and $m_1 > 0$

$$\frac{dW}{dt} = -2m_1 x_2^2$$

Hence, it can be said that for any value of x_1, $dW/dt = 0$ when $x_2 = 0$. The value of dW/dt is negative semi-definite. From this, one conclusion that can be drawn is that the system is stable for $m_2 = 1$ and $m_1 > 0$.

The method of calculation which we have just discussed is very much conservative. The reason will be understood shortly after calculation of the problem by the Liapunov's second method.

10.1.2 Liapunov's Second Method

Let us consider P a positive definite matrix such that

$$W(x) = x^T Px$$

Now,

$$\frac{\delta W(x)}{\delta t} = \dot{x}^T Px + x^T P\dot{x}$$

Since

$$\dot{x} = Ax, \quad \dot{x}^T = x^T A^T$$

$$\frac{\delta W(x)}{\delta t} = x^T A^T Px + x^T PAx$$

$$= x^T (A^T P + PA)x$$

Let

$$L = -(A^T P + PA)$$

Then,

$$\frac{\delta W(x)}{\delta t} = \dot{W}(x) = -x^T Lx$$

Now, our approach will be to select arbitrarily a positive definite or positive semi-definite matrix for L and solve the problem. For asymptotic stability, the necessary and sufficient condition is that the matrix P must be positive definite.

We know that any matrix is positive definite if its symmetric component is positive definite. We may consider L as a symmetric matrix. Now,

$$L = \frac{L + L^T}{2}$$

$$= -\left[\frac{1}{2}\left(A^T P + PA\right) + \frac{1}{2}\left(P^T A + A^T P^T\right)\right] \qquad [\because L^T = -(P^T A + A^T P^T)]$$

$$= -\left[A^T\left(\frac{P + P^T}{2}\right) + \left(\frac{P + P^T}{2}\right)A\right]$$

Let $S = \dfrac{P + P^T}{2}$, where S is symmetric. Therefore,

$$L = -[A^T S + SA]$$

Now if we apply this Liapunov's second method of calculation to the system of Example 10.1, that is,

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -m_1 x_2 - m_2 x_1$$

we get

$$\dot{x}_1 = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -m_2 & -m_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= Ax$$

Let us consider

$$L = I \text{ (unit matrix)}$$

then,

$$A^T S + SA = -I$$

or

$$\begin{bmatrix} 0 & -m_2 \\ 1 & -m_1 \end{bmatrix} \begin{bmatrix} S_{11} & S_{12} \\ S_{12} & S_{22} \end{bmatrix} + \begin{bmatrix} S_{11} & S_{12} \\ S_{12} & S_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -m_2 & -m_1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

or

$$\begin{bmatrix} -m_2 S_{12} & -m_2 S_{22} \\ S_{11} - m_1 S_{12} & S_{12} - m_1 S_{22} \end{bmatrix} + \begin{bmatrix} -m_2 S_{12} & S_{11} - m_1 S_{12} \\ -m_2 S_{22} & S_{12} - m_1 S_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

From the above, we get the following equations:

$$-2m_2 S_{12} = -1$$

$$S_{11} - m_1 S_{12} - m_2 S_{22} = 0$$

$$2S_{12} - 2m_1 S_{22} = -1$$

From the above equations, we have

$$S_{11} = \frac{m_1^2 + m_2^2 + m_2}{2m_1 m_2}$$

$$= \frac{m_1}{2m_2} + \frac{m_2 + 1}{2m_1}$$

$$S_{12} = \frac{1}{2m_2}$$

$$S_{22} = \frac{m_2 + 1}{2m_1 m_2}$$

Therefore,

$$S = \begin{bmatrix} \dfrac{m_1^2 + m_2^2 + m_2}{2m_1 m_2} & \dfrac{1}{2m_2} \\ \dfrac{1}{2m_2} & \dfrac{m_2 + 1}{2m_1 m_2} \end{bmatrix}$$

To make S positive definite

$$S_{11} = \frac{m_1}{2m_2} + \frac{m_2 + 1}{2m_1} > 0$$

$$S_{11} S_{22} - S_{12}^2 > 0$$

$$\left(\frac{m_1}{2m_2}+\frac{m_2+1}{2m_1}\right)\left(\frac{m_2+1}{2m_1 m_2}\right)-\frac{1}{4m_2^2}>0$$

or

$$\frac{1}{m_2}+\frac{m_2+1}{m_1^2}+\frac{m_2+1}{m_1^2 m_2}>0$$

or

$$\frac{1}{m_2 m_1^2}\left[m_1^2+(m_2+1)^2\right]>0$$

Hence, to make the above true, $m_2 > 0$ and $\dfrac{m_1^2+(m_2+1)^2}{m_1^2}$ is already greater than zero as per its mathematical structure.

Again from the condition

$$S_{11}=\frac{m_1}{2m_2}+\frac{m_2+1}{2m_1}>0$$

we get

$$\frac{m_1^2+m_2^2+m_2}{2m_1 m_2}>0$$

or

$$\frac{m_1^2+m_2^2+m_2}{m_1 m_2}>0$$

Since we have already considered $m_2 > 0$, m_1 is to be made greater than zero to make S positive definite. Hence, the condition of stability from the Liapunov's second method is found $m_1 > 0$ and $m_2 > 0$. Whereas in the case of the Liapunov's first method, the resutl was $m_2 = 1$ and $m_1 > 0$. Thus it is observed that the Liapunov's first method is highly conservative. The Liapunov's second method provides much better information.

EXAMPLE 10.2 Determine the stability range for the gain m of the system shown below:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ 0 & -2 & 1\\ -m & 0 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}0\\ 0\\ m\end{bmatrix}u$$

where u is the input.

Solution For determining the stability range for m, the input u is assumed to be zero. Thus,

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ 0 & -2 & 1\\ -m & 0 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -2x_2 + x_3\\ \dot{x}_3 &= -mx_1 - x_3\end{aligned}$$

Let us assume a positive semi-definite real symmetric matrix

$$L = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The above choice is permissible since $\dot{W}(x) = -x^T Lx$ cannot be identically equal to zero except at the origin.

$$\dot{W}(x) = -[x_1\ x_2\ x_3] \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = -x_3^2$$

Now, we know as per Liapunov's second method

$$A^T S + SA = -\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = -L = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Now, as per the data

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & 1 \\ -m & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Hence,

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & 1 \\ -m & 0 & -1 \end{bmatrix}$$

Therefore,

$$\begin{bmatrix} 0 & 0 & -m \\ 1 & -2 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{12} & S_{22} & S_{23} \\ S_{13} & S_{23} & S_{33} \end{bmatrix} + \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{12} & S_{22} & S_{23} \\ S_{13} & S_{23} & S_{33} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & 1 \\ -m & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

The S matrix is symmetric, that is why $S_{31} = S_{13}$ and $S_{32} = S_{23}$.

From the above matrix arrangement, we will get six equations and six unknowns

$$-2mS_{13} = 0$$

$$-mS_{23} + S_{11} - 2S_{12} = 0$$

$$-mS_{33} + S_{12} - S_{13} = 0$$

$$S_{12} - 2S_{22} = 0$$

$$S_{13} + S_{22} - 3S_{23} = 0$$

$$S_{23} - S_{33} = -\frac{1}{2}$$

Hence S_{11}, S_{12}, S_{13}, S_{22}, S_{23}, and S_{33} can be found out.

Therefore,

$$S = \begin{bmatrix} \dfrac{m^2+12m}{12-2m} & \dfrac{6m}{12-2m} & 0 \\ \dfrac{6m}{12-2m} & \dfrac{3m}{12-2m} & \dfrac{m}{12-2m} \\ 0 & \dfrac{m}{12-2m} & \dfrac{6}{12-2m} \end{bmatrix}$$

Now to make S positive definite, it is essential that

$$12 - 2m > 0 \qquad \text{and} \qquad m > 0$$

Therefore $0 < m < 6$ will be the range of values of m for stability.

EXAMPLE 10.3 How do you show a quadratic form positive definite?

Solution Any quadratic form can be represented by

$$x^T A x = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j$$

Suppose,

$$W(x) = 9x_1^2 + 4x_2^2 + x_3^2 + 2x_1x_2 - 2x_2x_3 - 4x_1x_3$$

The matrix A will be

$$\begin{bmatrix} 9 & 1 & -2 \\ 1 & 4 & -1 \\ -2 & -1 & 1 \end{bmatrix}$$

According to Sylvester's criterion, $W(x)$ will be positive definite if

$$9 > 0, \quad \begin{bmatrix} 9 & 1 \\ 1 & 4 \end{bmatrix} > 0$$

and

$$\begin{bmatrix} 9 & 1 & -2 \\ 1 & 4 & -1 \\ -2 & -1 & 1 \end{bmatrix} > 0$$

Now,

$$\begin{bmatrix} 9 & 1 \\ 1 & 4 \end{bmatrix} = 36 - 1 = 35 > 0$$

$$\begin{bmatrix} 9 & 1 & -2 \\ 1 & 4 & -1 \\ -2 & -1 & 1 \end{bmatrix} = 9(4 + 1) - 1(1 - 2) - 2(-1 + 8)$$

$$= 45 + 1 - 14 = 32 > 0$$

Hence $W(x)$ is positive definite.

EXAMPLE 10.4 A system is described as follows:

$$\ddot{p} + K_1 \dot{p} + K_2(\dot{p})^3 + p = 0$$

$K_1 > 0$ and $K_2 > 0$. Study the stability by the Liapunov's method.

Solution Let $x_1 = p$ and $x_2 = \dot{p}$.

Therefore,

$$\dot{x}_1 = x_2$$

Again from $\ddot{p} + K_1 \dot{p} + K_2(\dot{p})^3 + p = 0$, we get

$$\dot{x}_2 = -K_1 x_2 - K_2 x_2^3 - x_1$$

Let us choose $W = x_1^2 + x_2^2$ as the Liapunov's function.

There is no conventional procedure for selecting the Liapunov's function, but it is usually chosen from experience. Now

$$\begin{aligned}\frac{dW}{dt} &= 2x_1\dot{x}_1 + 2x_2\dot{x}_2 \\ &= 2x_1x_2 + 2x_2(-K_1x_2 - K_2x_2^3 - x_1) \\ &= 2x_1x_2 - 2K_1x_2^2 - 2K_2x_2^4 - 2x_2x_1 \\ &= -2(K_1x_2^2 + K_2x_2^4)\end{aligned}$$

This will be negative semi-definite and therefore the system is stable.

10.2 PHASE PLANE METHOD

This is one of the methods of solution of nonlinear control systems. As we have already seen, the linear control systems can be easily studied from their pole-zero configuration. To speak the truth, all the physical practical systems are hardly linear in reality. Hence, a generalized approach for the study of nonlinear systems is essential. In fact, there are no such general methods for the analysis and synthesis of nonlinear control systems. The nonlinear systems are simplified with certain approximations to a nonlinear second-order system. Graphical representation of phase trajectory can be easily applied to the above nonlinear second-order system for the study of the same. In other words, in the phase plane method, a general and convenient graphical method is adopted by utilizing the isoclines.

10.2.1 Method of Determining the Phase Trajectory for a Second-order System

A second-order system is described as follows.

$$\ddot{p} + f_1(p, \dot{p})\dot{p} + f_2(p, \dot{p})p = S$$

where $f_1(p, \dot{p})$ and $f_2(p, \dot{p})$ are functions of p and $\dot{p}$, respectively. Now putting $x_1 = p$ and $x_2 = \dot{p}$, we can write

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -f_1(x_1, x_2)x_2 - f_2(x_1, x_2)x_1 + S$$

Therefore,

$$\frac{dx_2}{dx_1} = \frac{\dot{x}_2}{\dot{x}_1} = -f_1(x_1, x_2) - f_2(x_1, x_2)\frac{x_1}{x_2} + \frac{S}{x_2}$$

The above is the expression of the slope of trajectory at the point (x_1, x_2). Let

$$\frac{dx_2}{dx_1} = m$$

When m is a constant, then the curve $\phi(x_1, x_2) = m$
where

$$\phi(x_1, x_2) = -f_1(x_1, x_2) - f_2(x_1, x_2)\frac{x_1}{x_2} + \frac{S}{x_2}$$

will describe a curve so that any trajectory crossing the curve will perform the same at a slope m. This type of curve is termed *isocline*.

10.2.2 Method of Developing Isocline and Trajectory

If the initial values of the variable and the derivatives are provided, the same is represented by the point on the phase plane. If the point is in the upper-half of the phase plane, the derivative has a positive value since $x_2 > 0$.

Hence, the increment of the variable x_1 will be positive for the positive increment in time. In the same way, when the initial point is in the lower-half of the phase plane, the increment of the variable will be negative for the positive increment in time. The above fact actually helps in understanding the direction of the movement of the solution point. Therefore, to draw the trajectory with a given set of initial conditions, the corresponding initial point is first located in the phase plane and then the solution point starts moving from this point onwards in a proper direction and crosses isoclines at the slope corresponding to the respective isocline.

First of all, let us consider a linear system and apply the above method to acquaint ourselves as to how the phase plane method is utilized. Then, we will apply this method to the nonlinear control system.

Let us say that a linear system is represented by the block diagram shown in Fig. 10.3.

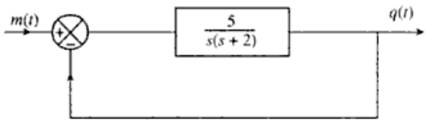


***Fig. 10.3** Linear second-order control system.*

It is a linear second-order control system with $m(t)$ input and $q(t)$ output. Suppose $m(t) = 2u(t)$ and the initial conditions are:

$$q(0) = -1$$

$$\dot{q}(0) = 0$$

From the above data, we can write

$$[M(s)-Q(s)]\frac{5}{s(s+2)} = Q(s)$$

or

$$M(s) - Q(s) = Q(s)\left[\frac{s(s+2)}{5}\right]$$

or

$$M(s) = Q(s)\left[1+\frac{s(s+2)}{5}\right]$$

or

$$\frac{Q(s)}{M(s)} = \frac{1}{1+\dfrac{s(s+2)}{5}} = \frac{5}{s^2+2s+5}$$

or

$$(s^2 + 2s + 5)Q(s) = 5M(s)$$

Now taking the Laplace inverse of the above equation, we have

$$\ddot{q} + 2\dot{q} + 5q = 5 \times 2 = 10 \qquad (\because \ m(t) = 2u(t))$$

Let

$$q = x_1$$

$$\dot{q} = x_2$$

Therefore,

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -2x_2 - 5x_1 + 10$$

We have already indicated in our general approach

$$\dot{x}_1 = x_2$$

$$\dot{x}_1 = -f_1(x_1, x_2)x_2 - f_2(x_1, x_2)x_1 + s$$

Hence

$$f_1(x_1, x_2) = 2$$

$$f_2(x_1, x_2) = 5$$

$$s = 10$$

$$\frac{dx_2}{dx_1} = -2 - 5\frac{x_1}{x_2} + \frac{10}{x_2}$$

or

$$m = -2 - 5\frac{x_1}{x_2} + \frac{10}{x_2}$$

Therefore, the equation of the family of isoclines will be

$$x_2 = -\frac{5x_1}{m+2} + \frac{10}{m+2}$$

The above equation indicates a family of straight lines. The slope of those lines will be

$$m_1 = -\frac{5}{m+2}$$

The lines intersect the x_1-axis at the point (Fig. 10.4) which can be determined as follows.

$$0 = -\frac{5x_1}{m+2} + \frac{10}{m+2}$$

or

$$x_1 = 2$$

Hence it is clear that all the isoclines will pass through the point $x_1 = 2$, $x_2 = 0$.

When $m = 0$, the equation of the isoclines can be expressed as

$$x_2 = -\frac{5}{2}x_1 + 5$$

Similarly for different values of m, the isoclines can be drawn. (See Fig. 10.4.)

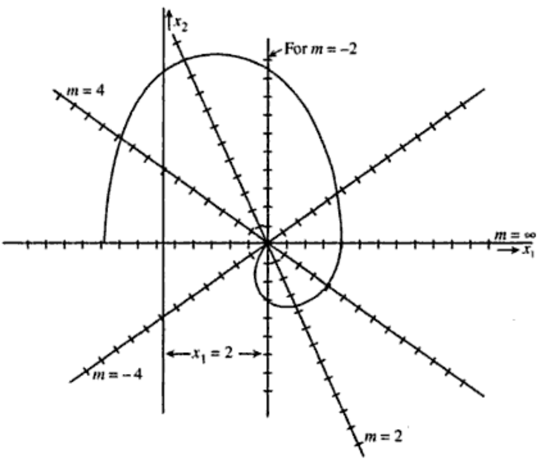


***Fig. 10.4** Isoclines and trajectory for a second-order control system.*

Now on each isocline, a series of parallel slopes are drawn having the value of gradient equal to the value of m for that particular isocline. For example, on isocline $m = -2$, parallel lines are drawn having slope $\tan\theta = -2$ and the parallel lines are drawn considering x_1 as the x-axis and x_2 as the y-axis since $dx_2/dx_1 = m$.

These parallel lines are drawn only to guide the trajectory about its direction.

The initial point is $Q(0) = -1$ and $\dot{Q}(0) = 0$. That means $x_1(0) = -1$ and $x_2(0) = 0$. Hence the trajectory will start from the point (–1, 0). Now, we have

$$m = -2 - 5\frac{x_1}{x_2} + \frac{10}{x_2}$$

At the point (–1, 0)

$$m = -2 - \frac{5(-1)}{0} + \frac{10}{0} = -2 + \infty$$

Hence, m is +ve and that is why the trajectory will move upwards.

The trajectory is drawn as shown in Fig. 10.4. The accuracy of this trajectory totally depends on the number of isoclines that have been drawn.

10.2.3 Phase Plane Trajectory for the Nonlinear System

We have just shown how the phase trajectory is drawn for a linear system. Now we will discuss the nature of the phase plane trajectory for a nonlinear system by applying the same principle of the phase plane method. For the nonlinear system, we will get a family of trajectories. That is why, such a diagram is also called *phase portrait*.

Let us take a nonlinear circuit as shown in Fig. 10.5, where r is a nonlinear resistance. The voltage and current characteristic of the nonlinear resistance r is described by the following relationship:

$$i = -AV + BV^3$$

$$E - V = L\frac{di}{dt}$$

$$= L\cdot\frac{d}{dt}(i' + i'') \qquad \text{where } i = i' + i''$$

$$= L\cdot\frac{d}{dt}\left[-AV + BV^3 + C\cdot\frac{dV}{dt}\right]$$

$$= -AL\frac{dV}{dt} + 3LBV^2\frac{dV}{dt} + LC\frac{d^2V}{dt^2}$$

or

$$E = V + \frac{dV}{dt}\left[3LBV^2 - AL\right] + LC\frac{d^2V}{dt^2}$$

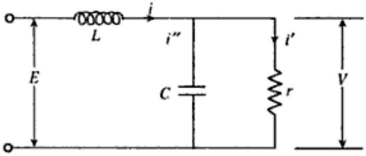


Fig. 10.5 Nonlinear circuit.

Let us assume $\tau = \dfrac{t}{\sqrt{LC}}$. Thus,

$$\frac{dV}{dt} = \frac{dV}{d\tau}\cdot\frac{d\tau}{dt} = \frac{dV}{d\tau}\frac{1}{\sqrt{LC}}$$

and

$$\frac{d^2V}{dt^2} = \frac{d}{dt}\left[\frac{dV}{d\tau}\frac{1}{\sqrt{LC}}\right] = \frac{d}{d\tau}\left[\frac{dV}{d\tau}\cdot\frac{1}{\sqrt{LC}}\right]\frac{d\tau}{dt}$$

$$= \frac{d^2V}{d\tau^2}\frac{1}{\sqrt{LC}}\frac{1}{\sqrt{LC}} = \frac{1}{LC}\frac{d^2V}{d\tau^2}$$

or

$$LC\frac{d^2V}{dt^2} = \frac{d^2V}{d\tau^2}$$

Again,

$$E = V + \frac{dV}{dt}\left[3LBV^2 - AL\right] + LC\frac{d^2V}{dt^2}$$

$$= V + \frac{dV}{d\tau}\frac{1}{\sqrt{LC}}\left[3LBV^2 - AL\right] + \frac{d^2V}{d\tau^2}$$

$$= \frac{d^2V}{d\tau^2} - \frac{dV}{d\tau}\left[\frac{AL}{\sqrt{LC}} - \frac{3LBV^2}{\sqrt{LC}}\right] + V$$

$$= \frac{d^2V}{d\tau^2} - \frac{dV}{d\tau}\left[A\sqrt{\frac{L}{C}} - 3B\sqrt{\frac{L}{C}}\frac{A}{A}V^2\right] + V$$

Let

$$a = A\sqrt{\frac{L}{C}}$$

and

$$b = \frac{3B}{A}$$

Therefore,

$$E = \frac{d^2V}{d\tau^2} - \frac{dV}{d\tau}\left[a - \frac{3BV^2}{A}a\right] + V$$

$$= \frac{d^2V}{d\tau^2} - \frac{dV}{d\tau}\left[a - bV^2a\right] + V$$

$$= \frac{d^2V}{d\tau^2} - \frac{dV}{d\tau}\left[a\left(1 - bV^2\right)\right] + V$$

Now, we have to develop the isoclines of the above nonlinear equation.
Let us develop the state equations by assuming

$$x_1(\tau) = V(\tau) \qquad \text{and} \qquad x_2(\tau) = \dot{V}(\tau)$$

For developing the isoclines, let us assume some standard values of a and b, say, $a = 0.2$ and $b = 1$. Now,

$$\dot{x}_1 = x_2$$

For making the solution easier, let the forcing function E be assumed to be zero. Hence the equation

$$E = \frac{d^2V}{d\tau^2} - \frac{dV}{d\tau}\left[a\left(1 - bV^2\right)\right] + V$$

is changed to

$$0 = \frac{d^2V}{d\tau^2} - \frac{dV}{d\tau}\left[0.2\left(1 - V^2\right)\right] + V$$

Since $x_1 = V$ and $x_2 = \dot{V}$, we have

$$0 = \dot{x}_2 - x_2\left[0.2\left(1 - x_1^2\right)\right] + x_1$$

or

$$\dot{x}_2 = 0.2(1 - x_1^2)x_2 - x_1$$

Since $\dot{x}_1 = x_2$, we have

$$\frac{\dot{x}_2}{\dot{x}_1} = \frac{0.2\left(1 - x_1^2\right)x_2 - x_1}{x_2}$$

or

$$\frac{dx_2}{dx_1} = 0.2\left(1 - x_1^2\right) - \frac{x_1}{x_2}$$

Thus the slope of the trajectory at the point (x_1, x_2) is given be

$$m = \frac{dx_2}{dx_1} = 0.2\left(1 - x_1^2\right) - \frac{x_1}{x_2}$$

Hence the equation of the isoclines will be as follows:

$$m = \frac{1}{5}\left(1 - x_1^2\right) - \frac{x_1}{x_2}$$

or

$$m = \frac{x_2 - x_2x_1^2 - 5x_1}{5x_2}$$

or

$$5x_2m = x_2 - x_2x_1^2 - 5x_1$$

or

$$x_2 = \frac{5x_1}{(1-5m)-x_1^2}$$

When $m = 0$, the isocline will be $x_2 = \dfrac{5x_1}{1-x_1^2}$.

The isoclines are drawn for various values of m, for example, $m = 0, 0.5, 1, 1.5, 2, 2.5, 3, -0.5, -1, -1.5, -2, -2.5, -3$, and so on (Fig. 10.6).

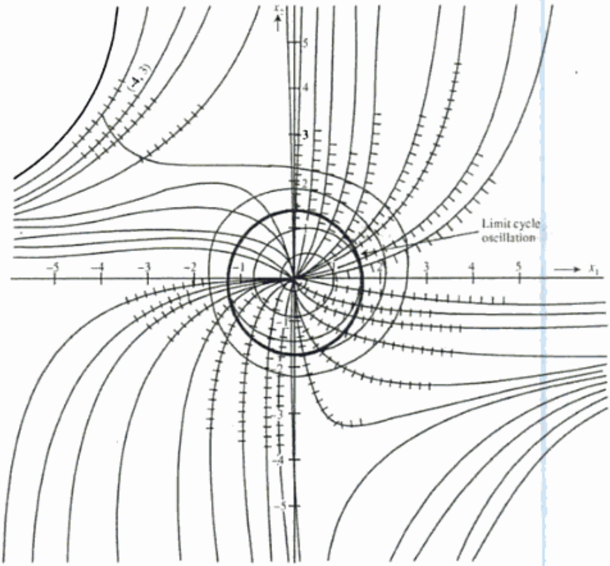


***Fig. 10.6** Isoclines and trajectories for a nonlinear system.*

Let us take the following two points from which the trajectories are to be developed.

(i) $V(0) = -4$ and $\dot{V}(0) = 3$

(ii) $V(0) = 0.5$ and $\dot{V}(0) = 0$

Hence both the initial points are $x_2(0) = 3$, $x_1(0) = -4$ and $x_2(0) = 0$, $x_1(0) = 0.5$.

First of all, the series of m gradients are drawn on the isoclines and the trajectories are drawn from the above two initial points deciding the direction of movement of the trajectories according to the principle already described.

It is being observed that both the trajectories are trapped in a closed curve as shown in Fig. 10.6 by the thick line. This indicates that a steady-state condition with zero velocity can never be obtained. Rather, it can be concluded that a stable steady state oscillation becomes the outcome with the voltage oscillating with fixed amplitude. This oscillation is termed the *limit cycle oscillation.*

10.2.4 Solution of Time

The time increases on the trajectory in the direction in which the solution point moves. Sometimes, it is required to determine the time between the two given points on the trajectory or to find out an explicit time solution. Now since,

$$\dot{x}_1 = x_2, \qquad x_2 = \frac{dx_1}{dt}$$

therefore,

$$t = \int \frac{1}{x_2} dx_1$$

This clearly indicates that the time may be obtained for a given phase trajectory by a simple graphical integration.

10.3 DESCRIBING FUNCTIONS

The nonlinear control system can also be dealt with by use of the describing function. This method is used for finding out the stability of a nonlinear system. Of course, the describing function approach is not based upon a solid mathematical basis. It is usually applicable to systems which have low-pass filter properties because the assumption of low-pass filter properties helps us to ignore the effect of harmonics in the system. The describing function approach is essentially a frequency response method. Let us take a nonlinear control system and try to implement the describing function approach.

Figure 10.7 shows a nonlinear control system, where A is a nonlinear device whose input R_1 and the output C_1 are related as follows:

$$C_1 = f(R_1)$$

Let the input R_1 be given by

$$R_1 = R_1 \sin \omega t$$

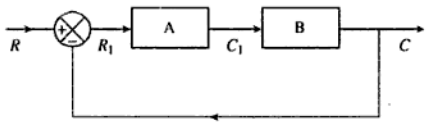


Fig. 10.7 Nonlinear control system.

The output of the nonlinear device A is given by

$$C_1 = f(R_1 \sin \omega t)$$

The preceding equation can be solved by Fourier series. The system B has low-pass filter properties. Hence the effects of higher harmonics in the output C_1 will be attenuated by B at its output C. Hence R_1 is assumed to remain unaffected by it.

Usually, the output of A will be of the form

$$C_1 = R_1^{\text{I}} \sin \omega t + R_1^{\text{II}} \cos \omega t + R_1^{\text{III}} \sin 2\omega t + R_1^{\text{IV}} \cos 2\omega t + \cdots$$

Eliminating the higher harmonics,

$$C_1 \approx R_1^{\text{I}} \sin \omega t + R_1^{\text{II}} \cos \omega t$$

Thus we have developed the harmonically linearized system. Our next approach in the describing function method is to find out the gain of the 'harmonically linearized' system. This harmonically linearized system is in the complex form. Usually, the general vector notations of R_1 and C_1 can be narrated as follows.

$$R_1 = R_1' \angle 0^\circ$$

$$C_1 = R_1' \angle 0^\circ + R_1'' \angle 90^\circ$$

Now,

$$\frac{C_1}{R_1} = \frac{R_1' \angle 0^\circ + R_1'' \angle 90^\circ}{R_1 \angle 0^\circ}$$

$$= \frac{R_1'}{R_1} + j\frac{R_1''}{R_1}$$

$$= K_1 + jK_2$$

$$= K_A \angle \theta$$

where

$$K_A = \sqrt{K_1^2 + K_2^2}$$

$$\theta = \tan^{-1} \frac{K_2}{K_1}$$

The describing function thus provides the ratio of the output to the input of the nonlinear device and is expressed in the vector form. Therefore, K_A and θ describe the describing function.

Now that we have an idea about the describing function approach, why not we apply it to a practical field problem?

Let us therefore find the describing function for an on-off system. The on-off system is described by a sinusoidal input $e = e_1 \sin \omega t$, (Fig. 10.8).

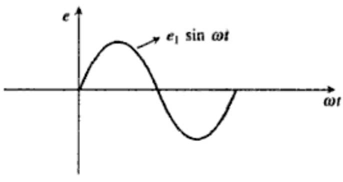


Fig. 10.8 Sinusoidal input.

The output C of the on-off device will develop a square wave with amplitudes $\pm C_m$ at the same frequency as that of the sinusoidal input.

The nonlinear device characteristic is shown in Fig. 10.9.

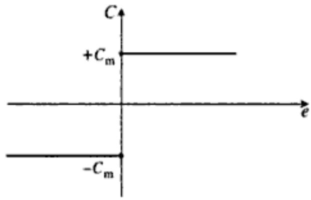


Fig. 10.9 Nonlinear characteristic.

From Figs. 10.8 and 10.9, it is very clear that the output voltage vs. ωt curve will develop a square wave as shown in Fig. 10.10.

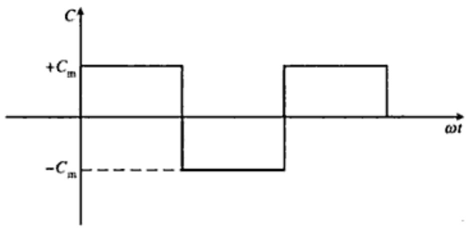


Fig. 10.10 Square wave.

Now applying Fourier analysis on the output waveform, we get

$$C = \frac{4C_m}{\pi}\left[\sin \omega t + \frac{1}{3}\sin 3\omega t + \cdots\right]$$

Hence the describing function can be approximated to the following expression by eliminating harmonics

$$C = \frac{4C_m}{\pi}\sin \omega t$$

and represented as

$$\frac{\frac{4C_m}{\pi}}{e_1} + j0$$

$$= \frac{4C_m}{\pi \cdot e_1} + j0$$

From the above illustration, it is also clear that an on-off system may work as a controller.

Let us now find out the solution of an open-loop system with an on-off controller. Assume that an on-off controller drives a system having a transfer function $1/s^2$. As we know the output of the on-off controller is

$$C = \frac{4C_m}{\pi}\left[\sin\omega t + \frac{1}{3}\sin 3\omega t + \frac{1}{5}\sin 5\omega t + \cdots\right]$$

Now if we apply the above output to the system having transfer function $1/s^2$, then the final output will be the double integration of C. Therefore:

$$\text{The final output} = \iint C\,dt\,dt$$

$$= \iint \frac{4C_m}{\pi}\left[\sin\omega t + \frac{1}{3}\sin 3\omega t + \frac{1}{5}\sin 5\omega t + \cdots\right] dt\,dt$$

$$= \int \frac{4C_m}{\pi}\left[-\frac{\cos\omega t}{\omega} - \frac{1}{3}\frac{\cos 3\omega t}{3\omega} - \frac{1}{5}\frac{\cos 5\omega t}{5\omega} - \cdots\right] dt$$

$$= \frac{4C_m}{\pi}\left[-\frac{\sin\omega t}{\omega^2} - \frac{1}{3}\frac{\sin 3\omega t}{9\omega^2} - \frac{1}{5}\frac{\sin 5\omega t}{25\omega^2} - \cdots\right]$$

$$= -\frac{4C_m}{\pi\omega^2}\left[\sin\omega t + \frac{\sin 3\omega t}{27} + \frac{\sin 5\omega t}{125} + \cdots\right]$$

Thus it is observed that the third harmonic is 3.7% and the fifth harmonic 0.8%.

Now, from the above illustration, it is also clear that the assumption of eliminating the third and fifth harmonics is quite justified.

Thus, the transfer function $1/s^2$ is acting like a filter in the above system. The above is also an example of the describing function.

Instead of an ideal on-off controller, if an on-off controller with dead zone is used (Fig. 10.11), then the describing function can be determined in the following manner.

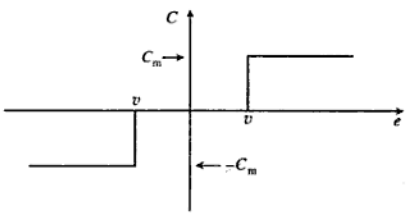


Fig. 10.11 On-off controller with dead zone.

Let us assume that $e = e_1 \sin\omega t$ is the sinusoidal input. On account of the dead-zone in the controller, the output will be zero for the angle ϕ. In other words, when the input voltage is less than v, no output will be observed. Figure 10.12 describes the input and output of the describing function.

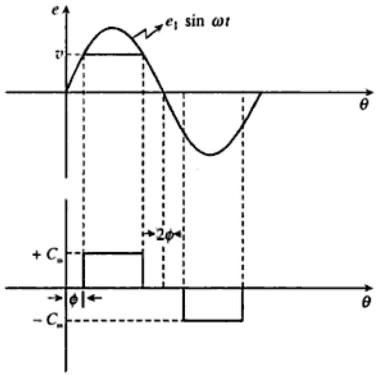


Fig. 10.12 Input and output of the describing function.

Applying Fourier series and considering the fundamental component at the output, we get the output as

$$\frac{4}{\pi}\int_{\phi}^{\pi/2} C_m \sin\theta \, d\theta = \frac{4C_m}{\pi}\left[-\cos\theta\right]_{\phi}^{\pi/2}$$

$$= \frac{4C_m}{\pi}\cos\phi$$

Now, $e_1 \sin \phi = v$, therefore, $\sin \phi = v/e_1$ or $\cos \phi = \sqrt{1 - v^2/e_1^2}$

Therefore,

$$\text{Output} = \frac{4C_m}{\pi}\sqrt{1-\frac{v^2}{e_1^2}}$$

and

$$\text{Describing function} = \frac{4C_m}{\pi e_1}\sqrt{1-\frac{v^2}{e_1^2}}$$

We now find the value of e_1, for which the describing function will be maximum. Thus,

$$\frac{d}{de_1}\left[\frac{4C_m}{\pi e_1}\sqrt{1-\frac{v^2}{e_1^2}}\right] = 0$$

or

$$\frac{4C_m}{\pi}\frac{d}{de_1}\left[\frac{1}{e_1}\sqrt{1-\frac{v^2}{e_1^2}}\right] = 0$$

or

$$-\frac{1}{e_1^2}\sqrt{1-\frac{v^2}{e_1^2}}+\frac{1}{e_1}\frac{1}{2}\frac{2v^2 e_1^{-3}}{\sqrt{1-\frac{v^2}{e_1^2}}}=0$$

or

$$-\frac{1}{e_1^2}\sqrt{\frac{e_1^2-v^2}{e_1^2}}+\frac{1}{e_1^4}\frac{v^2}{\sqrt{\frac{e_1^2-v^2}{e_1^2}}}=0$$

or

$$-\frac{1}{e_1^2}\left(\frac{e_1^2-v^2}{e_1^2}\right)+\frac{1}{e_1^4}v^2=0$$

or

$$\frac{-e_1^2+v^2+v^2}{e_1^4}=0$$

or

$$2v^2=e_1^2 \qquad \text{or} \qquad e_1=\sqrt{2}\,v$$

Hence, at $e_1 = \sqrt{2}\,v$, the describing function will be maximum. The maximum value of the describing function is thus given by

$$\frac{4C_m}{\pi e_1}\sqrt{1-\frac{v^2}{e_1^2}}=\frac{4C_m}{\pi\sqrt{2}v}\sqrt{1-\frac{v^2}{2v^2}}=\frac{2C_m}{\pi v}$$

For different values of e_1 if the magnitude of the describing function is plotted, then the outcome will be as shown in Fig. 10.13.

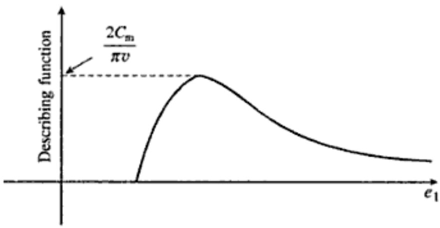


Fig. 10.13 Plot of the describing function versus e_1.

10.3.1 Application of the Describing Function Techniques

The describing function can be utilized to determine the system stability, both in the case of the closed-loop frequency response and transient response. The describing function describes the nonlinearity and then the stability can be easily assessed by the Nyquist diagram.

Suppose $D_1(j\omega)$ and $D_2(j\omega)$ indicate the linear part of the system and $D_N(e, j\omega)$ indicates the describing function of the nonlinear system (Fig. 10.14)

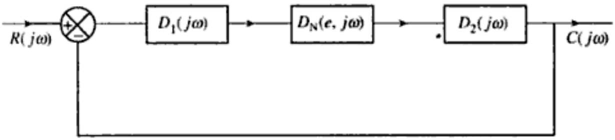


***Fig. 10.14** Describing function of nonlinear system.*

The describing function $D_N(e, j\omega)$ is the function of both the voltage and frequency. The transfer function of the closed-loop system is expressed as follows.

$$\frac{C(j\omega)}{R(j\omega)} = \frac{D_1(j\omega)\,D_N(e, j\omega)\,D_2(j\omega)}{1 + D_1(j\omega)\,D_N(e, j\omega)\,D_2(j\omega)}$$

Obviously, the characteristic equation to be examined for stability is

$$D_1(j\omega)D_N(e, j\omega)D_2(j\omega) = -1$$

The normal method will be to draw a polar plot with varying frequency. That means, the polar plot of $D_1(j\omega)D_N(e, j\omega)D_2(j\omega)$ is to be drawn. As per the Nyquist criterion, $-1 + j0$ will be the critical point for the stability analysis.

Since the describing function $D_N(e, j\omega)$ is not only frequency dependent but also amplitude e dependent, a modification to the above procedure is desired.

We can write $D_1(j\omega)D_N(e, j\omega)D_2(j\omega) = -1$ as

$$D_1(j\omega)D_2(j\omega) = \frac{-1}{D_N(e, j\omega)}$$

Thus we are making the left-hand side of the above equation amplitude independent. The stability in this case can be analyzed in the following manner.

(a) If there are no intersections between the curves of $-\dfrac{1}{D_N(e, j\omega)}$ and $D_1(j\omega)D_2(j\omega)$ and also if the curve of $-\dfrac{1}{D_N(e, j\omega)}$ is completely enclosed by the $D_i(j\omega)D_2(j\omega)$ curve, then the system is totally unstable.

(b) If there are no intersections between the curves of $-\dfrac{1}{D_N(e, j\omega)}$ and $D_1(j\omega)D_2(j\omega)$ and the curve $D_1(j\omega)D_2(j\omega)$ does not enclose the $-\dfrac{1}{D_N(e, j\omega)}$ curve, then the system, in the case, is termed absolutely stable.

(c) If there are intersections between the curves $D_1(j\omega)D_2(j\omega)$ and $-\dfrac{1}{D_N(e, j\omega)}$, then in that case, various possibilities exist from the point of view of stability.

Suppose, an on-off controller without dead zone is connected with the system having transfer function $\dfrac{K}{s(s+a)(s+b)}$ and the system is made closed loop as shown in Fig. 10.15.

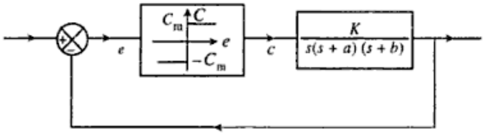


Fig. 10.15 System with on-off controller without dead zone.

We have already proved that the describing function of the on-off controller without dead zone is $\dfrac{4C_m}{\pi e_1}$, where input is $e = e_1 \sin \omega t$ and C_m is the output of the describing function.

Now, according to the stability analysis in the modified method,

$$\frac{K}{j\omega(j\omega+a)(j\omega+b)} = -\frac{\pi e_1}{4C_m}$$

Now the polar plots of $\dfrac{K}{j\omega(j\omega+a)(j\omega+b)}$ are made for three different values of K, say K_1, K_2, K_3. The plot of $-\dfrac{\pi e_1}{4C_m}$ will lie on the real negative-axis only since it possesses the variable real value of amplitude and no imaginary component.

The point of intersection of $\dfrac{K}{j\omega(j\omega+a)(j\omega+b)}$ and $-\dfrac{\pi e_1}{4C_m}$, i.e. M corresponding to the value $e_1 = e_1'$ (Fig. 10.16) is now analysed as follows. Any tendency for the oscillation amplitude e_1' to increase will certainly indicate the movement of the point on the negative real axis away to the left from the point M. That means $\dfrac{K}{j\omega(j\omega+a)(j\omega+b)}$ will not enclose $-\dfrac{\pi e_1}{4C_m}$ at that time.

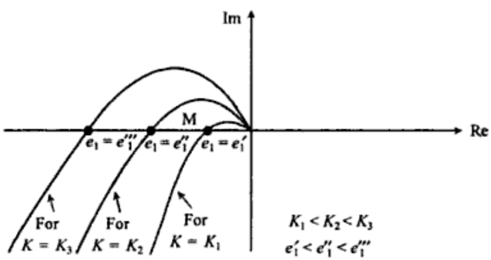


Fig. 10.16 Polar plots and plot of describing function for the system of Fig. 10.15.

In other words, the system will not enclose $(-1 + j0)$ at that condition and thus be termed stable. Similarly, for other values of K (i.e. $K = K_2$ and $K_2 = K_3$), the stability can be explained.

10.4 EXAMPLES OF NONLINEAR SYSTEMS

The nonlinearities are generally classified into two categories:

(a) Incidental
(b) Intentional

Incidental nonlinearities are those which are present in the system. For example, saturation, dead zone, coulomb friction, stiction, backlash, and so forth. The intentional nonlinearities, on the other hand, are those which are deliberately inserted in the system for modifying the system characteristics. A relay is the most important example of this.

Saturation. Saturation occurs due to the limitations of physical capabilities of the components. For example, amplifiers have output proportional to the input within a particular range of input signals. Figure 10.17 shows an example of the saturation nonlinearity.

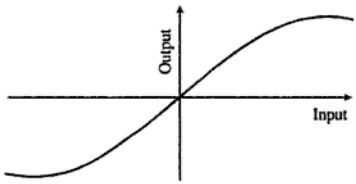


Fig. 10.17 *Saturation.*

Friction. The friction in a system has several components. The torque on the system experiencing the different types of friction is expressed as follows at the time of the start of rotation,

$$T = T_L + J \cdot \frac{d\omega_m}{dt} + T_M$$

where

$$T_L = T_{\text{viscous friction}} + T_{\text{coulomb friction}} + T_{\text{standstill}} + T_{\text{windage}}$$

Figure 10.18 describes the torque diagrammatically. At the stationary condition, i.e. at standstill, the torque is termed *stiction*

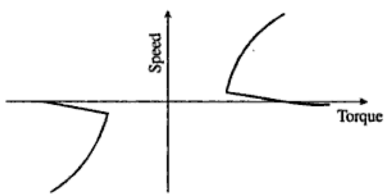


Fig. 10.18 *Stiction torque.*

Figure 10.19 describes the speed-torque characteristics for coulomb friction, viscous friction and standstill friction. Besides, there is also windage torque which is proportional to the square of the angular speed. Thus it is observed that the rotational motion in the presence of friction and windage provides a large nonlinearity.

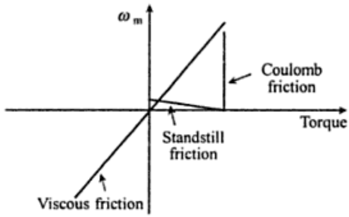


Fig. 10.19 Speed-torque characteristic in presence of torques.

Backlash. We have already studied magnetic hysteresis. But this term is also used in mechanical transmission system. Mechanical hysterisis is somewhat different from the magnetic one. It is termed *backlash.*

Figures 10.20 and 10.21 show the example of gear transmission. Here P is one of the teeth of the drive gear and Q is one of the slots of the driven gear. The tooth P of the drive gear is placed in between the two teeth R and S of the driven gear just at the middle position.

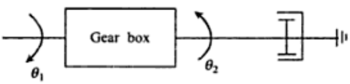


Fig. 10.20 Gear transmission.

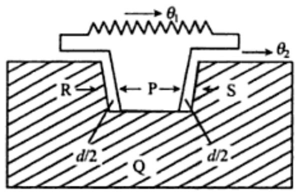


Fig. 10.21 Gear transmission.

Figure 10.22 describes the input and output motion characteristics for mechanical transmission.

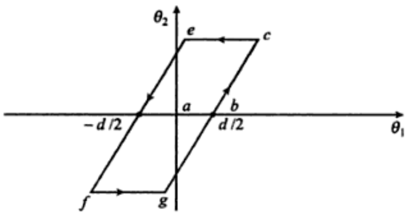


Fig. 10.22 Backlash.

The tooth P is driven clockwise from the position shown in Fig. 10.21. Until the tooth P touches the tooth S of the driven gear after traversing a distance $d/2$, no output motion will occur. That is why *ab* will be the locus of the input-output characteristics. As soon as the tooth P contacts the tooth S the driven gear starts rotating in the anticlockwise direction and the locus in Fig 10.22 will be *bc*. The slope of the line *bc* will depend on the gear ratio. If the ratio is one, then the angle will be 45°. Now the contact between P and S is lost and the driven gear will become stationary immediately.

Of course, if it is considered that the load is friction controlled with negligible inertia, then the above phenomenon will occur. Hence the output motion will remain zero till a distance $(d/2) + (d/2) = d$ is traversed in the reverse direction by the tooth P. Hence the locus will be *ce*.

As soon as the tooth P touches the tooth R, the driven gear starts moving in the clockwise direction and the locus will be *ef*. Again, the similar phenomenon will start, the input motion will be reversed and the locus will be *fg*. Thus, a complete cycle of the output motion will be the outcome. The width of the input-output curve is equal to the total backlash *d*.

This type of nonlinearity may produce sustained oscillations or the chattering phenomenon. Backlash is usually eliminated in the following manner.

(a) By using high quality gears
(b) By using spring-loaded split gear as the driven gear.

Dead zone. If the input is provided to the system but the system remains non-responsive, that means there is no output. The particular zone where this phenomenon occurs is termed *dead zone*. For example, when gears with backlash drive a torsional spring load, the driven gear will not move in a particular region since the spring remains untwisted (as shown in Fig. 10.23).

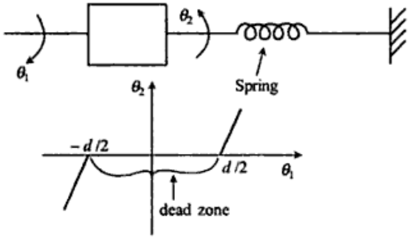


Fig. 10.23 Dead zone.

Relay. A relay is a nonlinear power amplifier. It is used intentionally in the control system. Usually a relay controlled system can be switched between several discrete states, for example, off, full forward, full reverse. A relay servosystem is the example.

From Fig. 10.24, it is quite clear that any error in the alignment of the load is fed back to the amplifier and the output of the amplifier is fed to the solenoid of the relay and that moves the contact in one direction or the other. The servo motor will then rotate in the desired direction as required for proper load alignment. But the transmission mechanism cannot be treated ideal because of the following reasons:

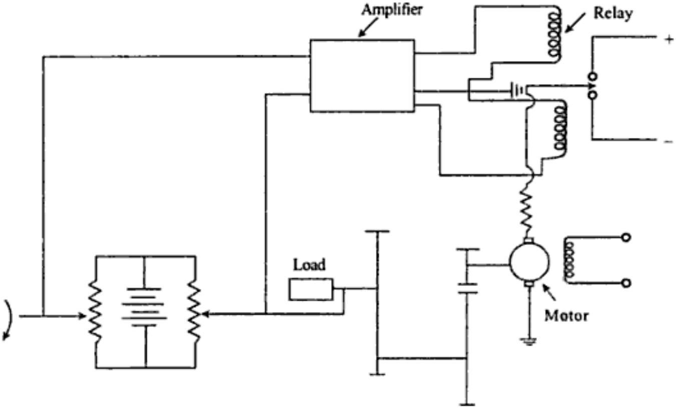
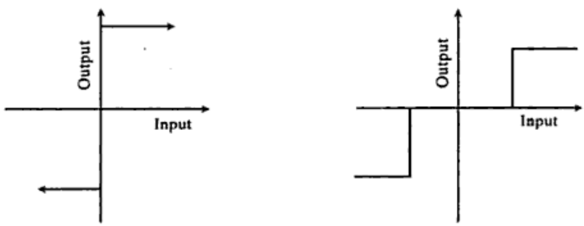


Fig. 10.24 Relay servosystem.

1. The relay has a definite amount of dead zone because the relay coil requires a finite amount of current to actuate the relay.
2. The relay characteristic also exhibits hysteresis because a larger value of coil current is needed to close the relay than the value of current at which the relay drops out. That is why several types of nonlinearity relay characteristics may appear as shown in Figs. 10.25(b), 10.26, and 10.27.



(a) *Relay at ideal condition.* (b) *Relay having dead zone.*

Fig. 10.25 Relay.

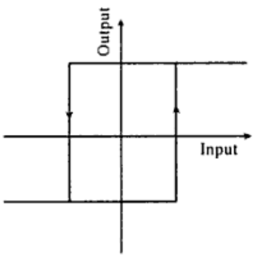


Fig. 10.26 Relay having hysteresis.

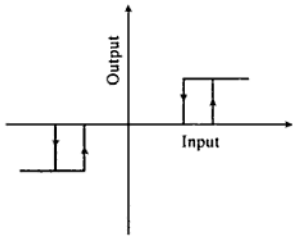


Fig. 10.27 Relay having both dead zone and hysteresis.

SUMMARY

The Liapunov's method for stability study is explained with an example. The Liapunov's second method is also described with an example. The phase plane method of solution of nonlinear control systems is given too. The method for determining the phase trajectory of a second-order system is then explained with examples from both linear and nonlinear systems. Isoclines are clearly defined. How to deal with nonlinear control systems by *describing functions* is also explained with examples. Application of the describing function techniques for determining system stability study is also discussed. Different examples of nonlinear systems involving saturation, friction, backlash, dead zone, and relay are described.

QUESTIONS

1. A system is described by the following equation:

$$\ddot{x} + \dot{x} + x^3 = 0$$

Its initial conditions are $x(0) = 1$, $\dot{x}(0) = 0$. Construct its trajectory on the phase-plane diagram.

2. Explain the limit cycle in the analysis of a nonlinear control system.
3. Describe the stability analysis of nonlinear systems using the describing function.
4. Explain the Liapunov's method for equilibrium states.
5. Determine the stability of the origin of the following system:

$$\dot{x}_1 = x_2$$
$$\ddot{x}_2 = -x_1 - x_2$$

6. Explain backlash with an example.
7. Give certain practical examples of nonlinearities in systems.

Chapter

11 Digital Control Systems

11.1 INTRODUCTION

With the development of computer technology, the digital computer is now utilized to design the controller. So far, we have dealt with the closed-loop and the open-loop control systems which used the analog controller. With the increasing complexity of the control systems, the digital controller is becoming the demand of the day. Microcomputers which use a 16-bit word or a 32-bit word with a speed as high as 300 MHz, can handle a large amount of data in any complex control process. That is why, a digital controller is an ideal choice for use in complex control systems. Since the computers have the capacity to receive and manipulate several inputs, a digital computer control system can even be a multivariable system.

Figure 11.1 shows the block diagram of a system with a digital controller. This system is also termed a sampled data control system.

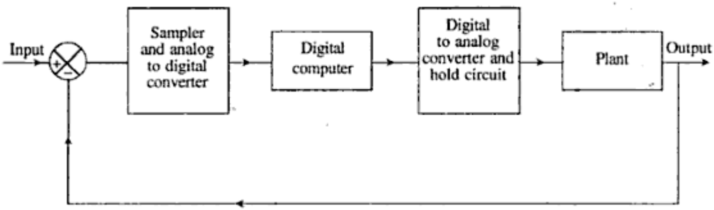


Fig. 11.1 Sampled data control system.

11.2 SAMPLED DATA CONTROL SYSTEM

The devices shown in Figure 11.1 serve the following purposes:

(a) **Sampler.** It is needed to convert the continuous time signal into a sequence of pulses.
(b) **Analog to digital converter** (ADC). The very purpose of the ADC is to transform the analog signal to digital. The digital signal is expressed in numerical code, e.g. binary code.
(c) **Digital computer.** All sorts of desired manipulations on the input signal are performed by the digital computer. These computations can even be made online as per requirement.

Digital to analog converter. Numerically coded output of the digital computer is decoded to continuous time signals by the digital-to-analog converter.

Hold circuit. It holds a signal for a desired time by utilizing a proper circuit.

Plant. The plant is controlled by the continuous time signal that is finally developed by the sampled data control system.

The implementation of a digital control system is not necessarily essential. It is used where a resolution of low value, say, $1/10^5$, is required. For example, if in the case of movement of table of a drilling machine, an accuracy of 0.015 mm over a total distance of 1 metre is required, then for a resolution of value $0.015/10^3$ (= 1.5×10^{-5}) the use of a digital controller is essential, converting the system into a sampled data control system.

11.3 SAMPLING

Sampling means that the signal at the output end of the sampler is in the form of short-duration pulses where each pulse is followed by a skip period during which no signal is observed and the control system operates as open loop during the skip period. The advantage of signal sampling is that it reduces the power demand made on the signal and is thereby helpful for signals of weak power origin. Hence, the sampling operation is sometimes purposely introduced. But sometimes, however, the signals are received in sampled form. Radar is one such example.

Usually, the uniform periodic type of sampling is applied, that means sampling is made at regular intervals of time. If the sampling period T_s is made very large, the sampling frequency $1/T_s$ would be too small and the information contained in the input signal may be lost in the output. That is why, the sampling rate must have a right value which will take care of the above.

Figures 11.2 and 11.3 describe how the uniform periodic sampling is made by a switching circuit in the sampler. Besides uniform periodic sampling, there exist multi-order sampling, multiple-rate sampling and random sampling. But, we will now concentrate on uniform periodic sampling.

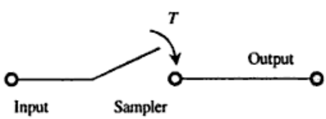


Fig. 11.2 Sampler.

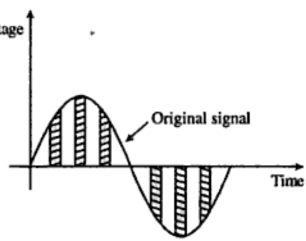


Fig. 11.3 Uniform periodic sampling.

The sampled signals are usually generated from the original signal by multiplying the original signal by a period pulse train already generated. Then only, the uniform periodic sampling is possible from an original signal.

Similarly, impulse sampling is also possible from the original signal by multiplying the original signal by the train of impulse pulses.

Figures 11.4 and 11.5 show the original signal and the resulting train of impulses, respectively. Both these signals are passed through the multiplier (Fig. 11.6). The multiplier value will be the same as $m(t)$ when the train of pulses occurs, otherwise the multiplier output will be zero (Fig. 11.7).

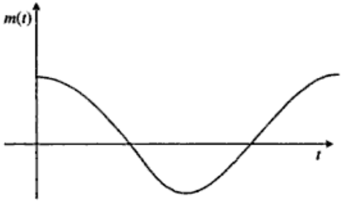


***Fig. 11.4** Original signal.*

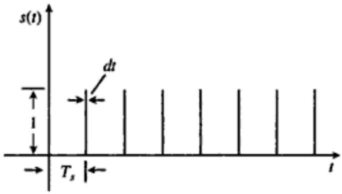


***Fig. 11.5** A train of impulses.*

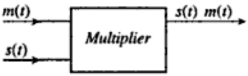


***Fig. 11.6** Multiplier.*

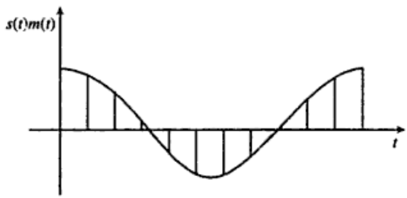


***Fig. 11.7** Multiplier output.*

Now, $s(t)$ can be represented by Fourier series analysis as

$$s(t) = \frac{dt}{T_s} + \frac{2dt}{T_s}\left(\cos 2\pi\frac{t}{T_s} + \cos 4\pi\frac{t}{T_s} + \cdots\right)$$

For example, the dc component is given by

$$\text{dc component} = \frac{1}{T_s}\int_0^{dt} 1 \cdot dt = \frac{dt}{T_s}$$

where T_s is the time period of the pulse and dt is the width of the pulse, with the value of $T_s = 1/2f_m$ where f_m is the original signal frequency. Now,

$$s(t)\, m(t) = \frac{dt}{T_s}m(t) + \frac{dt}{T_s}\left[2m(t)\cos 2\pi(2f_m)t + 2m(t)\cos 2\pi(4f_m)t + \cdots\right]$$

From the above equation, it is clear that the spectrum of the first term extends from 0 to f_m and the spectrum of the second term extends from $(2f_m - f_m)$ to $(2f_m + f_m)$. Figure 11.8 shows the magnitude of the spectral density of signal and Fig. 11.9 shows the amplitude of spectrum of the sampled signal. From Fig. 11.9 it is quite clear that, if the value of the sampling frequency is greater than $2f_m$, the sampled signal can be recovered exactly. On the other hand, if the sampling frequency is less than $2f_m$, there will occur an overlap of the double sideband suppressed carrier signal with the exact sampled signal. Thus, it is proved that if the sampling rate in any system exceeds twice the maximum signal frequency, the original signal can be developed with minimal distortion. This is also termed the *sampling theorem*.

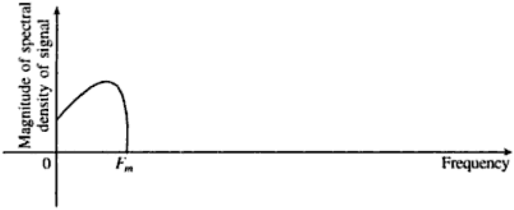


Fig. 11.8 Magnitude of spectral density of signal.

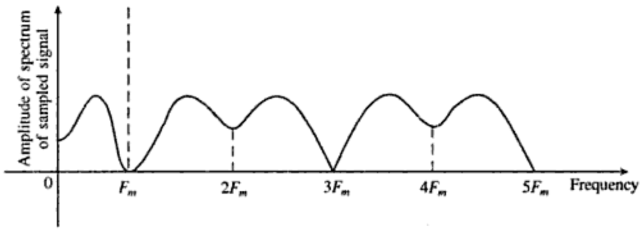


Fig. 11.9 Amplitude of spectrum of sampled signal.

11.4 SIGNAL RECONSTRUCTION

After the modification of the sampled-data signal by the digital controller or after the completion of transmission of the sampled data signal through a channel, it is essential to convert it to analog form for the utilization of the same in the continuous part of the system. This is usually done by different types of hold circuits. These hold circuits are also called *extrapolators*. The simplest hold circuit is termed *zero-order hold* (ZOH).

Zero-order hold means that the reconstructed signal has the same value as the last received sample of the entire sampling period.

Figure 11.10 shows the sampler with zero-order hold.

Figure 11.11 shows the output of the sampler after taking the sample from the original signal.

Figure 11.12 shows the output of the zero-order hold.

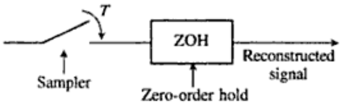


Fig. 11.10 Sampler with zero-order hold.

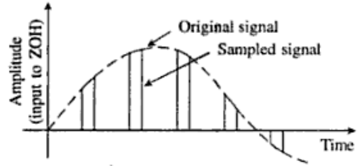


Fig. 11.11 Output of the sampler.

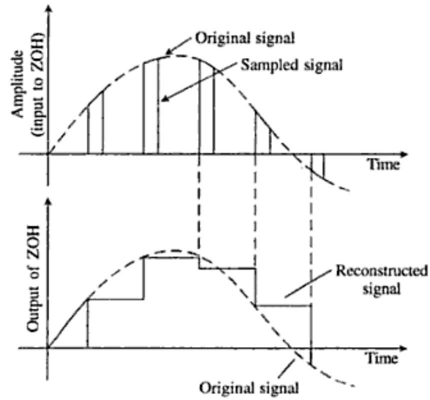


Fig. 11.12 Output of the zero-order hold.

11.5 LINEAR DISCRETE SYSTEM

We know that unit impulse response is represented by

$$\delta(K - m) = 1; \qquad K = m$$
$$= 0; \qquad K \neq m$$

An input sequence $r(K)$ is represented by the following series

$$r(K) = r(0)\delta(K) + r(1)\delta(K - 1) + r(2)\delta(K - 2) + \dots$$

Hence the system response to nth impulse $r(n)\delta(K - n)$ is

$$c_n(K) = r(n)h(K - n)$$

where the output sequence is represented by $c(K)$ for the input sequencer $r(K)$ and the system is usually characterized by its response $h(K)$ to unit discrete impulse. Therefore,

$$c(K) = \sum_{n=0}^{\infty} c_n(K)$$

$$= \sum_{n=0}^{\infty} r(n)h(K-n)$$

The above sum is termed *discrete convolution* and is defined as

$$c(K) = r(K) \cdot h(K)$$

When the system is causal

$$c(K) = \sum_{n=0}^{K} r(n)h(K-n) \qquad (\because h(K-n) = 0 \text{ for } n > K \text{ when the system is causal})$$

Let

$$i = K - n$$

then,

$$c(K) = \sum_{i=K}^{0} r(K-i)h(i)$$

or

$$c(K) = \sum_{i=0}^{K} h(i)r(K-i)$$

Therefore,

$$c(K) = r(K) \cdot h(K) = h(K) \cdot r(K)$$

Hence the convolution sum is also commutative.

Again the $\mathcal{Z}$-transform of $c(K)$ is

$$C(z) = \sum_{K=0}^{\infty} c(K)z^{-K}$$

Now,

$$c(K) = \sum_{n=0}^{\infty} r(n)h(K-n)$$

The $\mathcal{Z}$-transform of $\sum_{n=0}^{\infty} r(n)h(K-n)$ is

$$\sum_{K=0}^{\infty}\left[\sum_{n=0}^{\infty} h(K-n)r(n)\right]z^{-K}$$

Now interchanging the order of summation, we get

$$C(z) = \sum_{n=0}^{\infty} r(n)\sum_{K=0}^{\infty} h(K-n)z^{-K}$$

Substituting $i = K - n$, we have

$$C(z) = \sum_{n=0}^{\infty} r(n) \sum_{i=-n}^{\infty} h(i)\, z^{-i-n}$$

$$= \sum_{n=0}^{\infty} r(n)\, z^{-n} \sum_{i=0}^{\infty} h(i)\, z^{-i}$$

Since the system is causal, the impulse response $h(i) = 0$ for negative values of i. Therefore,

$$C(z) = R(z)H(z)$$

Hence the block diagram of a linear discrete time system in z-domain will be as shown in Fig. 11.13

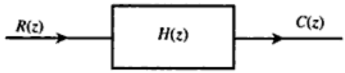


Fig. 11.13 Block representation of a linear discrete time system.

11.6 EQUIVALENT REPRESENTATION OF PULSE SAMPLER AND ZOH

Figure 11.14 describes the circuit of the pulse sampler with zero-order hold.

Figure 11.15 shows the ZOH input or the sampler output.

Figure 11.16 depicts the ZOH output.

Fig. 11.14 Pulse sampler with ZOH.

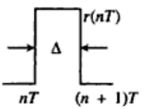


Fig. 11.15 ZOH input..

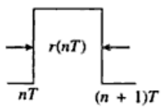


Fig. 11.16 ZOH output.

The ZOH output pulse observed at the nT instant can be written as follows:

$$r(nT)\left[u(t - nT) - u\left(t - \overline{n+1}T\right)\right]$$

At $t = nT$, the ZOH output will be

$$r(nT)\left[u(0) - u\left(nT - \overline{n+1}T\right)\right]$$

11.7 IMPULSE SAMPLING

We have studied that a pulse sampler with ZOH can be replaced by an impulse sampler and transfer function $\frac{1-e^{-sT}}{s}$.

Now, let us consider that a linear continuous signal $r(t)$ is passed through an impulse sampler as shown in Fig. 11.18(a). That is,

$$r^*(t) = \sum_{n=0}^{\infty} \gamma(nT)\delta(t-nT)$$

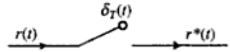


Fig. 11.18(a) Impulse sampler.

The impulse response of the linear continuous system is $h(t)$. Therefore, the output as shown in Fig. 11.18(b) is

$$c(t) = \sum_{n=0}^{\infty} r(nT)h(t-nT)$$

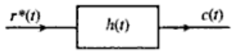


Fig. 11.18(b)

The output signal $c(t)$ is read off at discrete synchronous sampling instants (KT) with the help of a mathematical sampler. Therefore,

$$c(KT) = \sum_{n=0}^{\infty} r(nT)h(KT-nT) \quad \text{(as shown in Fig. 11.19)}$$

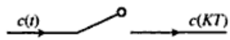


Fig. 11.19 Synchronous mathematical sampler

Again, when the above relation is expressed in Laplace transformation, it becomes

$$C(s) = H(s)R^*(s) \quad \text{(as shown in Fig. 11.20)}$$

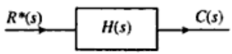


Fig. 11.20

where the Laplace inverse of $H(s)$ is $h(t)$.

Thus, we have already proved that in the case of $\mathcal{Z}$-transform

$$C(z) = R(z)H(z)$$

where $H(z)$ = $\mathcal{Z}$-transform of $h(nT)$

= $\mathcal{Z}$-transform of Laplace inverse of $H(s)$ when $t = nT$.

Hence the procedure of transformation of Laplace to $\mathcal{Z}$-transform will be:

1. Take the Laplace inverse of $H(s)$ to find,

$$h(t) = \mathcal{L}^{-1}H(s)$$

2. Determine $h(nT)$.
3. Then take the $\mathcal{Z}$-transform of $h(nT)$ to get $H(z)$.

This is therefore the procedure of deriving the $\mathcal{Z}$-transform of $H(s)$.

Table 11.1 Laplace and $\mathcal{Z}$-transform pairs corresponding to time functions

Time function	*Laplace transform*	*$\mathcal{Z}$-transform*
Unit step	$\frac{1}{s}$	$\frac{z}{z-1}$
t	$\frac{1}{s^2}$	$\frac{Tz}{(z-1)^2}$
$\frac{t^2}{2}$	$\frac{1}{s^3}$	$\frac{T^2 z(z+1)}{2(z-1)^3}$
$\frac{t^n}{n!}$	$\frac{1}{s^{n+1}}$	$\lim_{a\to 0} \frac{(-1)^n}{n!} \frac{\partial^n}{\partial a^n}\left[\frac{z}{z-e^{-aT}}\right]$
e^{-at}	$\frac{1}{s+a}$	$\frac{z}{z-e^{-aT}}$
te^{-at}	$\frac{1}{(s+a)^2}$	$\frac{Tze^{-aT}}{\left(z-e^{-aT}\right)^2}$
$1-e^{-at}$	$\frac{a}{s(s+a)}$	$\frac{\left(1-e^{-aT}\right)z}{(z-1)\left(z-e^{-aT}\right)}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$	$\frac{z \sin \omega T}{z^2 - 2z\cos \omega T + 1}$
$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$	$\frac{ze^{-aT}\sin \omega T}{z^2 - 2ze^{-aT}\cos \omega T + e^{-2aT}}$
$\frac{s}{s^2+\omega^2}$	$\cos \omega t$	$\frac{z(z-\cos \omega T)}{z^2 - 2z\cos \omega T + 1}$
$\frac{s+a}{(s+a)^2+\omega^2}$	$e^{-at}\cos \omega t$	$\frac{z^2 - ze^{-aT}\cos \omega T}{z^2 - 2ze^{-aT}\cos \omega T + e^{-2aT}}$

EXAMPLE 11.1 If we are interested to read the values of the continuous output at sampling instants when the system is continuous and the input is continuous, then the same can be represented as shown in Figs. 11.21, 11.22, and 11.23.

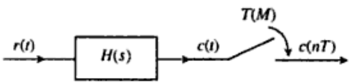


Fig. 11.21

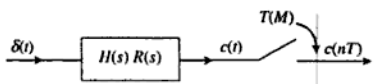


Fig. 11.22

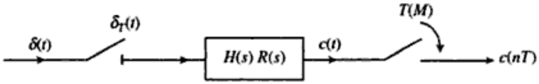


Fig. 11.23

where

$r(t)$ = continuous input
$c(t)$ = continuous output
$H(s)$ = continuous system
$T(M)$ = synchronous mathematical sampler
$\delta_T(t)$ = impulse sampler
$\delta(t)$ = impulse signal.

Solution Since

$$C(z) = R(z)H(z)$$

and

$$C(z) = \mathcal{Z}[H(s)\,R(s)]\;\mathcal{Z}[\delta(K)]$$

and

$$\mathcal{Z}[\delta(K)] = \sum_{K=0}^{\infty} \delta(K) z^{-K} = 1$$

Thus,

$$C(z) = HR(z)$$

$$C(s) = H(s)R(s)$$

$$C(z) = HR(z)$$

The $\mathcal{Z}$-transform of $[H(s)R(s)] = HR(z)$.

Again, the following two block diagrams shown in Figs. 11.24 and 11.25 are not the same.

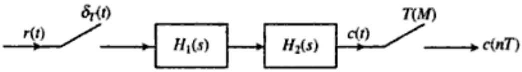


Fig. 11.24

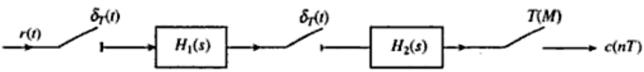


Fig. 11.25

For Fig. 11.24, the outcome of $\mathcal{Z}$-transform is Fig. 11.25(a). Hence, $H(z) = H_1H_2(z)$ [as shown in Fig. 11.25(a)]

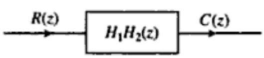


Fig. 11.25(a)

But in the case of Fig. 11.25 where the blocks $H_1(s)$ and $H_2(s)$ are separated by an impulse sampler,

$$H(z) = H_1(z)H_2(z)$$

Therefore,

$$H_1H_2(z) \neq H_1(z)H_2(z)$$

11.8 APPLICATION OF $\mathcal{Z}$-TRANSFORM TO SAMPLED DATA SYSTEMS

Let us consider an open-loop system. The input is $r(t)$. The transfer function is $G(s)$. The feeding to $G(s)$ is done by a pulse sampler and a zero-order hold. The output $c(t)$ is passed through the mathematical synchronous sampler to have the output $c(nT)$. Figure 11.25(b) describes this system. The equivalent sampled system for Fig. 11.25(b) will be that as shown in Fig. 11.26.

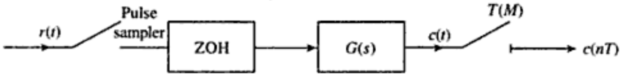


Fig. 11.25(b)

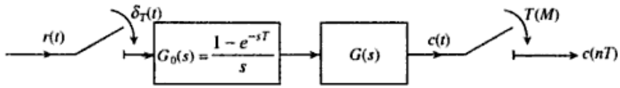


Fig. 11.26

Now the $\mathcal{Z}$-transform of $G_0(s)G(s)$ will be equal to the $\mathcal{Z}$-transform of $\left[\frac{1-e^{-sT}}{s}G(s)\right]$

$$= \mathcal{Z}\left[\frac{G(s)}{s} - \frac{e^{-sT}G(s)}{s}\right]$$

Suppose the Laplace inverse of $\left[\frac{G(s)}{s}\right] = f_1(t)$ and the Laplace inverse of $\left[\frac{e^{-sT}G(s)}{s}\right] = f_1(t - T)$.

Therefore,

$$\mathcal{Z}\left[\frac{e^{-sT}G(s)}{s}\right] = \mathcal{Z}[f_1(KT - T)]$$

$$= z^{-1}\mathcal{Z}[f_1(KT)]$$

$$= z^{-1}\mathcal{Z}\left[\frac{G(s)}{s}\right]$$

$$= \frac{(z-1)\left(1-e^{-bT}\right)}{(z-1)\left(z-e^{-bT}\right)} = \frac{1-e^{-bT}}{z-e^{-bT}}$$

Now let us consider a closed-loop system as shown in Fig. 11.29.

Figure 11.29 can be transferred to Fig. 11.30 by replacing the pulse sampler by the impulse sampler.

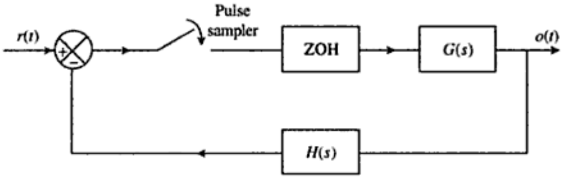


Fig. 11.29

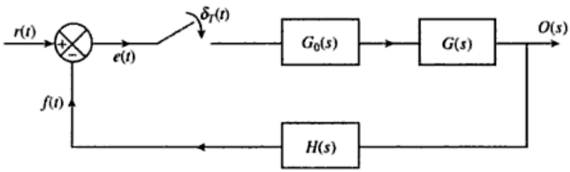


Fig. 11.30

Now,

$$O(z) = [\mathcal{Z}\text{-transform of } G_0(s)G(s)]E(z)$$

$$F(z) = [\mathcal{Z}\text{-transform of } G_0(s)G(s)H(s)]E(z)$$

$$e(t) = r(t) - f(t)$$

In case of sampling $t = nT$, i.e. $e(nT) = r(nT) - f(nT)$

Taking the $\mathcal{Z}$-transform of the above,

$$E(z) = R(z) - F(z)$$

or

$$E(z) = R(z) - \mathcal{Z}[G_0(s)H(s)H(s)]E(z)$$

or

$$E(z)\{1 + \mathcal{Z}\{G_0(s)G(s)H(s)\}\} = R(z)$$

Again,

$$O(z) = \mathcal{Z}[G_0(s)G(s)]E(z)$$

or

$$E(z) = \frac{O(z)}{\mathcal{Z}\left[G_0(s)G(s)\right]}$$

or

$$\frac{O(z)}{\mathcal{Z}\left[G_0(s)G(s)\right]}\left[1+\mathcal{Z}\left\{G_0(s)G(s)H(s)\right\}\right]=R(z)$$

or

$$\frac{O(z)}{R(z)}=\frac{\mathcal{Z}\left[G_0(s)G(s)\right]}{1+\mathcal{Z}\left[G_0(s)G(s)H(s)\right]}$$

$$=\frac{G_0G(z)}{1+G_0G\,H(z)}$$

If the pulse sampler and the zero-order hold are applied in the feedback path of the closed-loop system, then the $\mathcal{Z}$-transform analysis of the sampled data system will be as explained below.

Figure 11.31 shows that the pulse sampler with zero-order hold is applied in the feedback circuit. The equivalent circuit with impulse sampler is shown in Fig. 11.32.

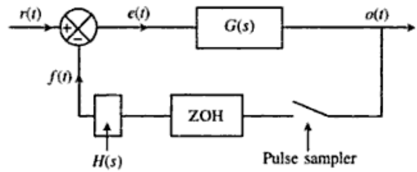


Fig. 11.31

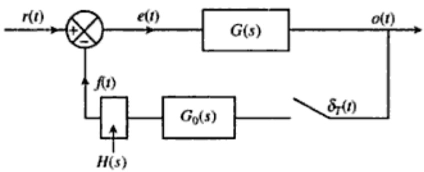


Fig. 11.32

Now,

$$R(s) - F(s) = E(s)$$

Again,

$$\frac{F(s)}{O^*(s)} = G_0(s)H(s)$$

or

$$F(s) = G_0(s)H(s)O^*(s)$$

Therefore,

$$E(s) = -G_0(s)H(s)O^*(s) + R(s)$$

where $O^*(s) \rightarrow$ sampled output $O(s)$.
Again,

$$\frac{O(s)}{E(s)} = G(s)$$

or

$$O(s) = E(s)G(s)$$

Multiplying $E(s) = -G_0(s)H(s)O^*(s) + R(s)$ by $G(s)$, we get

$$E(s)G(s) = -G(s)G_0(s)H(s)O^*(s) + G(s)R(s)$$

or

$$O(s) = -G_0(s)G(s)H(s)O^*(s) + G(s)R(s)$$

Applying $\mathcal{Z}$-transform, we get

$$O(z) = \mathcal{Z}\text{-transform of } [-G_0(s)G(s)H(s)O^*(s)] + \mathcal{Z}\text{-transform of } [G(s)R(s)]$$

Therefore,

$$O(z) = \mathcal{Z}[-G_0(s)G(s)H(s)]O(z) + \mathcal{Z}[R(s)G(s)]$$

($\because$ The $\mathcal{Z}$-transform of sampled $O(s)$ will be equal to $O(z)$)

or

$$O(z)\,[1 + G_0HG(z)] = RG(z)$$

or

$$O(z) = \frac{RG(z)}{1 + G_0HG(z)}$$

EXAMPLE 11.2 A closed-loop control system is described in Fig. 11.33. Determine the output in discrete form when a unit step is applied to the input.

Solution Figure 11.33 can be transformed to Fig. 11.34 where the impulse sampler is provided.

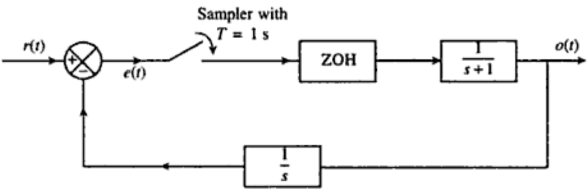


Fig. 11.33

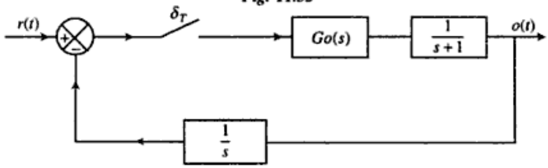


Fig. 11.34

Now, we know that for a closed-loop system,

$$\frac{O(z)}{R(z)} = \frac{\mathcal{Z}\text{-transform of } [G_0(s)G(s)]}{1 + \mathcal{Z}\text{-transform of } [G_0(s)G(s)H(s)]}$$

$$= \frac{\dfrac{1-e^{-1}}{z-e^{-1}}}{1 + \dfrac{e^{-1}z - 2e^{-1} + 1}{(z-1)(z-e^{-1})}}$$

$$= \frac{(1-e^{-1})(z-1)}{z^2 - z + 1 - e^{-1}}$$

Since the input is a unit-step input,

$$R(z) = \frac{z}{z-1}$$

Hence,

$$O(z) = \frac{z}{(z-1)} \frac{(1-e^{-1})(z-1)}{(z^2 - z + 1 - e^{-1})}$$

$$= \frac{z(1-e^{-1})}{z^2 - z + 1 - e^{-1}}$$

Putting the value of e, we will get the complex conjugate roots, say, $a + jb$ and $a - jb$. Thus,

$$O(z) = \frac{Az}{z-(a+jb)} + \frac{Bz}{z-(a-jb)}$$

where A and B are the coefficients.

Therefore, in discrete form by taking the $\mathcal{Z}$-inverse, we will get

$$O(n) = A \cdot (a + jb)^n + B(a - jb)^n$$

EXAMPLE 11.3 Find the output voltage in discrete form of the *RC* circuit as shown in Fig. 11.35 when the input voltage is applied as follows.

$$e(t) = e(nT) \quad \text{where } nT \le t \le (n+1)T \text{ and } T = 1 \text{ s}$$

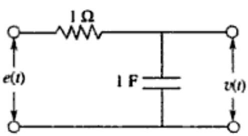


Fig. 11.35

Since in this problem $T = 1$ s, we have

$$e(nT) = n$$

Now,

$$V(z) = \frac{1-e^{-1}}{\left(z-e^{-1}\right)} \cdot E(z)$$

As

$$e(nT) = n$$

$$\mathcal{Z}\text{-transform of } e(nT) = \mathcal{Z}\text{-transform of } n$$

$$= \frac{Tz}{(z-1)^2} = \frac{z}{(z-1)^2}$$

Therefore,

$$V(z) = \frac{1-e^{-1}}{\left(z-e^{-1}\right)} \times \frac{z}{(z-1)^2}$$

$$= \left(1-e^{-1}\right)\frac{z}{\left(z-e^{-1}\right)(z-1)^2}$$

or

$$\frac{V(z)}{z} = \frac{1-e^{-1}}{\left(z-e^{-1}\right)(z-1)^2}$$

or

$$\frac{V(z)}{z} = \left(1-e^{-1}\right)\left[\frac{A}{(z-1)^2} + \frac{B}{z-1} + \frac{C}{z-e^{-1}}\right]$$

where

$$A = \lim_{z\to 1} \frac{1}{z-e^{-1}} = \frac{1}{1-e^{-1}}$$

$$B = \lim_{z\to 1} \frac{d}{dz}\left(\frac{1}{z-e^{-1}}\right) = \lim_{z\to 1}\left[-\left(z-e^{-1}\right)^{-2}\right]$$

$$= -\left(1-e^{-1}\right)^{-2}$$

$$C = \lim_{z\to e^{-1}} \frac{1}{(z-1)^2} = \frac{1}{\left(e^{-1}-1\right)^2}$$

Hence, the section $\left(-j\frac{\omega_S}{2}\right)$-0-$\left(+j\frac{\omega_S}{2}\right)$ on the $j\omega$-axis describes on the z-domain a unit circle in the anticlockwise direction by indicating the angle as $-\pi$, $-\pi/2$, 0, $\pi/2$, π.

When

$$s = \frac{-j\omega_S}{2}$$

then

$$z = \exp\left(-j\frac{\omega_S}{2}T\right)$$

$$= \exp\left(-j\frac{\omega_S}{2}\frac{2\pi}{\omega_S}\right)$$

$$= e^{-j\pi} = 1 \angle -\pi$$

Hence, the angle is $-\pi$. Similarly, the other angles are determined when the value on the $j\omega$-axis varies from $-j\frac{\omega_S}{2}$ to $+j\frac{\omega_S}{2}$

Figures 11.37 and 11.38 show the graphical representation in the s- and z-domains respectively.

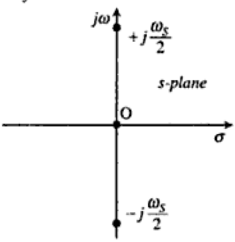


Fig. 11.37

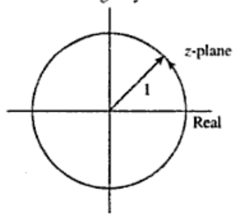


Fig. 11.38

11.9 STABILITY ANALYSIS OF DISCRETE SYSTEMS

A discrete system is generally described by the following $\mathcal{Z}$-transform.

$$O(z) = G(z)R(z)$$

where $O(z)$ is the output, $R(z)$ is the input, and $G(z)$ is the transfer function.

Suppose the input is impulse. Then the $R(z) = 1$. Therefore,

$$O(z) = G(z) = \frac{m_1}{z - a_1} + \frac{m_2}{z - a_2} + \cdots + \frac{m_K}{z - a_K}$$

The discrete time sequence of the above will be

$$O(nT) = m_1(a_1)^{n-1} + m_2(a_2)^{n-1} + \ldots + m_K(a_K)^{n-1}$$

where $n \geq 1$.

Now, from the above, it is very clear that if $|aj| < 1$ for $j = 1, 2, \dots n$, then the system response decays to zero. Thus it can be concluded that the poles of the system $\mathcal{Z}$-transfer function will be within the unit circle.

11.9.1 Methods of Stability Analysis of Discrete Systems

Jury's stability test

The procedure of Jury's stability analysis is as follows:

The characteristic polynomial needs to be found out first of all. That is,

$$F(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 = 0, \quad a_n > 0$$

The first test for Jury's stability test will be the test for necessity. If the necessity test is satisfied, then the sufficient condition tests need to be performed.

The necessary conditions for stability are the following:

$$F(1) > 0; \quad (-1)^n F(-1) > 0$$

If the above conditions are satisfied, then the sufficient conditions are tested by two methods.

Description of the first method for sufficient condition

Develop a table of coefficients of the characteristic polynomial as follows:

Row	z^0	z^1	z^2	...	z^{n-k}	...	z^{n-1}	z^n
1	a_0	a_1	a_2	...	a_{n-k}	...	a_{n-1}	a_n
2	a_n	a_{n-1}	a_{n-2}	...	a_k	...	a_1	a_0
3	b_0	b_1	b_2	...	...	...	b_{n-1}	
4	b_{n-1}	b_{n-2}	...	...	...	...	b_0	
5	c_0	c_1	c_2	...	...	...	c_{n-2}	
6	c_{n-2}	c_{n-3}	...	...	...	...	c_0	
...	...	...	...					
...	...	...	...					
$2n-5$	...	...						
$2n-4$	...	...						
$2n-3$	p_0	p_1	p_2					

where

$$b_k = \begin{vmatrix} a_0 & a_{n-k} \\ a_n & a_k \end{vmatrix}$$

Hence P_1 is positive innerwise.
Also,

$$P_2 = M - N$$

$$= \begin{bmatrix} -8 & 1 & 11 \\ -5 & \boxed{3} & 6 \\ -1 & 0 & 4 \end{bmatrix}$$

Hence 3 > 0 and

$$\begin{bmatrix} -8 & 1 & 11 \\ -5 & 3 & 6 \\ -1 & 0 & 4 \end{bmatrix} = -1(6 - 33) + 4[-8(3) + 5] = -49 < 0$$

Thus, P_2 is not positive innerwise; hence the system is unstable.

11.9.2 Analysis of Stability with the Help of Bilinear Transformation

We have just studied that to determine the stability in the z-domain, the roots of the characteristic equation should lie within the unit circle. Of course, the standard method of Routh and Nyquist criteria can also be applied if a complex transformation can be made so that the interior of the unit circle of the z-plane can be transferred to the left-half of a new plane. This transformation is termed the *bilinear transformation*. If the new plane is termed the r-plane, then

$$r = \frac{z-1}{z+1}$$

or

$$z = \frac{1+r}{1-r}$$

With $z = e^{j\theta}$ (where θ is varying from $-\pi$ to $+\pi$ via 0 in the anticlockwise direction), we have

$$r = \frac{e^{j\theta} - 1}{e^{j\theta} + 1}$$

$$= \frac{\cos\theta + j\sin\theta - 1}{\cos\theta + j\sin\theta + 1}$$

$$= \frac{\cos\theta - 1 + j\,2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2} + j\,2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$$

$$= \frac{-2\sin^2\frac{\theta}{2} + j\,2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + j\sin\frac{\theta}{2}\right)}$$

$$= \frac{2\sin\frac{\theta}{2}\left(-\sin\frac{\theta}{2} + j\cos\frac{\theta}{2}\right)}{2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + j\sin\frac{\theta}{2}\right)}$$

$$= j\tan\frac{\theta}{2}\frac{\left(-\sin\frac{\theta}{2} + j\cos\frac{\theta}{2}\right)}{\left(-\sin\frac{\theta}{2} + j\cos\frac{\theta}{2}\right)}$$

$$= j\tan\frac{\theta}{2}$$

When

$$\theta = -\pi$$

$$r = -j\tan\frac{\pi}{2} = -j\infty$$

When

$$\theta = 0$$

$$r = 0$$

When

$$\theta = +\pi$$

$$r = j\tan\frac{\pi}{2} = j\infty$$

Hence r varies from $-\infty$ to ∞ via 0 when θ varies from $-\pi$ to π via 0°. The shaded portions of Figs. 11.39 and 11.40 explain this.

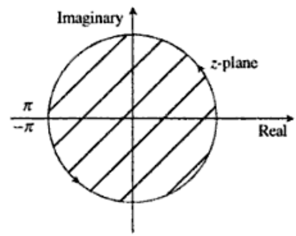


Fig. 11.39

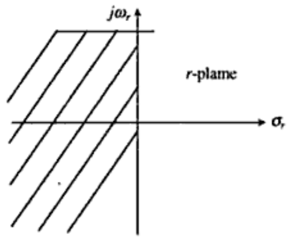


Fig. 11.40

Therefore the characteristic equation in the z-domain is to be determined, first of all. Say, that is

$$az^3 + bz^2 + cz + d = 0$$

or

$$\left(\frac{1+r}{1-r}\right)^3 - 0.5\left(\frac{1+r}{1-r}\right)^2 + 2.49\left(\frac{1+r}{1-r}\right) - 0.496 = 0$$

or

$$3.5r^3 - 2.5r^2 + 0.5r + 2.5 = 0$$

The changes of the sign in the characteristic equation indicate that the system is unstable. Even if we apply the Routh's criterion, the same result will appear. Thus,

r^3	3.5	0.5
r^2	–2.5	2.5
r	4	0
r^0	2.5	

The above Routh's criterion clearly indicates that the system is unstable. Whereas, if we check the stability as per a linear continuous system, we will find that the system is stable. The characteristic equation of the linear continuous system is

$$1 + G(s) = 0$$

or

$$1 + \frac{5}{s(s+1)(s+2)} = 0$$

or

$$s^3 + 3s^2 + 2s + 5 = 0$$

Applying the Routh's criterion, we get

s^3	1	2
s^2	3	5
s^1	1/3	0
s^0	5	

Hence the system is stable.

Thus, it is proved that in some cases, if sampling and ZOH are applied to a stable linear continuous system, the sampled system may be stable or may not be stable. That is why, there is immense importance of bilinear transformation to check the stability of the sampled system by Routh's criterion.

11.9.3 Schürcohn Stability Test

Schürcohn stability test is another method of testing the of stability of discrete systems. Suppose the transfer function in the z-domain is

$$G(z) = \frac{1}{1 - \frac{7}{4}z^{-1} - \frac{1}{2}z^{-2}}$$

Now, we have to study whether the system is stable or not. Let

$$A_2(z) = 1 - \frac{7}{4}z^{-1} - \frac{1}{2}z^{-2}$$

Let the coefficient of z^{-2} be assumed K_2; then $K_2 = -(1/2)$. We have to develop the reverse polynomial of the above, that is,

$$B_2(z) = -\frac{1}{2} - \frac{7}{4}z^{-1} + z^{-2}$$

Now,

$$A_1(z) = \frac{A_2(z) - K_2 B_2(z)}{1 - K_2^2}$$

$$= \frac{1 - \frac{7}{4}z^{-1} - \frac{1}{2}z^{-2} + \frac{1}{2}\left(-\frac{1}{2} - \frac{7}{4}z^{-1} + z^{-2}\right)}{1 - \left(-\frac{1}{2}\right)^2}$$

$$= 1 - \frac{7}{2}z^{-1}$$

The coefficient of z^{-1} is assumed K_1. Hence $K_1 = -(7/2)$.

Since $|K_1| > 1$, the system will be unstable. That means the magnitude of the coefficient of the highest inverse power in z in all the polynomial arrangements will be less than one, if the system is stable as a discrete system. Here $|K_2| < 1$ because $K_2 = -(1/2)$, but $|K_1| > 1$, because $K_1 = -(7/2)$.

That is why the system is unstable. This in a nut shell is the Schürcohn stability test for discrete systems. The generalized form of the Schürcohn stability test will be follows.

If the denominator polynomial of a transfer function is expressed as

$$A_m(z) = \sum_{K=0}^{m} a_m(K) z^{-K} \qquad a_m(0) = 1$$

then $B_m(z)$, i.e. the reverse polynomial $B_m(z)$ will be

$$B_m(z) = z^{-m} A_m(z^{-1})$$

$$= z^{-m} \sum_{K=0}^{m} a_m(K) z^{K}$$

$$= \sum_{K=0}^{m} a_m(K) z^{(-m+K)}$$

$$= \sum_{K=0}^{m} a_m(K) z^{-(m-K)}$$

Let $\quad m - K = n, \quad$ then $\quad K = m - n$

When $\quad K = 0, \quad n = m$

When $\quad K = m, \quad n = 0$

Thus,

$$B_m(z) = \sum_{n=m}^{0} a_m(m-n)z^{-n}$$

$$= \sum_{n=0}^{m} a_m(m-n)z^{-n}$$

Since n is an arbitrary number, n can also be expressed as K. Thus

$$B_m(z) = \sum_{K=0}^{m} a_m(m-K)z^{-K}$$

Now, in the Schürcohn stability test, the polynomial $A(z)$ should have all its roots inside the unit circle and for this we have to compute a set of coefficients termed ***reflection coefficients***, K_1, K_2, ..., K_N, from the polynomials $A_m(z)$.

For lower degree polynomials, $A_m(z)$, $m = N, N-1, N-2, ..., 1$, then according to the recursive equation

$$A_{m-1}(z) = \frac{A_m(z) - K_m B_m(z)}{1 - K_m^2}$$

where the coefficients K_m are expressed as

$$K_m = a_m(m)$$

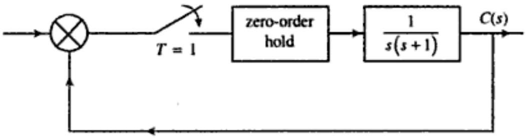
The Schürcohn stability finally narrates that the polynomial $A(z)$ will have all the roots inside the unit circle if and only if the coefficients K_m satisfy the condition $|K_m| < 1$ for all $m = 1, 2, ..., N$.

SUMMARY

The constituents of a sampled data control system such as sampler, analog to digital converter, digital computer, digital to analog converter, hold circuit, and plant are explained. The sampling procedure is described in detail. The signal reconstruction procedure is also explained. The idea of linear discrete systems is also provided. The equivalent representation of pulse sampler and zero-order hold is shown. The impulse sampling is explained. The Laplace and $\mathcal{Z}$-transform pairs corresponding to several time functions are tabulated. Application of the $\mathcal{Z}$-transform to sampled data systems is shown. Illustrative examples are also provided. Stability analysis in the discrete system is explained. The Jury's stability test is explained with examples. Analysis of stability with the help of bilinear transformation is discussed. Schürcohn-stability test is also described with an example.

QUESTIONS

1. Explain sample and hold. Obtain the frequency response of zero-order hold.
2. Obtain the unit-step response of the system shown below.



3. Define the stability of discrete control systems. Explain the Jury's test of stability.
4. Solve the following difference equation using the $\mathcal{Z}$-transform method.

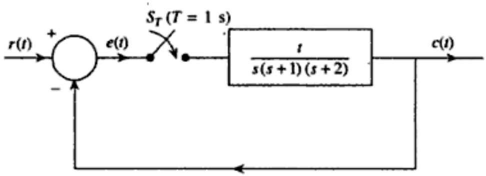
$$x(K+2) + 3x(K+1) + 2x(K) = 0$$

$$x(0) = 0, \qquad x(1) = 1.$$

5. Find the $\mathcal{Z}$-transform of $f(t)$ when

$$f(t) = \begin{cases} 0 \text{ for } t < 0 \\ \sin \omega t \text{ for } t \geq 0 \end{cases}$$

6. Find the characteristic equation of z-domain for the sample data system shown in the following figure and state whether the system is stable or not.



Chapter

12 Control System Devices

12.1 INTRODUCTION

Control system devices which make changes from one form to another are termed *transducers*. Actually, the control system devices convert process variables in one form into variables in another form. Suppose a potentiometer converts the angular position of a shaft to an output voltage, then the potentiometer will be termed transducer. Some of the devices that work as transducers are listed below:

(a) Potentiometer
(b) Synchro
(c) Differential transformer
(d) DC servomotor
(e) Tachogenerator
(f) Gyroscope
(g) Power amplifier
(h) Magnetic amplifier
(i) Stepper motor

12.2 POTENTIOMETERS

Potentiometers can be used as transducers for converting displacement or angular rotation into an output voltage. Figures 12.1 and 12.2 show the examples of linear potentiometers where linear displacement or angular position of a shaft is converted to a proportional voltage.

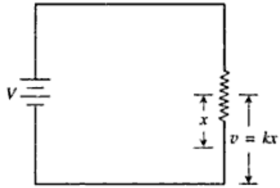


Fig. 12.1 Conversion of linear displacement to voltage.

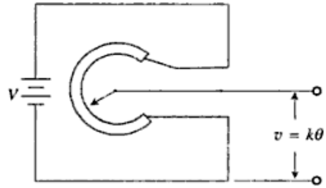


Fig. 12.2 Conversion of angular displacement to voltage.

Wire-wound potentiometer

Wire-wound potentiometers are used quite frequently in control systems. Here the wiper of the potentiometer moves from the top of one turn to the top of the consecutive turn by jumping. As a result the voltage does not change linearly, rather it provides a staircase type change as shown in Figs. 12.3 and 12.4. Thus, the output remains insensitive to variations of the wiper displacement between two consecutive steps. The measurement normally used is termed *resolution*. The resolution is defined as the minimum change ΔE in the output voltage in per cent of the total voltage found by rotating the shaft. For example, if there are N turns in the wire-wound potentiometer, then according to this definition,

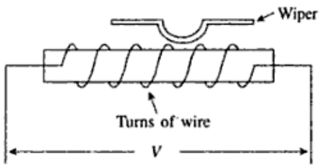


Fig. 12.3 Wire-wound potentiometer.

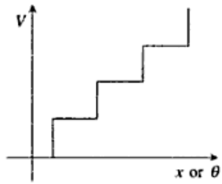


Fig. 12.4 Staircase type change in voltage.

$$\text{Resolution} = \frac{\Delta E}{E} \times 100 = \frac{\frac{E}{N}}{E} \times 100 = \frac{100}{N} \text{ per cent}$$

For wire-wound potentiometers, the resolution obtainable generally lies between 0.001 to 0.5 per cent. Linear wire-wound potentiometers are also used as error sensors. Figure 12.5 shows an arrangement of the error sensing transducer.

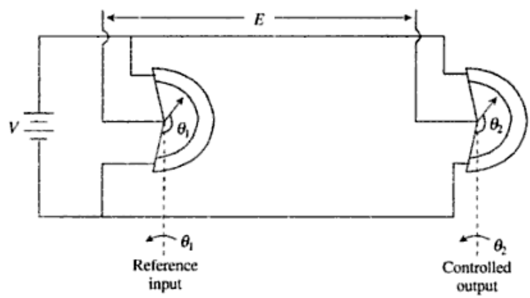


Fig. 12.5 Error sensing transducer.

Here the two potentiometers are arranged in a differential or bridge form. The output voltage E corresponds to the difference between the two angular input positions θ_1 and θ_2 at the wiper shafts. The error voltage E is, therefore, given by of laminated silicon steel and is slotted for accommodating a balanced three-phase winding which is generally of concentric coil type and is star connected. The rotor is of dumb-bell construction and is wound with a concentric coil. The ac voltage is applied to the rotor winding through the slip rings.

Figure 12.7 shows the constructional features of the synchro. The rotor is laminated. The brushes for the sliprings are metallic. The rotor may be of non-salient type structure, too, depending on the requirement.

The synchros are generally used for:

(a) Data transmission
(b) Error detection

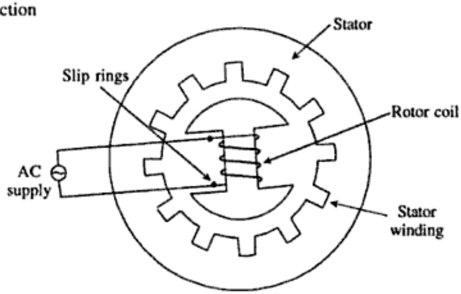


Fig. 12.7 Constructional features of synchro.

Synchros for data transmission

For data transmission, two synchros are used. Figure 12.8 depicts the same.

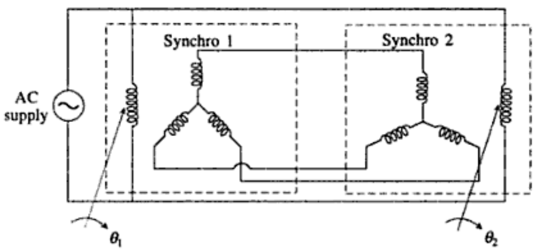


Fig. 12.8 Synchros for data transmission.

The synchro 1 is termed the *transmitter*. The synchro 2 is termed the *repeater*. The angular displacement of the rotor of one of the synchros produces an equal angular displacement of the rotor of the other synchro.

This type of synchronization takes place within one revolution. Such a system is termed the direct data transmission type synchro system. When ac supply is provided to the rotor of the first synchro, a pulsating magnetic field is developed. A voltage is induced in the stator coil of synchro 1 (transmitter) and current flows through the stator winding of synchro 2 since the stator windings of synchro 1 are connected with the stator windings of synchro 2. Again, the rotor winding of synchro 2 is connected to the ac supply. The flux developed by the currents flowing through the stator windings of synhro 2, will be in the same axis of the reference frame of synchro 2 as it is in synchro 1, but only in the reverse direction since the currents in the stator windings of both the synchros are equal. Thus the reference flux direction of synchro 2 is fixed. If there is misalignment of the rotor of synchro 2, then the flux developed by the rotor winding of synchro 2 due to the current flow in the rotor on account of the same ac supply, will not be in the reference frame. Hence a torque will be developed for proper alignment of the rotor of synchro 2. Thus any misalignment will be immediately rectified. For the data transmission, usually the rotors of both the transmitter and repeater are made salient. For preventing oscillations of the repeater, additional damping is also provided. Figure 12.9 describes the above scheme.

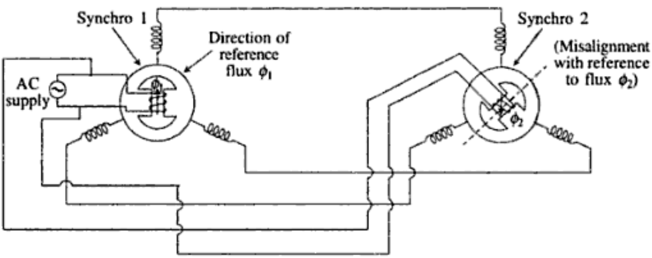


Fig. 12.9 *Direct data transmission type synchro system.*

Synchros for error detection

Figure 12.10 describes the synchro error detector. Here synchro 1 is termed the *synchro transmitter* and synchro 2 is termed the *control transformer*. When ac supply is connected to the rotor of synchro1, of voltage will be induced in the stator windings.

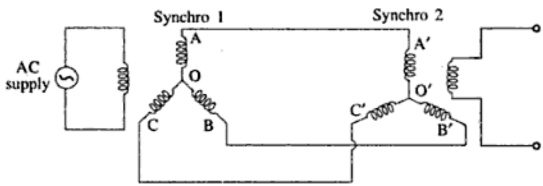


Fig. 12.10 *Synchros for error detection.*

Figure 12.16 shows an example of a differential transformer. With the movement of the core, the reluctance of the magnetic circuit will vary. This is the clear case of variable reluctance transformer.

Figure 12.17 shows another method of varying the reluctance of the differential transformer.

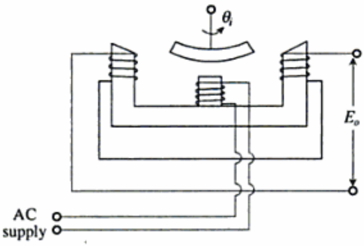


Fig. 12.16 *Differential transformer as variable reluctance transformer.*

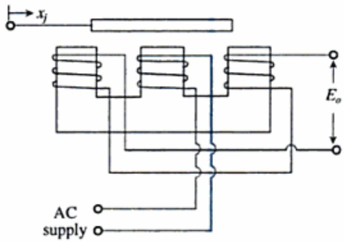


Fig. 12.17 *Variable reluctance transformer.*

It is also possible to develop a modulated wave at the output using a differential transformer if a high frequency carrier waveform is supplied to the primary, and the movable core is made to oscillate about the null position in a sinusoidal manner. The graphical representations are shown in Fig. 12.18.

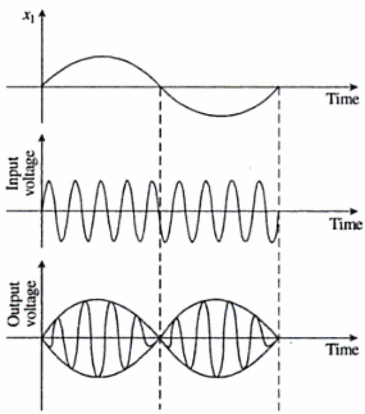


Fig. 12.18 *Modulated wave at the output of a variable reluctance transformer.*

Microsyn

This is a rotary differential transformer. Here both the stator and the rotor are made of iron laminations.

Figure 12.19 describes the mircosyn. Usually, there are four poles, each of which is provided with a coil for the primary winding and the other for the secondary winding. The rotor is made of a special shape as shown in Fig. 12.19. As a result, the reluctance of the magnetic circuit can be varied. The primary coils are connected in such a manner that for a particular direction of the primary current, the flux in poles A and B is inwards and that in poles C and D is outwards. The secondary coils are so arranged that the voltages induced in the secondary coils of poles A and C are in phase and oppose the voltages induced in the secondary coils of poles B and D. At the neutral position of the rotor, no output voltage will be produced. If the displacement of the rotor takes place from its neutral position in the clockwise direction, the reluctance in the poles A and C will increase while that in B and D will decrease. The outcome will be the development of voltage across the secondary terminals and that voltage will be proportional to the small angular displacement of the rotor. The range of rotation is generally limited to 10° to have a linear characteristic. When the angular displacement is transformed to voltage signal, the transducer is termed *microsyn*. In some cases, the change in alignment of the magnetic field is made for the difference in currents in the control winding. The device is then termed *torque microsyn*.

Figure 12.20 shows the schematic diagram of the torque microsyn.

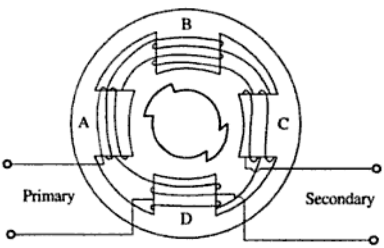


Fig. 12.19 Microsyn.

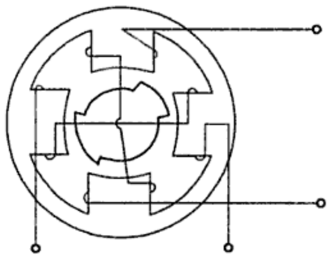


Fig. 12.20 Torque microsyn.

12.4 SERVOMOTORS

The power devices which are generally utilized in electrical control systems are ac and dc servomotors. Servomotors are constructed covering a wide range of power from fraction of a watt to kilowatts. The basic requirements of the control motors are as follows:

(a) The moment of inertia of the rotor should be small.

(b) The slope of the speed-torque curve should be negative and also not vary over the entire range of voltage regulation.

(c) Frequent starting operations should be withstood by the motor.

or

$$\frac{\theta}{V} = \frac{K_T}{s r_f \left(1 + s\frac{L_f}{r_f}\right) F\left(1 + \frac{sJ}{F}\right)}$$

or

$$\frac{\theta}{V} = \frac{K_T}{r_f \cdot Fs(1 + s\tau_f)(1 + s\tau_m)}$$

where $$\tau_f = \frac{L_f}{r_f} \quad \text{and} \quad \tau_m = \frac{J}{F}$$

Thus,

$$\frac{\theta}{V} = \frac{K'}{s(\tau_m s + 1)(\tau_f s + 1)} \qquad \left(\text{where } K' = \frac{K_T}{F r_f}\right)$$

Figure 12.22 describes the block diagram representation of a field-controlled dc servomotor.

Figure 12.23 describes the armature-controlled dc servomotor. Here, the field is excited by a constant field current I_f and the input voltage is V. The following assumptions are made.

(a) The field current is constant.
(b) The armature reaction is eliminated and the developed torque is expressed as, $T_d = KI_a$.
(c) The armature inductance is neglected.

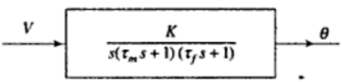


Fig. 12.22 Block diagram of a field-controlled dc servomotor.

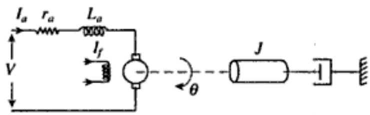


Fig. 12.23 Armature-controlled dc servomotor.

The back emf developed due to rotation of the armature is $E_b(s) = K_b s\theta(s)$, where K_b is the constant of proportionality. Thus,

$$V(s) = r_a I_a(s) + K_b s\theta(s)$$

Again,

$$T_d = J\frac{d^2\theta}{dt^2} + F\frac{d\theta}{dt}$$

Taking the Laplace transform of the above,

$$T_d(s) = Js^2\theta(s) + Fs\theta(s)$$

$$= (Js^2 + Fs)\theta(s)$$

One of the input windings is excited by a fixed line voltage V_r and is termed the reference winding. The second winding is placed in quadrature with the first one and it is connected to the control voltage from the source *via* a servo amplifier. The reference winding is usually fed with a voltage having an approximate phase shift of 90° with respect to the control voltage.

The voltages applied to the windings are not balanced. The direction of rotation of the motor reverses when the control phase voltage and hence the voltage signal input of the servo amplifier changes sign.

Figure 12.26 compares the speed-torque characteristics of the conventional induction motor with the ac servomotor. Curves 1 and 2 are the speed-torque characteristics of the conventional induction motors. Curves 3 and 4 are the speed-torque characteristics of induction motors for servo operation. The *X*/*R* ratio of servomotor is usually small compared to that of the conventional induction motor. Curve 1 is the speed-torque characteristic of a conventional induction motor and curve 2 is the speed-torque characteristic of a conventional induction motor at reduced voltage. Now for the conventional motor, the stable operation can only occur when the speed lies between ω_1 to ω. For stability, it is essential that $\delta T/\delta\omega$ should be negative for all control voltages. Hence, to increase the stable zone, it is essential to construct a motor having *X*/*R* ratio that is small enough for utilization as an ac servomotor.

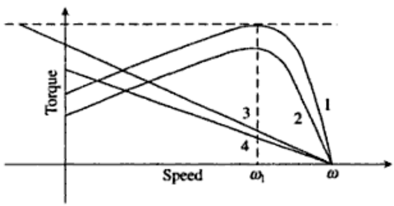


***Fig. 12.26** Speed-torque characteristics of induction motor.*

Figure 12.27 describes the speed-torque characteristic curves of an ac two-phase servomotor for various control voltages. The negative sign of the control voltage indicates the characteristic of the control signal with reversed phase. The curves are usually nonlinear. The general equation for the torque developed in a two-phase motor can be obtained from linear analysis by approximating the characteristic curve into a linear form. The following assumptions are therefore made:

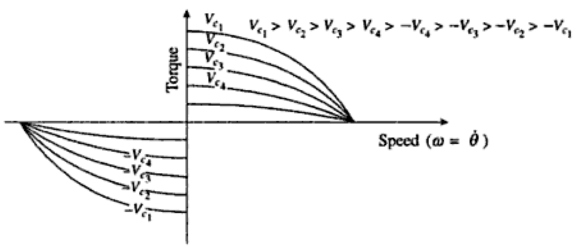


***Fig. 12.27** Speed-torque characteristics of an ac two-phase servomotor.*

or

$$\frac{\theta(s)}{V_c(s)} = \frac{K_2}{Js^2 + Fs - K_1 s}$$

$$= \frac{K_2}{Js^2 + s(F - K_1)} = \frac{K_2}{s(Js + F - K_1)}$$

or

$$= \frac{K_2}{(F - K_1)s\left(\frac{J}{F - K_1}s + 1\right)} = \frac{K}{s(\tau_m s + 1)}$$

where

$$K = \frac{K_2}{F - K_1} \quad \text{and} \quad \tau_m = \frac{J}{F - K_1}$$

Since $\frac{\delta T_d}{\delta \dot{\theta}} = K_1$ is negative inherently for stable operation, this in turn, develops additional damping. This additional damping is known as the *electrical damping*. When K_1 is positive, then $F - K_1$ will be negative for some K_1 tending to make the operation unstable.

The inertia of a squirrel cage motor is made small by providing the drag cup rotor since the servomotor needs to have very small inertia. Here, a metallic cup is used as the rotor as shown in Fig. 12.29. A stationary iron core, like a plug inside the cup, completes the magnetic circuit.

The principal disadvantage of the two-phase control motor is the inherent inefficiency of a squirrel cage induction motor running at a large slip.

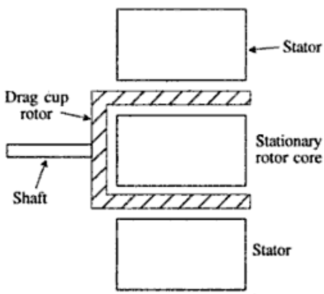


Fig. 12.29 Servomotor with drag cup rotor.

12.5 TACHOGENERATORS

The purpose of the tachogenerator is to transform the mechanical angular speed into a directly dependent voltage signal. Tachogenerators are used for instrumentation in the control process, for computers, and for many other purposes. The operating principle of the tachogenerator is that its magnetic flux is constant and the emf induced is proportional to the angular speed.

DC tachogenerator

A dc tachogenerator is a separately excited machine. Sometimes, the permanent magnet is also used. The demerit of the permanent magnetic field is that it deteriorates with aging, mechanical jolts, and shocks. If magnetic materials such as Alnico, Magnico are used, the aging can be avoided to a great extent.

The design of the armature of a dc tachogenerator is similar to that of a conventional dc machine. The generators are usually provided with more than two poles (even number) for a smooth output. A low-pass filter is sometimes connected to reduce high frequency noise. For reducing the voltage drop across the brushes, metal brushes with silver tips are generallty used. Figure 12.30 shows a dc tachogenerator with permanent magnet field.

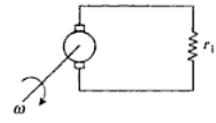


Fig. 12.30 DC tachogenerator.

AC tachogenerator

In ac tachogenerators, provision of three-phase windings is the usual practice. The number of poles are generally more than two. The output of the tachogenerator is connected to a three-phase bridge rectifier. The reasons for use of polyphase tachogenerators are the following:

(a) More power output per unit weight of the tachogenerator is obtained.
(b) Ripple content is decreased and the frequency is increased.

Figures 12.31 and 12.32 show the single-phase and the three-phase tachogenerator respectively.

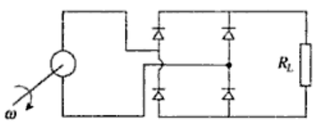


Fig. 12.31 Single-phase tachogenerator.

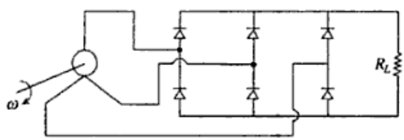


Fig. 12.32 Three-phase tachogenerator.

Figure 12.33 shows the ac tachometer. Here two stator field coils are mounted at quadrature. The tachometer rotor is usually a thin aluminium cup that rotates in an air-gap between a fixed magnetic structure. The light inertia rotor of highly conducting material provides a uniformly short-circuited secondary current. Suppose the voltage is applied to the reference coil and it is $V_r \cos \omega_c t$. The flux which will be produced on the coil is $\phi_r \sin \omega_c t$. Let the speed be expressed mathematically as

$$\dot{\theta}(t) = \dot{\theta}_m \cos K\omega_c t$$

Figure 12.36 shows the external characteristics of a cross-field generator for various degrees of compensation. Here, a indicates the fraction of the d-axis armature mmf compensated or neutralized.

$$\phi_d = K[N_f I_f - N_a I_d(1 - a)]$$

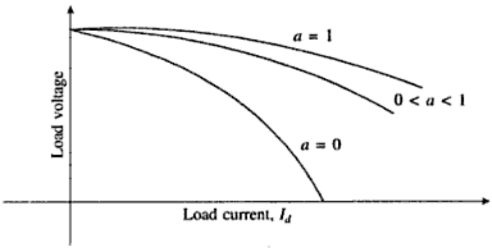


Fig. 12.36 External characteristics of the cross-field generator.

where ϕ_d is the flux in the direct axis. N_f and N_a are the effective turns of the control field coils and armature respectively. Therefore, finally, the voltage induced in the quadrature axis will be

$$e_q = K_f n I_f - (1 - a) K_q n I_d.$$

where n is the rpm of the machine. Thus,

$$I_q = \frac{K_f n I_f - (1-a) K_q n I_d}{R_q}$$

where R_q is the resistance of the quadrature axis.

Flux in the quadrature axis will be

$$\phi_q = K_2 N_a I_q$$

Therefore, induced emf

$$e_d = K_d n I_q$$

$$= K_d n \left[\frac{1}{R_q} \left\{ K_f n I_f - (1-a) K_q n I_d \right\} \right]$$

$$= \frac{K_d K_f}{R_q} n^2 I_f - \frac{K_d K_q}{R_q} (1-a) n^2 I_d$$

At no load, $I_d = 0$. Therefore,

$$e_{do} = \frac{K_d K_f}{R_q} n^2 I_f$$

$$= \frac{K_d K_f}{R_q} n^2 \frac{V_f}{R_f} \qquad \left(\because I_f = \frac{V_f}{R_f} \right)$$

Hence the voltage amplification is $\frac{K_d K_f n^2}{R_q R_f}$ at no load. Let the amplification on no-load be G, then

$$e_d = GV_f - K(1-a)I_d$$

where

$$K = \frac{K_d K_q}{R_q} n^2 \text{ and } G = \frac{K_f K_d}{R_q R_f} n^2$$

If the armature resistance drop is considered, then

$$V_d = e_d - R_a I_d$$

where R_a is the armature resistance including the resistance of the compensating winding. Thus,

$$\begin{aligned} V_d &= e_d - R_a I_d \\ &= GV_f - K(1-a)I_d - R_a I_d \\ &= GV_f - [R_a + K(1-a)]I_d \end{aligned}$$

If the load resistance is R_L, then $V_d = R_L I_d$. Thus,

$$V_d = R_L I_d = GV_f - \left[R_a + \frac{K_d K_q}{R_q} n^2 (1-a)\right] I_d$$

or

$$R_L I_d + I_d\left[R_a + \frac{K_d K_q}{R_q} n^2 (1-a)\right] = GV_f$$

or

$$I_d = \frac{GV_f}{R_L + R_a + \frac{K_d K_q n^2}{R_q}(1-a)}$$

$$= \frac{\frac{K_d K_f}{R_q R_f} n^2 V_f}{R_L + R_a + \frac{K_d K_q}{R_q} n^2 (1-a)}$$

$$= \frac{\frac{n^2}{R_q}\left(K_d K_f I_f\right)}{R_L + R_a + \frac{K_d K_q}{R_q} n^2 (1-a)}$$

$$= \frac{K_d K_f I_f}{\frac{R_q}{n^2}(R_L + R_a) + K_d K_q (1-a)}$$

Figure 12.37 shows the load current I_d vs. the speed characteristics of an uncompensated cross-field generator for various values of the load resistance, i.e. r_{L_1}, r_{L_2}, and r_{L_3}, where $r_{L_1} < r_{L_2} < r_{L_3}$ and the current I_f is kept constant.

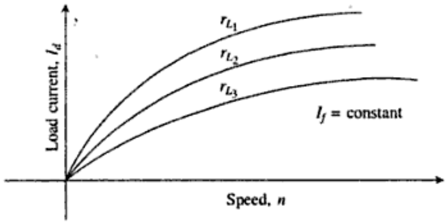


Fig. 12.37

Figure 12.38 shows the external characteristics of a cross-field generator for various degrees of compensation and at constant speed n.

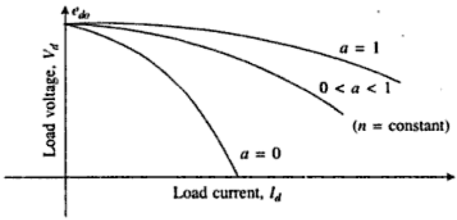


Fig. 12.38

12.7 MAGNETIC AMPLIFIERS

A magnetic amplifier is a device for obtaining power amplification. In this device a saturable reactor is used. Static magnetic amplifiers have a time constant ranging from a few tens of milliseconds for low power applications to a few seconds for higher powers. Magnetic amplifiers are used in industrial control drives and in airborne applications such as guided missiles.

The merits of a magnetic amplifier are the following:

(a) Its reliability is high.

(b) Its life is long.

(c) It has high overload capacity.

(d) Its impedance can be matched to a wider range of values and the mixing of different signals is easy, particularly on account of the ease with which such signals are isolated.

(e) Its gain is relatively high.

(f) The utility of the magnetic amplifier increases with increase in supply frequency since the voltage drop across the reactor depends directly on its frequency of operation.

Saturable core reactors

The fundamental element of the magnetic amplifier is the saturable reactor. It consists of a laminated core of some magnetic material. The performance of saturable reactors depends on the properties of the cores. It is imperative to design the hysteresis loop of the reactor core as narrow and steep as possible. In other words, it is necessary to keep the coercive H_C as small as possible.

Figure 12.39 shows the hysteresis loop. The value of H_C is of the order of 1 to 10 AT/metre. The retentivity B_r is very near to the saturation flux density.

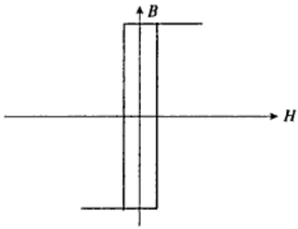


***Fig. 12.39** Hysteresis loop.*

Figure 12.40 shows the schematic diagram of a saturable core reactor. The iron core carries the following two windings as shown in this figure.

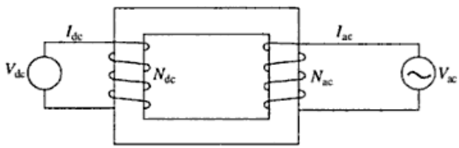


***Fig. 12.40** Saturable core reactor.*

(a) An ac winding of N_{ac} turns excited by an ac source voltage V_{ac}.

(b) A dc winding of N_{dc} turns excited by a dc source voltage V_{dc}.

The main objective of the dc supply is to develop a control mmf that will produce a flux, in order to vary the degree of saturation. The effective permeability decreases with a higher degree of saturation in a magnetic circuit.

From Fig. 12.41, it is very much clear that for a change in B in the saturation region, the change in H is very high in comparison to the similar changes in the linear region of the B-H curve. Since $\Delta B/\Delta H = \mu_r \mu_0$, with the increase in ΔH in the saturation region, the effective relative permeability μ_r decreases. The decrease in effective permeability means that the effective reactance of the winding decreases. The supply voltage v_{ac} as shown in Fig. 12.40 is also termed the *gate voltage*. Here,

$$v_{ac} = i_{ac}r + N_{ac}A \cdot \frac{dB}{dt}$$

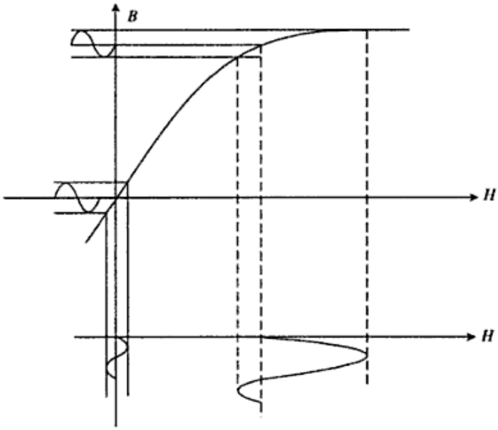


Fig. 12.41

where

r = resistance of the ac or gate winding
A = cross-sectional area of the magnetic core
B = flux density in the core

If r is small, then

$$v_{ac} = N_{ac}A \cdot \frac{dB}{dt}$$

Again, if the dc excitation is present, then

$$B = B_{dc} + B_{ac}$$

If the gate voltage $v_{ac} = v_m \sin \omega t$, then flux density B_{ac} will also be sinusoidal. Therefore,

$$B = B_{dc} + B_m \sin \omega t$$

and

$$v_{ac} = N_{ac} A \cdot \frac{d}{dt}\left(B_{dc} + B_m \sin \omega t\right)$$

$$= N_{ac} A\, B_m\, \omega \cos \omega t$$

The rms value of V_{ac} will be

$$= \frac{N_{ac} A\, B_m\, 2\pi f}{\sqrt{2}}$$

$$= \sqrt{2}\pi N_{\alpha\chi} A\, B_\mu f = 4.44 f\, N_{\alpha\chi} A B_\mu$$

Again, the maximum amplitude of the magnetizing force

$$H_m = \sqrt{2}\,\frac{N_{ac} I_{ac}}{l}$$

where

l = mean magnetic path in metres

I_{ac} = the rms current in the ac winding.

Therefore, equivalent reactance,

$$X_{eq} = \frac{V_{ac}}{I_{ac}} = \frac{\sqrt{2} \times 4.44 f\, N_{ac}^2 A}{l}\,\frac{B_m}{H_m}$$

$$= \frac{\sqrt{2} \times 4.44 f\, N_{ac}^2 A}{l}\,\mu_{eff} = \frac{6.28 f\, N_{ac}^2 A}{l}\,\mu_{eff}$$

where μ_{eff} is the effective permeability.

Demerits of the ordinary saturable core reactor

The saturable core reactor shown in Fig. 12.40 has the following disadvantages:

(a) A high voltage is induced in the control winding due to transformer action. Therefore, proper insulation is needed for protecting this winding from such overvoltage. This voltage may circulate current in the dc circuit and that is why, a high inductance is placed in series with the control winding. Automatically, the response of the control system will become slow.

(b) The harmonic content also becomes very high in the ac waveform as even harmonic distortion is present.

For the above reasons, the design and construction of the saturable core reactor are modified as in the case of magnetic amplifiers.

Figures 12.42 and 12.43 show how the design and construction of two saturable core reactors have been modified to act as magnetic amplifiers.

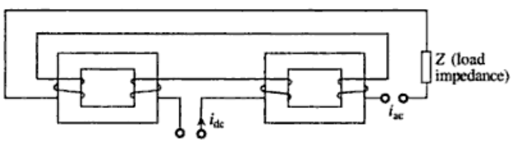


Fig. 12.42 Two saturable core reactors connected as a magnetic amplifier.

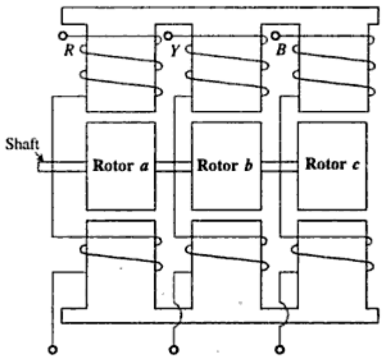


Fig. 12.48 Three-stack variable reluctance stepper motor.

The inductance of the stator winding can be expressed in two parts. One part is fixed and the other part depends on the angular movement of the rotor. For example, the phase a will have $L_a = L_0 + L \cos T\theta$ where T is the number of teeth and θ is the mechanical angle. Again, the torque is equal to the rate of change of energy. Therefore,

$$\text{Torque} = \frac{d}{d\theta}\frac{1}{2}L_a i_a^2 = \frac{1}{2}i_a^2\frac{dL_a}{d\theta}$$

$$= -\frac{1}{2}i_a^2 TL \sin T\theta$$

Figure 12.49 shows the relative angular displacement of the stator phases for the movement of the rotor. Suppose the positions of the stator phase a, phase b, phase c with respect to the rotor are as shown in the Fig. 12.49. That means the stator phase c teeth are totally aligned with the teeth of the rotor. Now, if a pulse is applied to phase a of stator, the teeth of the stator phase a will be aligned with the rotor teeth. It means that a progressive angular displacement of the rotor will take place which will be equal to $(360/KT)^\circ$ where K is the number of stacks and T is the number of teeth.

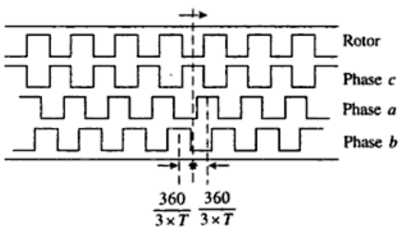


Fig. 12.49 Relative angular displacement of the stator phases with respect to the rotor.

The arrangement of the system is made such that the angular difference of the alignment of each phase will be (360/*KT*). Therefore, any pulse to any phase will shift that stack by an angle (360/*KT*) and align the system with the rotor and the corresponding phase. The motor will again stop. It means that any positional error will develop a pulse which will set right the alignment again. Similarly, for pulse excitation to the phase *b*, the rotor will shift 2 × 360/*KT* in the forward direction to have itself aligned with phase *b*. On account of the inertia of the driven mechanism, the motor will exhibit oscillatory behaviour at each step of movement and that is suppressed to an acceptable value by providing additional mechanical or electrical damping. In the case of aligned position, special precaution needs to be taken so that no load torque may develop on the motor shaft as that would create alignment error unnecessarily. The direction of movement will change (i.e. become anticlockwise) if the pulsing sequence is changed from *abc* to *acb*.

If the two phases are excited simultaneously, then a more powerful system will be developed. It means that the sequence will be *ab*, *bc*, *ca* instead of *abc*. In this case, the torque expression will be as follows.

$$L_a = L_0 + L\cos T\theta$$

$$L_b = L_0 + L\cos\left(T\theta - \frac{360}{KT}\right)$$

Therefore,

$$T_{ab} = \frac{1}{2}\frac{d(L_a + L_b)}{d\theta}i^2$$

$$= \frac{1}{2}\frac{d}{d\theta}\left[L\cos T\theta + L\cos\left(T\theta - \frac{360}{KT}\right) + 2L_0\right]i^2$$

$$= -\frac{1}{2}TLi^2\sin T\theta - \frac{1}{2}TLi^2\sin\left(T\theta - \frac{360}{KT}\right)$$

$$= -\frac{1}{2}TLi^2\left[\sin T\theta + \sin\left(T\theta - \frac{360}{KT}\right)\right]$$

$$= -\frac{1}{2}Li^2T\,2\sin\frac{T\theta + T\theta - \frac{360}{KT}}{2}\cos\frac{T\theta - T\theta + \frac{360}{KT}}{2}$$

$$= -Li^2\,T\sin\left(T\theta - \frac{180}{KT}\right)\cos\frac{180}{KT}$$

From the above expression, it is clear that the rotor movement will be 180/*KT*, that is, half of the angle, which was 360/*KT*, for *abc* or *acb* pulse sequence.

Figure 12.50 shows the characteristic of torque vs. pulse rate of stepper motor.

The slew range is that in which the load velocity follows the pulse rate without loosing steps, but cannot start, stop, or reverse on command. The start range is that in which the load position follows the pulses without loosing steps. The maximum torque point is the maximum holding torque of the excited motor to a steady load. As the stepping rate is increased, the motor provides less torque since the rotor has less time to drive the load from one position to the next because of the stator winding current pattern having been shifted.

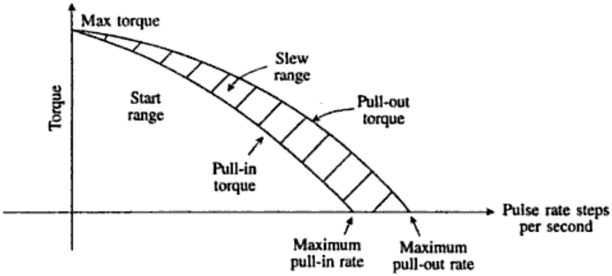


Fig. 12.50 Torque vs. pulse rate of stepper motor.

Permanent magnet stepper motor

Figures 12.51 and 12.52 show the two-phase four-pole permanent magnet stepper motor. Figure 12.51 is for phase *a* and Fig. 12.52 for phase *b*. The rotor is made of ferrite material. It is permanently magnetized. The stator stack of phase *b* is electrically at quadrature with that of phase *a*. When phase *a* is excited by a pulse, the rotor will be aligned with phase *a* as shown in Fig. 12.51. When the phase *b* is also excited, the effective stator poles will be shifted anticlockwise by an angle 221/2°. Obviously, the rotor will move 221/2° anticlockwise for aligning with the effective magnetic axis. If the excitation of phase *a* is withdrawn, the rotor will shift another 221/2° anticlockwise to align itself with the new effective magnetic axis which is the magnetic axis of the phase *b* only. Now, if the phase *a* is the reverse excited, then the rotor will move another 221/2° in the anticlockwise direction.

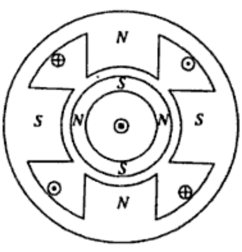


Fig. 12.51 Two-phase four-pole permanent magnet stepper motor—phase a.

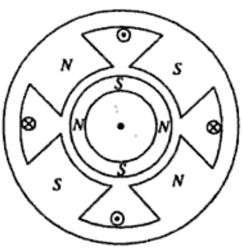


Fig. 12.52 Two-phase four-pole permanent stepper motor—phase b.

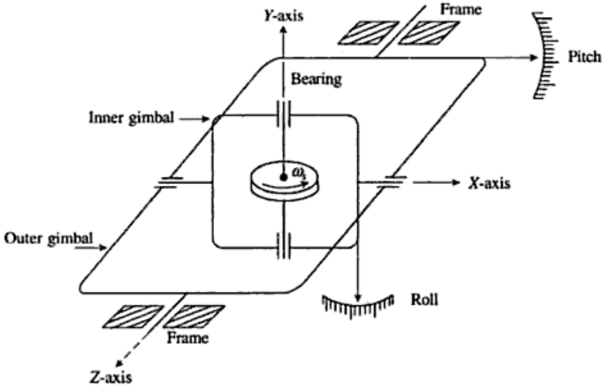


***Fig. 12.55** Vertical gyroscope*

Figure 12.56 shows the pitch, roll and yaw motions of the aeroplane.

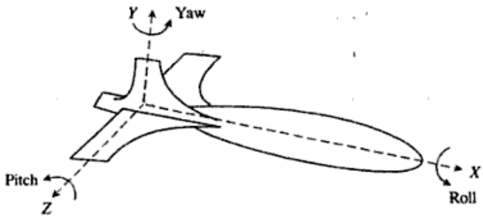


Fig. 12.56

In the gyroscope, the spinning wheel rotates in bearings located in the inner gimbal. The inner gimbal is free to rotate in bearings placed in the outer gimbal which in the long run can rotate in the frame to which the gyro is fixed. The gyro wheel is brought to spinning condition with the help of a synchronous motor whose rotor is mounted on the axis of the spinning wheel. When the spinning axis is in the horizontal position, the gyroscope is termed *directional gyro*. When the axis is in the vertical position, it is termed *vertical gyro*. Directional and vertical gyros are used to measure the three rotational motions of the aeroplane. Pitch is the rotational motion about the lateral or Z-axis, roll is the about the longitudinal or X-axis and yaw is the rotational motion about the normal or Y-axis. A vertical gyro measures the pitch and roll. A directional gyro measures yaw. If the electrical signals are needed, then pitch, roll and yaw scales are replaced by synchros.

Now to determine the torque, the torque needed for accelerating the wheel about the Z-axi is given by

$$T = -H\dot{\alpha} + I_1 \frac{d^2\beta}{dt^2} = -H\dot{\alpha} + I_1\ddot{\beta}$$

where β is the angle through which the wheel turns about the Z-axis.

An undesired disturbance torque may be present at the Y-axis, i.e. the output axis. The gyr response to this torque can be found independently as follows when the angular movements of th spin-axis are small. Thus,

$$\Delta H = T_1 \Delta t$$

$$\tan \Delta\beta = \frac{\Delta H}{H} = \frac{T_1}{H}\Delta t \qquad \text{(See Fig. 12.59)}$$

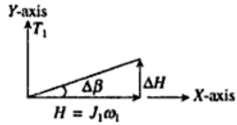


Fig. 12.59

Therefore,

$$\frac{d\beta}{dt} = \dot{\beta} = \frac{T_1}{H}$$

or

$$T_1 = H\dot{\beta}$$

With the moment of inertia, the above equation will turn to

$$T_1 = H\dot{\beta} + J\frac{d^2\alpha}{dt^2}$$

$$= H\dot{\beta} + J\ddot{\alpha}$$

where α is the angle through which the wheel moves about the Y-axis.

If damping and spring restoring torques are considered, then

$$T_1 = H\dot{\beta} + J\ddot{\alpha} + F\dot{\alpha} + K\alpha$$

where F is the damping coefficient and K is the spring constant. Therefore, finally we get two equations

$$T = -H\dot{\alpha} + I_1\ddot{\beta}$$

$$T_1 = H\dot{\beta} + J\ddot{\alpha} + F\dot{\alpha} + K\alpha$$

Taking the Laplace transform of the above equations, we get

$$T(s) = -sH\alpha(s) + s^2 I_1 \beta(s) \qquad (1)$$

$$= \frac{T(s)\left[\dfrac{Js^2 + Fs + K}{I_1}\right] + \dfrac{SH}{I_1} T_1(s)}{s^2\left[Js^2 + Fs + K + \dfrac{H^2}{I_1}\right]}$$

When the disturbing torque $T_1(s)$ does not exist, then

$$\alpha(s) = \frac{-\dfrac{H}{I_1} T(s)}{s\left(Js^2 + Fs + K + \dfrac{H^2}{I_1}\right)}$$

or

$$\frac{\alpha(s)}{T(s)} = \frac{-\dfrac{H}{I_1}}{s\left(Js^2 + Fs + K + \dfrac{H^2}{I_1}\right)}$$

Free Gyro

In the case of free gyro, the values of J, F, K and $T_1(s)$ will be zero because there will be no restraining force in any direction. Hence at the time of application of the input torque $T(s)$, the disturbance torque $T_1(s)$ will be zero. Therefore,

$$\alpha(s) = \frac{-\dfrac{H}{I_1} T(s)}{s\dfrac{H^2}{I_1}}$$

$$= -\frac{H}{I_1} T(s) \times \frac{I_1}{sH^2}$$

$$= -\frac{T(s)}{sH}$$

or

$$s\alpha(s) = -\frac{T(s)}{H}$$

or

$$\frac{d\alpha}{dt} = -\frac{T}{H}$$

Similarly, at the time of calculation of disturbance $T_1(s)$ at the output Y-axis, the input torque $T(s)$ will be zero. Over and above, for free gyro, the restraining torque will not be present, hence J, F and K will also be zero. Therefore, J, F, K and $T(s)$ are to be made zero in the preceding relation for $\beta(s)$. Thus,

$$\beta(s) = \frac{T(s)\left[\frac{Js^2 + Fs + K}{I_1}\right] + \frac{SH\,T_1(s)}{I_1}}{s^2\left[Js^2 + Fs + K + \frac{H^2}{I_1}\right]}$$

$$= \frac{\frac{SH}{I_1}T_1(s)}{s^2\frac{H^2}{I_1}}$$

$$= \frac{sH}{I_1}T_1(s) \times \frac{I_1}{s^2H^2}$$

$$= \frac{T_1(s)}{sH}$$

or

$$s\beta(s) = \frac{T_1(s)}{H}$$

or

$$\frac{d\beta}{dt} = \frac{T_1}{H}$$

Rate gyro

When the disturbance torque is zero and the spring constant is very large in the output axis, the gyro system is termed *rate gyro*. The rate gyro is used for the purpose of providing rate feedback from roll, pitch or yaw for damping out vehicle oscillations about these axes.

Therefore, $T_1(s)$, J and F are to be put zero in the mathematical expression of $T_1(s) = sH\beta(s) + (Js^2 + Fs + K)\alpha(s)$, i.e.

$$0 = sH\beta(s) + K\alpha(s)$$

or

$$sH\beta(s) = -K\alpha(s)$$

or

$$\alpha(s) = -\frac{H}{K}s\beta(s)$$

Therefore,

$$\alpha = \frac{-H}{K}\frac{d\beta}{dt}$$

Restrained gyro

The restrained gyro has a single degree of freedom with respect to the frame when only one axis is free to move. In this case, $T_1 = 0$ and K will be negligible in comparison to the damping coefficient F. Hence putting these in the expression of $T_1(s) = sH\beta(s) + (Js^2 + Fs + K)\alpha(s)$, we get

$$0 = sH\beta(s) + (Js^2 + Fs)\alpha(s)$$

or

$$(Js^2 + Fs)\alpha(s) = -sH\beta(s)$$

or

$$\alpha(s) = -\frac{sH\beta(s)}{s(Js + F)}$$

$$= -\frac{H\beta(s)}{F\left(\frac{J}{F}s + 1\right)}$$

Since F is very large, $\frac{J}{F} \to 0$. Therefore,

$$\alpha(s) = \frac{-H}{F}\beta(s)$$

or

$$\alpha = \frac{-H}{F}\beta$$

This gyro is also called the integrating gyro, since α is proportional to the integral of the rate of β.

Accuracy

The free gyro provides an inertial frame of relatively low accuracy. The rate gyro has low accuracy as well. The restrained or intergrating gyro is very rugged and extremely accurate.

SUMMARY

Potentiometers, synchros, differential transformers, servomotors, tachogenerators, magnetic amplifiers, stepper motors and gyroscopes are described in this Chapter.

The wire wound potentiometer is explained. The merits and demerits of potentiometers are also discussed. The use of synchro is presented both for data transmission and error detection. The operation of the differential synchro is also discussed. The principle of operation of the differential transformer is also explained. Microsyn is also covered. DC servomotor, ac servomotor are explained with diagrams. The principles of dc tachogenerator and ac tachogenerator are explained. The principles of metadyne and amplidyne are described with diagrams. Magnetic amplifier is described with saturable reactor. The operation of variable reluctance stepper motor and permanent magnet stepper motors is also explained.

The principle of operation of the gyroscope is explained. Pitch, roll and yaw are discussed with diagrams. The mathematical analysis of gyro is made. Free gyro, rate gyro, and restrained gyro are explained with mathematical deductions.

QUESTIONS

1. What are the merits and demerits of using the potentiometer as a control device?
2. Describe the principle of operation of synchro.
3. What is differential transformer? Explain its operation as a control device.
4. What are the uses of the servomotor?
5. Describe the principle of operation of the tachogenerator.
6. Explain the principle of operation of the rotating amplifier.
7. What is magnetic amplifier? What are the merits of using the magnetic amplifier as a control device?
8. Where do you use the stepper motor? Explain its principle of operation.
9. Describe the principle of operation of the gyroscope.
10. What are free gyro, rate gyro, and restrained gyro?

Chapter 13 Optimal Control Theory

13.1 INTRODUCTION

Before proceeding to know what the optimal control theory is, we will start from mathematics and gradually move towards the control system problem. Suppose two curves shown in Fig. 13.1 are respectively designated by $C + \Delta C = f(x + \Delta x, \Delta y)$ and $C = f(x, y)$.

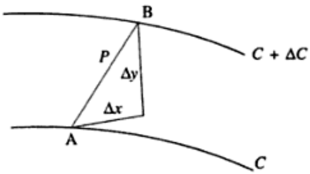


Fig. 13.1 Curves C and C + ΔC.

Now ΔC is the change in the value of the function for all small increments in x and y. Let AB, which is the shortest distance between the points A and B, be divided into components Δx and Δy along the coordinate axes.

Accordingly, $(P)^2 = (\Delta x)^2 + (\Delta y)^2$ should be minimum, where P is the magnitude of the vector AB. Now $\Delta C = f(x + \Delta x, y + \Delta y) - f(x, y) = (\delta C/\delta x)\Delta x + (\delta C/\delta y)\Delta y$ for very small value of ΔC, the steepest distance between the two curves may be taken to be a straight line. Now, $\delta C/\delta x$ and $\delta C/\delta y$ will remain constant according to the principle of calculus. If we want to move from C to $C + \Delta C$ by the shortest path, then $\delta p/\delta x = 0$ and $\delta p/\delta y = 0$, since P denotes the distance. Therefore,

$$(P)^2 = (\Delta x)^2 + (\Delta y)^2 \tag{13.1}$$

$$\Delta C = \frac{\delta C}{\delta x}\Delta x + \frac{\delta C}{\delta y}\Delta y \tag{13.2}$$

Thus,

$$P = \left[(\Delta x)^2 + \left\{ \frac{\Delta C - \dfrac{\delta C}{\delta x}\Delta x}{\dfrac{\delta C}{\delta y}} \right\}^2 \right]^{1/2}$$

or

$$P = \frac{\sqrt{(\Delta x)^2\left(\frac{\delta C}{\delta y}\right)^2 + \Delta C^2 - 2\frac{\delta C}{\delta x}\Delta x + \left(\frac{\delta C}{\delta x}\Delta x\right)^2}}{\left(\frac{\delta C}{\delta y}\right)^2}$$

Now, as per the theory already discussed, $\frac{\delta P}{\delta x} = 0.$

Therefore,

$$2(\Delta x)\left\{\left(\frac{\delta C}{\delta x}\right)^2 + \left(\frac{\delta C}{\delta y}\right)^2\right\} - 2\Delta C\frac{\delta C}{\delta x} = 0$$

or

$$2\Delta x\left\{\left(\frac{\delta C}{\delta x}\right)^2 + \left(\frac{\delta C}{\delta y}\right)^2\right\} = 2\Delta C\frac{\delta C}{\delta x}$$

Since $\frac{\delta C}{\delta x}$ and $\frac{\delta C}{\delta y}$ are constant, $\Delta x \propto \frac{\delta C}{\delta x}$.

Similarly, we can show $\Delta y \propto \frac{\delta C}{\delta y}$.

This is also called the gradient method. If the function C is to be minimized, then the procedure of optimization is also termed optimization by the steepest descent method. The function C to be optimized is also called the *objective function*.

13.2 OPTIMIZATION BY STEEPEST DESCENT METHOD

Suppose there are a large number of independent variables and C is the objective function of the independent variables $x_1, x_2, \ldots, x_n$, that is, the function is $C(x_1, x_2, \ldots, x_n)$, then

$$\Delta C = \frac{\delta C}{\delta x_1}\Delta x_1 + \frac{\delta C}{\delta x_2}\Delta x_2 + \cdots + \frac{\delta C}{\delta x_n}\Delta x_n \tag{13.3}$$

If the distance from the multidimensional surface C to the multidimensional surface $C + \Delta C$ is P, then

$$P^2 = \Delta x_1^2 + \Delta x_2^2 + \cdots + \Delta x_n^2 \tag{13.4}$$

Differentiating Eqs. (13.3) and (13.4), we get

$$d(\Delta C) = \frac{\delta C}{\delta x_1}d(\Delta x_1) + \frac{\delta C}{\delta x_2}d(\Delta x_2) + \cdots + \frac{\delta C}{\delta x_n}d(\Delta x_n) \tag{13.5}$$

and

$$2P\,dP = 2\Delta x_1 d(\Delta x_1) + 2\Delta x_2 d(\Delta x_2) + \cdots + 2\Delta x_n d(\Delta x_n) \tag{13.6}$$

As per the minimum-maximum theorem of calculus, when optimization is done,

$$dP = 0$$

or

$$2\Delta x_1 d(\Delta x_1) + 2\Delta x_2(\Delta x_2) + \cdots + 2(\Delta x_n)d(\Delta x_n) = 0 \tag{13.7}$$

Again ΔC being constant, $d(\Delta C) = 0$. Thus,

$$\frac{\delta C}{\delta x_1}d(\Delta x_1) + \frac{\delta C}{\delta x_2}d(\Delta x_2) + \cdots + \frac{\delta C}{\delta x_n}d(\Delta x_n) = 0 \tag{13.8}$$

From Eqs. (13.7) and (13.8), we get

$$\Delta x_1 \propto \frac{\delta C}{\delta x_1}, \quad \Delta x_2 \propto \frac{\delta C}{\delta x_2}, \ldots, \Delta x_n \propto \frac{\delta C}{\delta x_n}$$

Since $x_1, x_2, \ldots, x_n$ are independent variables, hence

$$\Delta x_n = K\frac{\delta C}{\delta x_n}$$

where $n = 1, 2, 3, \ldots, n$.

In the case of minimization,

$$\Delta x_n = -K\frac{\delta C}{\delta x_n} \tag{13.9}$$

where $n = 1, 2, \ldots, n$, because C will gradually reduce as $x_1, x_2, \ldots, x_n$ increase.

Since the independent variables are being changed in proportion to the gradient of the objective function with respect to the said independent variables, the technique of optimization is termed *gradient technique*. Over and above, the steepest path is followed for optimization and that is why it is termed the *steepest descent method*. The principle of optimization by the gradient method can be understood graphically by referring to Fig. 13.2.

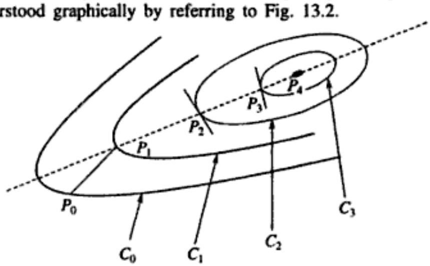


Fig. 13.2 ***Steepest descent method.***

Here C_0 is the value of the function to be optimized at the starting point P_0. In the first step, the gradients of the function at the point P_0 with respect to the independent variables are determined. Then the independent variables are changed according to Eq. (13.9) and a new point P_1 is obtained where P_0P_1 shows the direction of the gradient. The value of the function at this point P_1 is C_1. In the next step, the gradient at P_1 is determined and similarly a new point P_2 is obtained. This procedure is continued till the function is minimized. In principle, the steepest descent method will not reach an optimum point in a finite number of steps. The steps are progressively shortened through each iteration.

Partical approach of gradient method

When the values of $\delta C/\delta x_1$, $\delta C/\delta x_2$, ..., $\delta C/\delta x_n$ are widely varying, it is not wise to take the value of the constant of proportionality the same for all the independent variables although the mathematical analysis desires that. We have to slightly deviate from the principle of ideal mathematics for obtaining the *global optimum point* (or *ideal optimum point*) quickly. We have to determine all the gradients, $\delta C/\delta x_1$, $\delta C/\delta x_2$, ..., $\delta C/\delta x_n$; then according to the absolute values of the gradients, we choose the values of K. In this case, the values of K are different from one another, but to provide proper weightage to each variable, we make the deviation by not accepting the minimum-most value of K for all the independent variables. Of course, now, there is every probability to reach local optimum instead of global optimum. But, the same cane be taken care of when the local optimum is achieved. Once the local optimum is obtained, the values of K are taken equal to the minimum-most. Then the same procedure of optimization is followed to approach towards the global optimum.

When optimization is being done on a system, then the zone at which the optimization is to be made is also specified and that is termed *feasible zone*. The unspecified zone is termed *infeasible zone*. The feasible zone is identified by the constraints. The constraints are broadly classified into two types.

(a) Equality constraint
(b) Inequality constraint

Suppose a constraint T, a function of x_1, x_2, and x_3 is given by, $T = 5x_1 + 3x_2 + 4x_3$, then T is called the equality constraint. But, if $T \geq 5x_1 + 3x_2 + 4x_3$, then the constraint T is called the inequality constraint.

13.3 OPTIMIZATION WITH CONSTRAINT BY GRADIENT METHOD

Equation (13.9) shows the method of optimization of a function C where there is no imposition of any constraint. When there exists any constraint having certain limiting value, the method is to be modified. Let $T(x_1, x_2, \ldots, x_n)$ be a constraint whose value does not cross a limiting magnitude T_{limit} or in other words, the value of T is not allowed to increase after it reaches T_{limit}. This means that ΔT is to be made zero. That is,

$$\Delta T = \frac{\delta T}{\delta x_1}\Delta x_1 + \frac{\delta T}{\delta x_2}\Delta x_2 + \cdots + \frac{\delta T}{\delta x_n}\Delta x_n = 0$$

If there are six independent variables, then the equation will take the form

$$\frac{\delta T}{\delta x_1}\Delta x_1 + \frac{\delta T}{\delta x_2}\Delta x_2 + \cdots + \frac{\delta T}{\delta x_6}\Delta x_6 = 0$$

or

$$\Delta x_1 = \frac{-\left(\dfrac{\delta T}{\delta x_2}\Delta x_2 + \cdots + \dfrac{\delta T}{\delta x_6}\Delta x_6\right)}{\dfrac{\delta T}{\delta x_1}}$$

Thus x_1 is now changed to a dependent variable because the change in the variable x_1 depend on the values of x_2, x_3, x_4, x_5, and x_6 when T becomes a constant. The general equation will take the form

(b) The remaining elements in the resolvement column (the sth column) remain unchanged.
(c) The remaining elements in the resolvement row (the rth row) change sign.
(d) The "ordinary elements" $b_{ij}(i \neq r, j \neq s)$ (i.e. the elements that do not belong either to the resolvement row or to the column) are given by the formula

$$b_{ij} = a_{ij}a_{rs} - a_{is}a_{rj}$$

All the entries in the new matrix are divided by the element a_{rs}.

Proof of the elimination technique

It will be shown that the marices (13.10) and (13.11) are identical.

From matrix (13.10), it is found that

$$y_1 = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1s}x_s + \cdots + a_{1n}x_n \quad (13.12)$$

From matrix (13.11), it is found that

$$y_1 = \frac{b_{11}}{a_{rs}}x_1 + \frac{b_{12}}{a_{rs}}x_2 + \cdots + \frac{b_{1(s-1)}}{a_{rs}}x_{(s-1)} + \left(\frac{a_{1s}}{a_{rs}}y_r\right) + \frac{b_{1(s+1)}}{a_{rs}}x_{(s+1)} + \cdots + \frac{b_{1n}}{a_{rs}}x_n \quad (13.13)$$

We have to prove that y_1 in Eq. (13.12) is equal to y_1 in Eq. (13.13).

Since $b_{ij} = a_{ij}a_{rs} - a_{is}a_{rj}$, we get

$$b_{11} = a_{11}a_{rs} - a_{1s}a_{r1}$$

or

$$\frac{b_{11}}{a_{rs}}x_1 = \left(\frac{a_{11}a_{rs} - a_{1s}a_{r1}}{a_{rs}}\right)x_1$$

$$= a_{11}x_1 - \frac{a_{1s}}{a_{rs}}a_{r1}x_1$$

Similarly,

$$\frac{b_{12}x_2}{a_{rs}} = a_{12}x_2 - \frac{a_{1s}}{a_{rs}}a_{r2}x_2$$

Equation (13.13) will take the form,

$$y_1 = a_{11}x_1 + a_{12}x_2 + \cdots + \frac{a_{1s}}{a_{rs}}y_r + \cdots + a_{1n}x_n - \frac{a_{1s}}{a_{rs}}a_{r1}x_1 - \frac{a_{1s}}{a_{rs}}ar_2x_2 - \cdots - \frac{a_{1s}}{a_{rs}}\ a_{rn}x_n \quad (13.14)$$

From matrix (13.11), it is found that

$$x_s = \frac{a_{r1}}{a_{rs}}x_1 - \frac{a_{r2}}{a_{rs}}x_2 + \cdots + \frac{y_r}{a_{rs}} - \cdots - \frac{a_{rn}}{a_{rs}}x_n$$

Multiplying the above equation by a_{1s}, we have

$$a_{1s}x_s = -\frac{a_{1s}}{a_{rs}}a_{r1}x_1 - \frac{a_{1s}}{a_{rs}}a_{r2}x_2 + \cdots + \frac{a_{1s}}{a_{rs}}y_r - \cdots - \frac{a_{1s}}{a_{rs}}a_{rn}x_n$$

Incorporating the above equation in the right-hand side of Eq. (13.14), we get

$$y_1 = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1s}x_s + \ldots + a_{1n}x_n$$

Thus it is proved that y_1 in Eq. (13.12) is equal to y_1 in Eq. (13.13). Similarly, a generalized conclusion can be made for $y_2, y_3, \ldots, y_m$.

13.4 MINIMIZATION OF FUNCTIONS BY NUMERICAL METHODS

We have already studied the gradient technique. This is nothing but one of the numerical methods of minimization. The steepest descent method is the simplest example of the gradient technique. In the steepest descent method, the search direction for minimization is made by guessing an initial value of the independent variable.

Suppose the initial value of the independent variable is $x^{(0)}$, then the next value of the independent variable will be

$$x^{(1)} = x^{(0)} + K^*\left(-\frac{\delta f}{\delta x}\right)_{\text{at } x = x^{(0)}}$$

where K^* the constant of proportionality is also called the *optimum step size* which is greater than zero. The K^* will satisfy

$$f\left(x^{(0)} + K^*\left(-\frac{\delta f}{\delta x}\right)_{\text{at } x = x^{(0)}}\right) \le f\left(x^{(0)} + K\left(-\frac{\delta f}{\delta x}\right)_{\text{at } x = x^{(0)}}\right)$$

Usually, for fast operation of the steepest descent method, the value of K_i is taken such that there will be a 1% change of the function. The value of K_i is made to change in doubling fashion till the function decreases. But this method creates a problem when the function is elongated valley type. Then some modified form of the numerical method is applied for minimization that is called the Fletcher–Powell method. This is one of the best methods for unconstrained optimization.

13.4.1 Fletcher–Powell Method

In the Fletcher–Powell method, a positive definite matrix H is used to find the search direction. Here,

$$x^{(1)} = x^{(0)} + K^*\,(-H)\left(\frac{\delta f}{\delta x}\right)_{\text{at } x = x^{(0)}}$$

In general,

$$x^{(i+1)} = x^{(i)} + K_i^*\,(-H_i)\left(\frac{\delta f}{\delta x}\right)_{\text{at } x = x^{(i)}}$$

when

$$f\left[x^{(i)} + K_i^*(-H_i)\left(\frac{\delta f}{\delta x}\right)_{\text{at } x = x^{(i)}}\right] \le f\left[x^{(i)} + K_i(-H_i)\left(\frac{\delta f}{\delta x}\right)_{\text{at } x = x^{(i)}}\right]$$

Over and above, the matrix H_i is also improved. That is,

$$H_{i+1} = H_i + A_i + B_i$$

where

$$A_i = K_i^*\,\frac{\left(\frac{\delta f}{\delta x}\right)_{\text{at } x = x^{(i)}}\cdot\left(\frac{\delta f}{\delta x}\right)^T_{\text{at } x = x^{(i)}}}{\left(\frac{\delta f}{\delta x}\right)^T_{\text{at } x = x^{(i)}}\left[\left.\frac{\delta f}{\delta x}\right|_{\text{at } x = x^{(i-1)}} - \left.\frac{\delta f}{\delta x}\right|_{\text{at } x = x^{(i)}}\right]}$$

$$B_i = -\frac{H_i\left[\left.\frac{\delta f}{\delta x}\right|_{\text{at } x=x^{(i+1)}} - \left.\frac{\delta f}{\delta x}\right|_{\text{at } x=x^{(i)}}\right]\left[\left.\frac{\delta f}{\delta x}\right|_{\text{at } x=x^{(i+1)}} - \left.\frac{\delta f}{\delta x}\right|_{\text{at } x=x^{(i)}}\right]^T H_i^T}{\left[\left.\frac{\delta f}{\delta x}\right|_{\text{at } x=x^{(i+1)}} - \left.\frac{\delta f}{\delta x}\right|_{\text{at } x=x^{(i)}}\right]^T H_i \left[\left.\frac{\delta f}{\delta x}\right|_{\text{at } x=x^{(i+1)}} - \left.\frac{\delta f}{\delta x}\right|_{\text{at } x=x^{(i)}}\right]}$$

13.4.2 Newton–Raphson Method

This is another numerical method for error minimization. Usually, in the control system, the error from the desired value is minimized using the feedback principle. See Fig. 13.3. Suppose there is a set of nonlinear equations,

$$f_1(x_1, x_2, \ldots, x_n) = y_1$$

$$f_2(x_1, x_2, \ldots, x_n) = y_2$$

$$\vdots$$

$$f_n(x_1, x_2, \ldots, x_n) = y_n$$

and the initial estimate for the independent variables are $x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)}$.

Now $\Delta x_1, \Delta x_2, \ldots, \Delta x_n$ are the corrections required. Hence,

$$f_1(x_1^{(0)} + \Delta x_1, x_2^{(0)} + \Delta x_2, \ldots, x_n^{(0)} + \Delta x_n) = y_1$$

$$f_2(x_1^{(0)} + \Delta x_1, x_2^{(0)} + \Delta x_2, \ldots, x_n^{(0)} + \Delta x_n) = y_2$$

$$\vdots$$

$$f_n(x_1^{(0)} + \Delta x_1, x_2^{(0)} + \Delta x_2, \ldots, x_n^{(0)} + \Delta x_n) = y_n$$

Applying the Taylor's Theorem,

$$f_1(x_1^{(0)} + \Delta x_1, x_2^{(0)} + \Delta x_2, \ldots, x_n^{(0)} + \Delta x_n)$$

$$= f_1(x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)}) + \Delta x_1 \left.\frac{\delta f_1}{\delta x_1}\right|_{x_1^{(0)}} + \Delta x_2 \left.\frac{\delta f_1}{\delta x_2}\right|_{x_2^{(0)}} + \cdots + \Delta x_n \left.\frac{\delta f_1}{\delta x_n}\right|_{x_n^{(0)}}$$

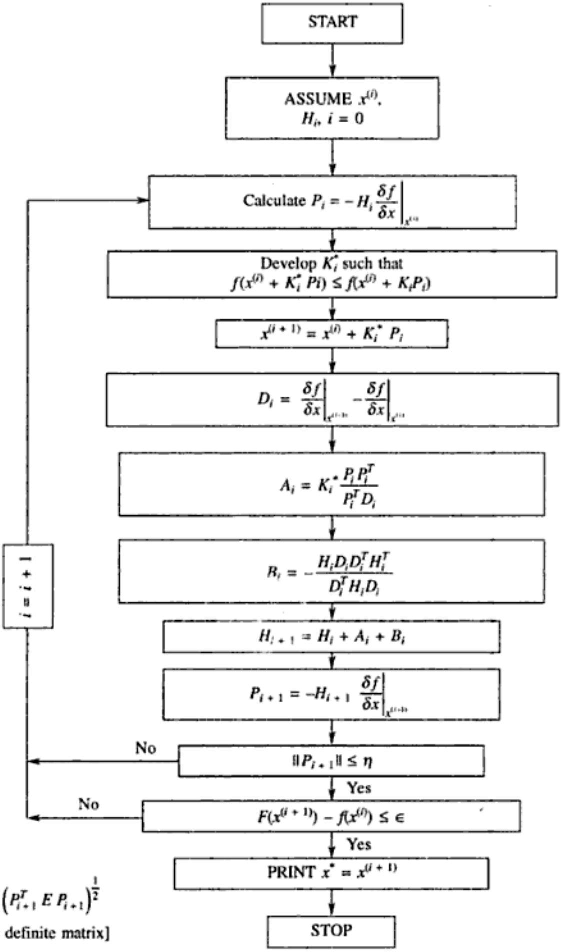
Neglecting higher powers of $\Delta x_1, \Delta x_2, \ldots, \Delta x_n$, we obtain

$$y_1 = f_1(x_1^{(0)}, x_2^{(0)}, x_n^{(0)}) + \Delta x_1 \left.\frac{\delta f_1}{\delta x_1}\right|_{x_1^{(0)}} + \Delta x_2 \left.\frac{\delta f_1}{\delta x_2}\right|_{x_2^{(0)}} + \cdots + \Delta x_n \left.\frac{\delta f_1}{\delta x_n}\right|_{x_n^{(0)}}$$

Similarly,

$$y_2 = f_2(x_1^{(0)}, x_2^{(0)}, x_n^{(0)}) + \Delta x_1 \left.\frac{\delta f_2}{\delta x_1}\right|_{x_1^{(0)}} + \Delta x_2 \left.\frac{\delta f_2}{\delta x_2}\right|_{x_2^{(0)}} + \cdots + \Delta x_n \left.\frac{\delta f_2}{\delta x_n}\right|_{x_n^{(0)}}$$

and so on.



NOTE:

$$\|P_{i+1}\| = \left(P_{i+1}^T\, E\, P_{i+1}\right)^{\frac{1}{2}}$$

[E is a positive definite matrix]

Fig. 13.3 Flow chart of the Fletcher–Powell method.

or

$$2P - Q = 2$$

or

$$P - \frac{Q}{2} = 1$$

or

$$P + \frac{Q}{-2} = 1 \text{ (which represents straight line)}$$

Let us assume the initial point $P^{(0)} = 1$ and $Q^{(0)} = -1$. Therefore,

$$f_1(P^{(0)}, Q^{(0)}) = P^2 - 4Q - 4 = 1 + 4 - 4 = 1$$

$$f_2(P^{(0)}, Q^{(0)}) = 2P - Q - 2$$

$$= 2 + 1 - 2 = 1$$

$$\frac{\delta f_1}{\delta P} = 2P = 2, \quad \frac{\delta f_1}{\delta Q} = -4$$

$$\frac{\delta f_2}{\delta P} = 2, \quad \frac{\delta f_2}{\delta Q} = -1$$

Therefore,

$$f_1(P^{(0)}, Q^{(0)}) + \Delta P \frac{\delta f_1}{\delta P} + \Delta Q \frac{\delta f_1}{\delta Q} = 0$$

or

$$1 + \Delta P(2) + \Delta Q(-4) = 0$$

or

$$1 + 2\Delta P - 4\Delta Q = 0 \tag{13.15}$$

Again,

$$f_2(P^{(0)}, Q^{(0)}) + \Delta P \frac{\delta f_2}{\delta P} + \Delta Q \frac{\delta f_2}{\delta Q} = 0$$

or

$$1 + \Delta P(2) - \Delta Q = 0$$

or

$$1 + 2\Delta P - \Delta Q = 0 \tag{13.16}$$

From Eqs. (13.15) and (13.16), we get

$$-3\Delta Q = 0 \quad \text{or} \quad \Delta Q = 0$$

Putting $\Delta Q = 0$ in Eq. (13.15), we get

$$1 + 2\Delta P = 0 \quad \text{or} \quad \Delta P = -\frac{1}{2}$$

The new set of values of P and Q will be

$$P^{(1)} = P^{(0)} + \Delta P = 1 - 0.5 = 0.5$$

$$Q^{(1)} = Q^{(0)} + \Delta Q = -1 + 0 = -1$$

The above process is repeated with the new set of values of $P^{(1)}$ and $Q^{(1)}$.

$$f_1(P^{(1)}, Q^{(1)}) = P^2 - 4Q - 4$$

$$= 0.25 + 4 - 4 = 0.25$$

$$f_2(P^{(1)}, Q^{(1)}) = 2P - Q - 2$$
$$= 2.(0.5) + 1 - 2 = 0$$
$$\frac{\delta f_1}{\delta P} = 2P = 2 \times 0.5 = 1, \quad \frac{\delta f_1}{\delta Q} = -4$$
$$\frac{\delta f_2}{\delta P} = 2, \quad \frac{\delta f_2}{\delta Q} = -1$$

Therefore, the equations developed are as follows:

$$f_1(P^{(1)}, Q^{(1)}) + \Delta P \frac{\delta f_1}{\delta P} + \Delta Q \frac{\delta f_1}{\delta Q} = 0$$

or

$$0.25 + \Delta P - 4\Delta Q = 0 \tag{13.17}$$

or

$$f_2(P^{(1)}, Q^{(1)}) + \Delta P \frac{\delta f_2}{\delta P} + \Delta Q \frac{\delta f_2}{\delta Q} = 0$$

or

$$0 + \Delta P(2) + \Delta Q(-1) = 0$$
$$2\Delta P - \Delta Q = 0 \tag{13.18}$$

Solving Eqs. (13.17) and (13.18), we get

$$\Delta Q = \frac{0.5}{7} = 0.07143$$
$$\Delta P = 0.03571$$

Thus, the new set of values of P and Q will be

$$P^{(2)} = P^{(1)} + \Delta P = 0.5 + 0.03571 = 0.53571$$
$$Q^{(2)} = Q^{(1)} + \Delta Q = -1 + 0.07143 = -0.92857$$

Therefore,

$$f_1(P^{(2)}, Q^{(2)}) = P^2 - 4Q - 4$$
$$= (0.53571)^2 - 4(-0.92857) - 4$$
$$= 0.28699 + 3.71428 - 4 = 0.00127$$
$$f_2(P^{(2)}, Q^{(2)}) = 2P - Q - 2$$
$$= 2(0.53571) + 0.92857 - 2$$
$$= -\ 0.00001$$

The same procedure is repeated.

$$f_1(P^{(2)}, Q^{(2)}) + \Delta P \frac{\delta f_1}{\delta P} + \Delta Q \frac{\delta f_1}{\delta Q} = 0$$

state vector, $u(t)$ is the $p \times 1$ input vector, and f is a vector-valued function. In the problem of control system, we have to specify properly the objective function which is to be optimized.

The objective function to be optimized in control systems is termed *performance index*. A number of performance indices are used in practice. Those are as follows:

(a) Integral square error (ISE)
(b) Integral time absolute error (ITAE).
(c) Integral of the absolute magnitude of error (IAE)
(d) Integral time-square error (ITSE)

The error signal in a closed-loop control system is the difference between the input signal and the feedback signal. The objective of the control system is to reduce the absolute value of the error. That is why, the error signal in this different form is taken as the performance index which is to be optimized. The integral square error is

$$\text{ISE} = \int_0^{\infty} e^2(t)dt$$

and

$$\text{ITAE} = \int_0^{T} t|e(t)|dt$$

(Performance index for a specified period from 0 to time T)

$$\text{ITSE} = \int_0^{T} te^2(t)\,dt$$

$$\text{IAE} = \int_0^{T} |e(t)|\,dt$$

We know that a control system is usually described by the block diagram as shown in Fig. 13.5.

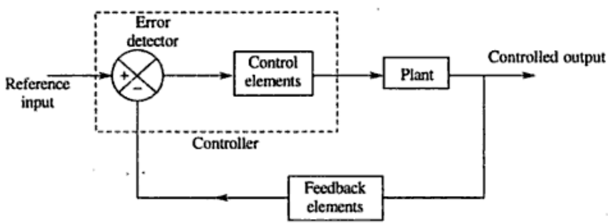


Fig. 13.5 Block diagram of the control system.

Hence for the solution of an optimal control problem, we have to perform the following:

(a) An optimum function U that will act upon the plant is to be found out.

(b) The optimum control function U is to be realized by the controller.

Hence, the question of optimum design of the controller arises. The design of the optimum controller depends on the following factors:

(a) Characteristics of the plant

(b) Requirements of the plant

(c) Data of the plant received by the controller

13.5.1 Characteristics of the Plant

This actually describes the limitations of the system components. On account of the limitations, there will be constraints on state variables and control variables. For example, the plant inputs $u_1(t)$, $u_2(t)$, . . ., $u_p(t)$ would have some restricted values.

Thus an admissible control is defined as the control which satisfies the control constraints during the entire control interval from t_0 to t_f. In general, the admissible control is denoted by the capital letter U. Then input $u(t)$ will belong to the admissible control capital U. Similarly, the state trajectory which satisfies the state variable constraints from t_0 and t_f is termed *admissible trajectory*.

13.5.2 Requirements of the Plant

It is usally expressed by the performance index. To optimize mathematically, the performance index means that the plant requirements are fulfilled. In other words, we are to design a control system that would run perfectly in optimum condition. Therefore, the proper choice of the performance index according to the need of the system is the main factor for optimization. Hence, selection of the proper performance index according to need of the problem is the main criterion. The control problems can therefore be visualized as follows.

(a) Minimum time problem

(b) Minimum energy problem

(c) Minimum fuel problem

(d) Minimum regulator problem

(e) State regulator problem

(f) Output regulator problem

(g) Servomechanism or tracking problem

Minimum time problem

The objective of the minimum time problem is to transfer a system from its initial state to the specified target in the minimum time. The interception of attacking aircraft and missiles is an example of such a control problem. Mathematically, It is expressed as

$$PI = \int_{t_0}^{t_1} dt$$

$$PI = \int_{t_0}^{t_1} \left[\sum_{i=1}^{n} \{x_i(t)\}^2 \right] dt$$

The matrix form is

$$PI = \int_{t_0}^{t_1} [X^T(t)X(t)]\, dt$$

and the generalized form is

$$PI = \int_{t_0}^{t_1} [X^T(t)GX(t)]\, dt$$

where G is the real symmetric positive semi-definite matrix. The positive semi-definite matrix means that $X^TGX \geq 0$, with

$$G = \begin{bmatrix} G_1 & 0 & 0 \\ 0 & G_2 & 0 \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ 0 & \vdots & G_n \end{bmatrix}$$

as a diagonal matrix.

For minimizing the deviation of the final state $X(t_1)$ of the system from the desired state, the performance index is additionally modified by incorporating the final state as well. That means,

$$PI = X^T(t_1)LX(t_1) + \int_{t_0}^{t_1} [X^T(t)GX(t)]dt$$

where L is also a positive semi-definite, real symmetric, constant matrix. A more practically feasible performance index is obtained by incorporating the input which is added as the penalty term for the physical constraints on it. Therefore, the final form of the performance index is

$$PI = \frac{1}{2} X^T(t_1)LX(t_1) + \frac{1}{2}\left[\int_{t_0}^{t_1} [X^T(t)GX(t) + U^T(t)FU(t)]dt \right]$$

When the state regulator problem becomes that of infinite time,

$$PI = \frac{1}{2}\left[\int_{0}^{\infty} [X^T(t)GX(t) + U^T(t)FU(t)]dt \right]$$

Output regulator problem

When the output regulator problem is considered, X will be replaced by Y. The performance index will then be

$$PI = \frac{1}{2}Y^T(t_1)LY(t_1) + \frac{1}{2}\left[\int_{t_0}^{t_1} [Y^T(t)GY(t) + U^T(t)FU(t)]dt\right]$$

Servomechanism or tracking problem

The servomechanism or tracking problem keeps the system state $x(t)$ very near to the desired state in a particular interval of time.

Suppose the desired state is $d(t)$. Then the error is $e(t) = [x(t) - d(t)]$ and the performance index will be

$$PI = \frac{1}{2}e^T(t_1)Le(t_1) + \frac{1}{2}\int_{t_0}^{t_1} [e^T(t)Ge(t) + U^T(t)FU(t)]dt$$

The main objective of servomechanism or tracking problem is to maintain the error small.

EXAMPLE 13.2 Suppose in a control system a parameter is to be optimized. The block diagram of the system is shown in Fig. 13.6.

Now,

$$R(s) - O(s) = E(s)$$

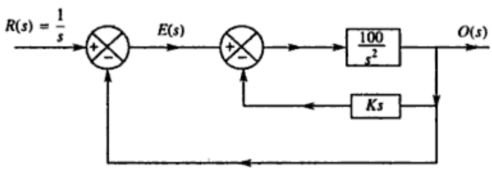


Fig. 13.6 Example 13.2.

or

$$[R(s) - O(s)]\,\frac{\dfrac{100}{s^2}}{1 + \dfrac{100K}{s}} = O(s)$$

which gives the block diagram as shown in Fig. 13.7.

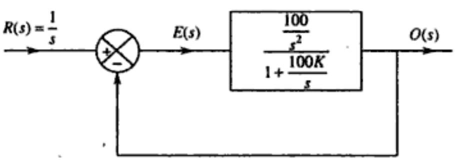


Fig. 13.7 Modified block diagram of Fig. 13.6.

or

$$K = \frac{1}{10}$$

Also,

$$\frac{d^2(PI)}{dK^2} = \frac{200K \cdot 200K^2 - 400K(100K^2 - 1)}{(200\,K^2)^2} = \frac{2K}{(200K^2)^2} > 0 \quad (\because\ K \text{ is positive})$$

Some important formulae used in derivation of *PI*

$$\int e^{ax} \sin(bx + c)dx = \frac{e^{ax}}{a^2 + b^2}[a \sin(bx + c) - b \cos(bx + c)]$$

$$\int e^{ax} \cos(bx + c)dx = \frac{e^{ax}}{a^2 + b^2}[a \cos(bx + c) + b \sin(bx + c)]$$

Laplace transform of $e^{at} \sin bt = \dfrac{b}{(s-a)^2 + b^2}$

Laplace trasform of $e^{at} \cos bt = \dfrac{s-a}{(s-a)^2 + b^2}$

To derive *PI*, in Example 13.2 we have seen that

$$E(s) = \frac{s + 100K}{s^2 + 100Ks + 100} \qquad \text{and} \qquad PI = \int_0^\infty e^2(t)dt$$

Therefore,

$$e(t) = \mathcal{L}^{-1}E(s) = \mathcal{L}^{-1}\frac{s + 100K}{s^2 + 2 \cdot s \cdot 50K + (50K)^2 - (50K)^2 + 100} = \mathcal{L}^{-1}\frac{s + 100K}{(s + 50K)^2 + \left(\sqrt{100 - 2500K^2}\right)^2}$$

or

$$e(t) = \mathcal{L}^{-1}\frac{s + 50K}{(s + 50K)^2 + \left(\sqrt{100 - 2500K^2}\right)^2} + \mathcal{L}^{-1}\frac{50K}{(s + 50K)^2 + \left(\sqrt{100 - 2500K^2}\right)^2}$$

$$= e^{-50Kt} \cos\sqrt{100 - 2500K^2}\, t + \frac{50K}{\sqrt{100 - 2500K^2}} e^{-50Kt} \sin\sqrt{100 - 2500K^2}\, t$$

Therefore,

$$e^2(t) = e^{-100Kt} \cos^2\sqrt{100 - 2500K^2}\, t + \frac{2500K^2}{100 - 2500K^2} e^{-100Kt} \sin^2\sqrt{100 - 2500K^2}\, t +$$

$$2e^{-100Kt} \cdot \sin\sqrt{100 - 2500K^2}\, t \times \cos\sqrt{100 - 2500K^2}\, t \times \frac{50K}{\sqrt{100 - 2500K^2}}$$

$$= e^{-100Kt}\left(\frac{1+\cos 2\sqrt{100-2500K^2}\,t}{2}\right)+\frac{2500K^2}{100-2500K^2}e^{-100Kt}\left(\frac{1-\cos 2\sqrt{100-2500K^2}\,t}{2}\right)$$

$$+ e^{-100Kt}\sin 2\sqrt{100-2500K^2}\,t\cdot\frac{50K}{\sqrt{100-2500K^2}}$$

$$= \frac{e^{-100Kt}}{2}+\frac{e^{-100Kt}}{2}\cos 2\sqrt{100-2500K^2}\,t+\frac{1250K^2}{100-2500K^2}e^{-100Kt}-\frac{1250K^2\,e^{-100Kt}}{100-2500K^2}\times$$

$$\cos 2\sqrt{100-2500K^2}\,t+e^{-100Kt}\sin 2\sqrt{100-2500K^2}\,t\,\frac{50K}{\sqrt{100-2500K^2}}$$

$$= e^{-100Kt}\left[\frac{1}{2}+\frac{1250K^2}{100-2500K^2}\right]+e^{-100Kt}\cos 2\sqrt{100-2500K^2}\,t\left[\frac{1}{2}-\frac{1250K^2}{100-2500K^2}\right]+$$

$$e^{-100Kt}\sin 2\sqrt{100-2500K^2}\,t\,\frac{50K}{\sqrt{100-2500K^2}}$$

Now to determine $\int_0^\infty e^2(t)dt$, we proceed step-by-step as below:

Now,

$$\int e^{-100Kt}dt = \frac{e^{-100Kt}}{-100K}$$

and

$$I = \int e^{-at}\cos bt\; dt = +e^{-at}\frac{\sin bt}{b}-\int\frac{(-a)\,e^{-at}(+\sin bt)}{b}dt$$

$$= \frac{e^{-at}\sin bt}{b}+\frac{a}{b}\int e^{-at}\sin bt\,dt$$

$$= \frac{e^{-at}\sin bt}{b}+\frac{a}{b}\left[e^{-at}\left(\frac{-\cos bt}{b}\right)-\int(-a)e^{-at}\left(\frac{-\cos bt}{b}\right)dt\right]$$

or

$$I = e^{-at}\frac{\sin bt}{b}-\frac{a}{b^2}e^{-at}\cos bt-\frac{a^2}{b^2}\int e^{-at}\cos bt\,dt \quad \left(\text{where } I = \int e^{-at}\cos bt\,dt\right)$$

Therefore,

$$I\left[1+\frac{a^2}{b^2}\right] = \frac{e^{-at}}{b}\left[\sin bt-\frac{a}{b}\cos bt\right]$$

$$= \frac{1}{200K(1-25K^2)} + \frac{K-50K^3}{8-200K^2} + \frac{K}{4}$$

$$= \frac{1}{200K(1-25K^2)} + \frac{K-50K^3}{8(1-25K^2)} + \frac{K}{4}$$

$$= \frac{1+25K^2-1250K^4+50K^2(1-25K^2)}{200K(1-25K^2)}$$

$$= \frac{1+25K^2-1250K^4+50K^2-1250K^4}{200K(1-25K^2)}$$

$$= \frac{1+75K^2-2500K^4}{200K(1-25K^2)}$$

$$= \frac{1}{200K}\left[\frac{(1-25K^2)(1+100K^2)}{(1-25K^2)}\right] = \frac{1+100K^2}{200K}$$

Therefore,

$$PI = \frac{1+100K^2}{200K}$$

Alternatively,

$$PI = \int_0^\infty e^2(t)\,dt = \frac{1}{2\pi j}\int_{-j\infty}^{+j\infty} E(s)E(-s)\,ds$$

where

$$E(s) = \frac{s+100k}{s^2+100ks+100}$$

$$\text{General form of } E(s) = \frac{N_{n-1}s^{n-1}+\cdots+N_1s+N_0}{D_ns^n+D_{n-1}s^{n-1}+\cdots+D_1s+D_0} \qquad \text{(A)}$$

So PI can be written as

$$PI = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty}\frac{N(s)N(-s)}{D(s)D(-s)}ds$$

As $E(s) = \dfrac{N(s)}{D(s)} = \dfrac{s+100K}{s^2+100Ks+100}$ is a second-order system, n being 2,

the values of PI integral (calculated by using the Hurwitz determinants) are

$n = 1$; $\quad PI_1 = \dfrac{N_0^2}{2D_0D_1}$

$n = 2;\qquad PI_2 = \dfrac{N_1^2 D_0 + N_0^2 D_2}{2 D_0 D_1 D_2}$

$n = 3;\qquad PI_3 = \dfrac{N_2^2 D_0 D_1 + (N_1^2 - 2N_0 N_2) D_0 D_3 + N_0^2 D_2 D_3}{2 D_0 D_3 (-D_0 D_3 + D_1 D_2)}$

Since $E(s) = \dfrac{s + 100K}{s^2 + 100Ks + 100}$, comparing with (A), we get

$$D_0 = 100,\ D_1 = 100K;\ D_2 = 1,\ N_0 = 100K,\ N_1 = 1$$

Therefore,

$$PI_2 = \frac{1 \times 100 + 10^4 K^2 \times 1}{2 \times 100 \times 100K \times 1}$$

$$= \frac{10^4 K^2 + 100}{2 \times 10^4 K}$$

Hence,

$$PI = \frac{1 + 100K^2}{200K}$$

13.5.3 Plant Data Supplied to the Controller

Suppose the system is closed-loop control. The controller obtains information by way of feedback lines on the actual state $x(t)$ of the process. If the functional relationship for the optimal control at time t is

$$U^*(t) = f(x(t), t)$$

then the optimal control will be the closed-loop control, whereas if the optimal control is found as function of time for a specified initial state value, then the optimal control will be the open-loop control. For example, $U^*(t) = f(x(t_0), t)$.

Thus, it can be understood whether an open-loop controller or a closed-loop controller is required. The closed-loop controller can store information about the plant during its operation and reduce the effects of disturbance and compensate for the variations in plant parameters. The open-loop controller has no way to know any information about the plant except for that which is available at the time the control starts.

13.6 MATHEMATICAL PROCEDURES FOR OPTIMAL CONTROL DESIGN

The following mathematical procedures are widely used in the design of optimal control systems.

(a) Calculus of variations
(b) Pontryagin's minimum/maximum principle
(c) Minimum time problems
(d) Hamilton–Jacobi approach
(e) Dynamic programming

13.6.1 Calculus of Variations

Before studying the calculus of variations, we have to know clearly what is the main difference between a function and a functional. A functional is a kind of function whose independent variable is a function rather than a number or a set of numbers. For example, the performance index (*PI*) is a functional. The calculus of variations actually deals with the functional that extremises in order to find out its maximum or minimum value. In calculus, the maximum or minimum value of the function is determined by making $dy/dx = 0$, when y is the function having an independent variable and the maximum or minimum value is further tested by finding d^2y/dx^2, i.e. whether negative or positive. But in the case of calculus of variations, generally the necessary condition is tested but the sufficient condition is not tested.

Calculus of variations is usually applied to the two-point boundary-value problem. Sometimes it is termed the TPBVP problem

The TPBVP is generally classified as follows:

(a) Fixed-end problem

(b) Variable-end point problem

Before studying the above in detail, let us go back to some mathematical concepts necessary to have a clear view of calculus of variations.

From our study of calculus and matrix, we know that if $f(x)$ is continuous for all x and its gradient vector

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \dfrac{\partial f}{\partial x_1} \\ \dfrac{\partial f}{\partial x_2} \\ \vdots \\ \vdots \\ \dfrac{\partial f}{\partial x_n} \end{bmatrix}$$

is continuous for all x, and

$$\frac{\partial^2 f}{\partial x^2} = \begin{bmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \dfrac{\partial^2 f}{\partial x_1 \partial x_n} \\ \dfrac{\partial^2 f}{\partial x_2 \partial x_1} & \dfrac{\partial^2 f}{\partial x_2^2} & \cdots & \dfrac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \dfrac{\partial^2 f}{\partial x_n \partial x_1} & \dfrac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \dfrac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

is continuous for all x, then the Taylor's series says

$$f(x) = f(x_1) + \left.\frac{\delta f}{\delta x}\right|_{x_1}(x - x_1) + \frac{1}{2}(x - x_2)^T \left.\frac{\delta^2 f}{\delta x^2}\right|_{x_1}(x - x_1) + \cdots$$

and the necessary condition for vector x_1 to be minimum is

$$\left.\frac{\delta f}{\delta x}\right|_{x_1} = 0$$

The sufficient condition is that $\left.\frac{\partial^2 f}{\partial x^2}\right|_{x_1}$ is positive definite. It means that all the values of the matrix will be greater than zero.

Let us take a simple case,

$$\Delta f(x, \Delta x) = \frac{\delta f}{\delta x_1}\Delta x_1 + \frac{\delta f}{\delta x_2}\Delta x_2$$

In case of extremum condition, $\Delta f(x, \Delta x)$ as per definition is

$$\frac{\delta f}{\delta x_1}\Delta x_1 + \frac{\delta f}{\delta x_2}\Delta x_2 = 0$$

Suppose, the constraint equation is $\Delta T = \frac{\delta T}{\delta x_1}\Delta x_1 + \frac{\delta T}{\delta x_2}\Delta x_2$

If the constraint is not allowed to increase, then

$$\frac{\delta T}{\delta x_1}\Delta x_1 + \frac{\delta T}{\delta x_2}\Delta x_2 = 0$$

Two equations are therefore obtained:

$$\frac{\delta f}{\delta x_1}\Delta x_1 + \frac{\delta f}{\delta x_2}\Delta x_2 = 0$$

$$\frac{\delta T}{\delta x_1}\Delta x_1 + \frac{\delta T}{\delta x_2}\Delta x_2 = 0$$

or

$$\frac{\Delta x_1}{\Delta x_2} = -\frac{\dfrac{\delta f}{\delta x_2}}{\dfrac{\delta f}{\delta x_1}}$$

$$\frac{\Delta x_1}{\Delta x_2} = -\frac{\dfrac{\delta T}{\delta x_2}}{\dfrac{\delta T}{\delta x_1}}$$

This is the condition for the interior extremum, that means the maximum or minimum value within certain boundary at which the function is defined.

The $\Delta x_1/\Delta x_2$ is nothing but the proportionality constant which is also called the Lagrange multiplier. It is usually denoted by λ.

The Lagrangian is defined by

$$L(x, \lambda) = f(x) + \lambda T(x)$$

$$\frac{\partial L}{\partial x} = \frac{\delta f(x)}{\delta x} + \lambda \frac{\delta T(x)}{\delta x}$$

If the constraint equation

$$T(x) = 0$$

then

$$\frac{\delta L}{\delta \lambda} = T(x) = 0$$

If x^* is a constrained extremum, then λ^* will be the optimum value such that

$$\left.\frac{\delta L}{\delta x}\right|_{x^*, \lambda^*} = 0$$

$$\left.\frac{\delta L}{\delta \lambda}\right|_{x^*, \lambda^*} = 0$$

Proof

Suppose,

$$dL(x, \lambda) = df(x) + \lambda dT(x)$$

$$= \frac{\delta f}{\delta x_1}\Delta x_1 + \frac{\delta f}{\delta x_2}\Delta x_2 + \lambda \frac{\delta T}{\delta x_1}\Delta x_1 + \lambda \frac{\delta T}{\delta x_2}\Delta x_2$$

Now,

$$\frac{\Delta x_2}{\Delta x_1} = \frac{-\dfrac{\delta f(x)}{\delta x_1}}{\dfrac{\delta f(x)}{\delta x_2}}$$

or

$$\Delta x_2\left(\frac{\delta f(x)}{\delta x_2}\right) + \Delta x_1\left(\frac{\delta f(x)}{\delta x_1}\right) = 0$$

Similarly,

$$\Delta x_1 \frac{\delta T(x)}{\delta x_1} + \Delta x_2 \frac{\delta T(x)}{\delta x_2} = 0$$

Hence,

$$dL(x, \lambda) = 0$$

Again, for extremum condition $dL(x, \lambda) = 0$

$$\frac{\delta L}{\delta x} = 0, \quad \frac{\delta L}{\delta \lambda} = 0$$

or

$$x_3 - x_1 - x_2 = 1$$

Again from (13.23)

$$x_1 + x_2 + x_3 = 1$$

or

$$2x_3 = 2 \quad \text{or} \quad x_3 = 1$$

From (13.22)

$$x_1 x_2 + 4 - x_3 = 0$$

or

$$x_1 x_2 + 4 - 1 = 0$$

or

$$x_1 x_2 = -3$$

Now from (13.23)

$$x_1 + x_2 + x_3 = 1$$

or

$$x_1 + x_2 + 1 = 1$$

or

$$x_1 + x_2 = 0$$

or

$$x_1 = -x_2$$

Therefore,

$$x_1 x_2 = -3$$

or

$$-x_2^2 = -3$$

or

$$x_2 = \pm \sqrt{3}$$

or

$$x_1 = \mp \sqrt{3}$$

Therefore,

$$\text{either } x_1 = \sqrt{3},\ x_2 = -\sqrt{3},\ x_3 = 1$$

$$\text{and} \quad x_1 = -\sqrt{3},\ x_2 = \sqrt{3},\ x_3 = 1$$

$$\lambda_1 = 2$$

Since

$$2x_3 - \lambda_1 + \lambda_2 = 0$$

Therefore,

$$2 - 2 + \lambda_2 = 0$$

or

$$\lambda_2 = 0$$

Use of calculus of variations in three standard forms

Calculus of variations is usally applied to the two-point boundary value problems in the following three standard forms.

$$PI = \int_{t_0}^{t_f} L(x, u, t)\, dt$$

$$PI = \left[G(x, u, t)\right]_{t_0}^{t_f}$$

$$PI = \int_{t_0}^{t_f} L(x, u, t) dt + |G(x, u, t)|_{t_0}^{t_f}$$

First of all let us concentrate on

$$PI = \int_{t_0}^{t_f} L(x, u, t) dt$$

This form is called the Lagrange problem. Hence, t_0 and t_f are the initial and final time respectively.

Lagrange fixed end problem

Since $\dot{x} = Ax(t) + Bu(t)$, the PI can be defined as follows.

$$PI = \int_{t_0}^{t_f} L[x(t), \dot{x}(t), t] dt$$

The end points $t = t_0$ and $t = t_f$ are fixed. Say $x(t_0) = x_0$, $x(t_f) = x_f$. Hence,

$$\Delta x(t_0) = 0$$

$$\Delta x(t_f) = 0$$

If PI is denoted by J, then

$$\Delta J = \int_{t_0}^{t_f} L(x + \Delta x, \dot{x} + \Delta \dot{x}, t) dt - \int_{t_0}^{t_f} L(x, \dot{x}, t) dt$$

$$= \int_{t_0}^{t_f} [L(x + \Delta x, \dot{x} + \Delta \dot{x}, t) - L(x, \dot{x}, t)] dt$$

Applying Taylor's theorem, approximately, we can write

$$\Delta J = \int_{t_0}^{t_f} \left[\frac{\delta L}{\delta x}(x, \dot{x}, t)\Delta x + \frac{\delta L}{\delta \dot{x}}(x, \dot{x}, t)\Delta \dot{x}\right] dt$$

$$= \left[\int_{t_0}^{t_f} \frac{\delta L}{\delta x}(x, \dot{x}, t)\Delta x\right] dt + \left[\int_{t_0}^{t_f} \frac{\delta L}{\delta \dot{x}}(x, \dot{x}, t)\Delta \dot{x}\right] dt$$

$$= \left[\int_{t_0}^{t_f} \frac{\delta L}{\delta x}(x, \dot{x}, t)\Delta x\right] dt + \left[\frac{\delta L}{\delta \dot{x}}(x, \dot{x}, t)\Delta x\right]_{t_0}^{t_f}$$

or

$$= \int_{t_0}^{t_f} \frac{d}{dt}\frac{\delta L}{\delta \dot{x}}(x, \dot{x}, t)\Delta x\, dt$$

or

$$\Delta J = \int_{t_0}^{t_f} \left[\frac{\delta L}{\delta x} - \frac{d}{dt}\frac{\delta L}{\delta \dot{x}}\right]\Delta x \cdot dt + \left[\frac{\delta L}{\delta \dot{x}}\Delta x\right]_{t_0}^{t_f}$$

Since

$$\Delta x(t_f) = 0,\ \Delta x(t_0) = 0$$

$$\Delta J = \int_{t_0}^{t_f} \left[\frac{\delta L}{\delta x} - \frac{d}{dt}\frac{\delta L}{\delta \dot{x}}\right]\Delta x\, dt$$

For optimum value of J, $\Delta J = 0$. Thus,

$$\int_{t_0}^{t_f} \left[\frac{\delta L}{\delta x} - \frac{d}{dt}\frac{\delta L}{\delta \dot{x}}\right]\Delta x\, dt = 0$$

That means,

$$\frac{\delta L}{\delta x} - \frac{d}{dt}\frac{\delta L}{\delta \dot{x}} = 0$$

for any arbitrary value of x. The above equation is termed the *Euler's equation*.

An example related to Euler's equation

If

$$J = \int_{t_2}^{t_1} \sqrt{1 + \dot{x}^2}\, dt$$

with boundary condition $x(t_2) = m$ and $x(t_1) = n$, then the optimal curve, that is, the extremal can be determined as follows.

As per Euler's equation,

$$\frac{\delta L}{\delta x} - \frac{d}{dt}\left(\frac{\delta L}{\delta \dot{x}}\right) = 0$$

In this problem, $L = \sqrt{1 + \dot{x}^2}$. As x and $\dot{x}$ are separately dealt with in the Lagrange problem,

$$\frac{\delta L}{\delta x} = 0 \text{ (} L \text{ does not contain any } x \text{ term)}$$

$$\frac{d}{dt}\left(\frac{\delta L}{\delta \dot{x}}\right) = 0$$

$$\frac{\delta L}{\delta \dot{x}} = K \quad (K = \text{constant})$$

or

$$\frac{\delta}{\delta \dot{x}}\sqrt{1+\dot{x}^2} = K$$

or

$$\frac{1}{2}(1+\dot{x}^2)^{-\frac{1}{2}} \cdot 2\dot{x} = K$$

or

$$\frac{\dot{x}}{(1+\dot{x}^2)^{1/2}} = K \quad \text{or} \quad \frac{\dot{x}^2}{1+\dot{x}^2} = K^2$$

or

$$\dot{x}^2 = \frac{K^2}{1-K^2}$$

or

$$\dot{x} = \frac{K}{\sqrt{1-K^2}} = K_1 \quad \text{or} \quad \frac{dx}{dt} = K_1$$

or

$$x = K_1 t + K_2$$

This is nothing but the equation of a straight line.

Now, $n = K_1 t_1 + K_2$, $m = K_1 t_2 + K_2$

Hence K_1 and K_2 can be calculated.

EXAMPLE 13.3 Determine the optimal integral curves when the performance index is expressed as follows

$$J = \int_0^{\pi/2} \left(\dot{x}_1^2 + \dot{x}_2^2 + 2x_1x_2\right)dt$$

The boundary conditions are, $x_1(0) = 0$, $x_2(0) = 0$, $x_1(\pi/2) = -1$, and $x_2(\pi/2) = 1$

Solution Since there are two variables, x_1 and x_2

$$\frac{\delta L}{\delta x_1} - \frac{d}{dt}\left(\frac{\delta L}{\delta \dot{x}_1}\right) = 0 \quad \text{and} \quad \frac{\delta L}{\delta x_2} - \frac{d}{dt}\left(\frac{\delta L}{\delta \dot{x}_2}\right) = 0$$

where $L = \dot{x}_1^2 + \dot{x}_2^2 + 2x_1x_2$. Therefore

$$\frac{\delta L}{\delta x_1} = 2x_2$$

$$\frac{d}{dt}\left(\frac{\delta L}{\delta \dot{x}_1}\right) = \frac{d}{dt}(2\dot{x}_1) = 2\ddot{x}_1$$

Therefore,

$$2x_2 - 2\ddot{x}_1 = 0 \qquad \text{or} \qquad \ddot{x}_1 = x_2$$

Also,

$$\frac{\delta L}{\delta x_2} = 2x_1$$

Therefore,

$$\frac{d}{dt}\left(\frac{\delta L}{\delta \dot{x}_2}\right) = \frac{d}{dt}(2\dot{x}_2) = 2\ddot{x}_2$$

$$2x_1 - 2\ddot{x}_2 = 0 \qquad \text{or} \qquad x_1 = \ddot{x}_2$$

or

$$x_1 = \frac{d^2}{dt^2}(x_2) = \frac{d^2}{dt^2}(\ddot{x}_1) = \frac{d^2}{dt^2}\left(\frac{d^2}{dt^2}(x_1)\right) = \frac{d^4}{dt^4}(x_1)$$

Putting $x_1 = ae^{mt}$, we have

$$am^4e^{mt} = ae^{mt}$$

or

$$m^4 = 1 \qquad \text{or} \qquad (m^2 + 1)(m^2 - 1) = 0$$

or

$$m = \pm 1, m = \pm j$$

Hence,

$$x_1 = a_1e^t + a_2e^{-t} + a_3 \sin t + a_4 \text{cost } t$$

Again,

$$x_2 = \ddot{x}_1 \qquad \text{or} \qquad x_2 = \frac{d^2}{dt^2}(a_1e^t + a_2e^{-t} + a_3 \sin t + a_4 \cos t)$$

$$= a_1e^t + a_2e^{-t} - a_3 \sin t - a_4 \cos t$$

Now, $x_1(0) = 0$, $x_2(0) = 0$, $x_1(\pi/2) = -1$, $x_2(\pi/2) = 1$. Therefore, we obtain

$$0 = a_1 + a_2 + a_4 \tag{1}$$

$$-1 = a_1e^{\pi/2} + a_2e^{-\pi/2} + a_3 \tag{2}$$

$$0 = a_1 + a_2 - a_4 \tag{3}$$

$$1 = a_1e^{\pi/2} + a_2e^{-\pi/2} - a_3 \tag{4}$$

(1) + (3) : $2(a_1 + a_2) = 0$ or $a_1 + a_2 = 0$ or $a_1 = -a_2$

(1) – (3): $2a_4 = 0$ or $a_4 = 0$

(2) – (4): $-2 = 2a_3$ or $a_3 = -1$

From (2), we get

$$-1 = a_1e^{\pi/2} - a_1e^{-\pi/2} \qquad (\because a_1 = -a_2)$$

Thus,

$$\frac{\delta}{\delta \dot{x}}\left(\sqrt{1+\dot{x}^2}\right) = K \qquad \text{or} \qquad \frac{1}{2}(1+\dot{x}^2)^{-\frac{1}{2}} \cdot 2\dot{x} = K$$

or

$$\frac{\dot{x}^2}{1+\dot{x}^2} = K^2 \qquad \text{or} \qquad \dot{x}^2 = K^2 + K^2\dot{x}^2$$

or

$$\dot{x} = \frac{K}{\sqrt{1-K^2}} = K_1 \qquad \text{or} \qquad \frac{dx}{dt} = K_1$$

or

$$x = K_1 t + K_2$$

When $t = 0$, $x = 0$, therefore, $K_2 = 0$. Thus,

$$x = K_1 t$$

Now,

$$L + (\dot{y} - \dot{x})\frac{\delta L}{\delta \dot{x}}\bigg|_{t=t_f} = 0 \qquad \text{(transversality condition)}$$

or

$$\sqrt{1+\dot{x}^2} + (\dot{y} - \dot{x})\frac{\delta L}{\delta \dot{x}}\bigg|_{t=t_f} = 0$$

Now,

$$y(t) = 3 - t, \;\; \dot{y} = -1$$

$$x = K_1 t, \;\; \dot{x} = K_1$$

$$\frac{\delta L}{\delta \dot{x}} = \frac{\delta}{\delta \dot{x}}\sqrt{1+\dot{x}^2} = \frac{1}{2}\frac{2\dot{x}}{\sqrt{1+\dot{x}^2}} = \frac{\dot{x}}{\sqrt{1+\dot{x}^2}} = \frac{K_1}{\sqrt{1+K_1^2}}$$

Thus, the transversality condition reduces to

$$\sqrt{(1+K_1^2)} + (-1-K_1)\cdot\frac{K_1}{\sqrt{1+K_1^2}} = 0$$

or

$$\sqrt{(1+K_1^2)} - \frac{K_1(1+K_1)}{\sqrt{1+K_1^2}} = 0$$

or

$$(1+K_1^2) - K_1(1+K_1) = 0$$

or

$$1 - K_1 = 0 \qquad \text{or} \qquad K_1 = 1$$

Therefore, the solution is

$$x(t) = K_1 t = t$$

EXAMPLE 13.5 Optimize $\int_0^3 [\dot{x}^2(t) + 2x(t)\dot{x}(t) + 2x^2(t)]dt$, such that $x(0) = 1$ and $x(3) =$ free.

Solution Since the performance index is $\int_0^3 [\dot{x}^2(t) + 2x(t)\dot{x}(t) + 2x^2(t)]dt$, Lagrangian L will be

$$L = \dot{x}^2 + 2x\dot{x} + 2x^2$$

According to the Euler-Lagrange equation,

$$\frac{\delta L}{\delta x} - \frac{d}{dt}\frac{\delta L}{\delta \dot{x}} = 0$$

or

$$(2\dot{x} + 4x) - \frac{d}{dt}(2\dot{x} + 2x) = 0$$

or

$$2\dot{x} + 4x - 2\ddot{x} - 2\dot{x} = 0$$

or

$$\ddot{x} = 2x$$

Say, $x = Ae^{mt}$. Then, $\dot{x} = Ame^{mt}$

or

$$\frac{d}{dt}(A \cdot m e^{mt}) = 2Ae^{mt} \qquad (\because \ddot{x} = 2x)$$

or

$$A \cdot m^2 e^{mt} = 2Ae^{mt}$$

or

$$m^2 = 2 \qquad \text{or} \qquad m = \pm\sqrt{2}$$

Hence

$$x(t) = A_1 e^{\sqrt{2}t} + A_2 e^{-\sqrt{2}t}$$

Now, $x(0) = 1$; thus, $1 = A_1 + A_2$.

Again, at the other end $t = 3$, $x(t)$ is free and t is fixed at the end point. That is,

$$\left.\frac{\delta L}{\delta \dot{x}}\right|_{t=3} = 0$$

or

$$(2\dot{x} + 2x)_{t=3} = 0$$

or

$$(\dot{x} + x)_{t=3} = 0$$

Now,

$$\dot{x} = \sqrt{2}\, A_1 e^{\sqrt{2}t} - \sqrt{2} A_2 e^{-\sqrt{2}t}$$

Therefore,

$$\left(\sqrt{2}\, A_1 e^{\sqrt{2}t} - \sqrt{2} A_2 e^{-\sqrt{2}t} + A_1 e^{\sqrt{2}t} + A_2 e^{-\sqrt{2}t}\right)_{t=3} = 0$$

or

$$\left[A_1 e^{\sqrt{2}t}\left(\sqrt{2}+1\right) + A_2 e^{-\sqrt{2}t}(1-\sqrt{2})\right]_{t=3} = 0$$

When $t = 3$, we have

$$A_1 e^{3\sqrt{2}}\left(\sqrt{2}+1\right) + A_2 e^{-3\sqrt{2}}\left(1-\sqrt{2}\right) = 0$$

or

$$(1 - A_2) e^{3\sqrt{2}}\left(\sqrt{2}+1\right) + A_2 e^{-3\sqrt{2}}(1-\sqrt{2}) = 0 \quad (\because A_1 = 1 - A_2)$$

or

$$(1 - A_2) e^{3\sqrt{2}} \times 2.414 + A_2 e^{-3\sqrt{2}} \times (-0.414) = 0$$

or

$$A_2\left(-e^{-3\sqrt{2}} \times 0.414 - 2.414 e^{3\sqrt{2}}\right) = -\,2.414 e^{3\sqrt{2}}$$

or

$$A_2 = \frac{2.414 e^{3\sqrt{2}}}{2.414 e^{3\sqrt{2}} + 0.414 e^{-3\sqrt{2}}}$$

Therefore,

$$A_1 = 1 - \frac{2.414 e^{3\sqrt{2}}}{2.414 e^{3\sqrt{2}} + 0.414 e^{-3\sqrt{2}}}$$

$$= \frac{0.414 e^{-3\sqrt{2}}}{2.414 e^{3\sqrt{2}} + 0.414 e^{-3\sqrt{2}}}$$

Hence, the outcome is

$$x(t) = A_1 e^{\sqrt{2}t} + A_2 e^{-\sqrt{2}t}$$

$$= \frac{0.414 e^{-3\sqrt{2}}}{2.414 e^{3\sqrt{2}} + 0.414 e^{-3\sqrt{2}}} e^{\sqrt{2}t} + \frac{2.414 e^{3\sqrt{2}}}{2.414 e^{3\sqrt{2}} + 0.414 e^{-3\sqrt{2}}} e^{-\sqrt{2}t}$$

EXAMPLE 13.7 Optimize the following performance index

$$J = \int_0^3 \left(\frac{d^2\phi}{dt^2}\right)^2 dt$$

when the equality constraints are

$$\phi(0) = 1, \quad \phi(3) = 0$$

$$\dot{\phi}(0) = 1, \quad \dot{\phi}(3) = 0$$

Solution Let us consider

$$u(t) = \ddot{\phi}(t) = \frac{d^2\phi}{dt^2}$$

then,

$$J = \int_0^3 u^2(t)\,dt$$

Let

$$x_1(t) = \phi(t)$$

$$x_2(t) = \phi(t)$$

$$\dot{x}_1(t) = x_2(t)$$

$$\dot{x}_2(t) = \ddot{\phi}(t) = u(t)$$

According to the state space equation,

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u(t)$$

or

$$\dot{X} = AX + Bu$$

The modified performance index (optimization with constraints)

$$J' = \int_0^3 \left[u^2(t) + \lambda^T \left\{AX + Bu - \dot{x}\right\}\right]dt$$

Now

$$\lambda^T [AX + Bu - \dot{x}]$$

$$= [\lambda_1 \quad \lambda_2]\left\{\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u\right\} - \begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}$$

$$= [\lambda_1 \quad \lambda_2]\begin{bmatrix}x_2 - \dot{x}_1\\ u - \dot{x}_2\end{bmatrix}$$

$$= \lambda_1(x_2 - \dot{x}_1) + \lambda_2(u - \dot{x}_2)$$

Therefore,

$$J' = \int_0^3 [u^2(t) + \lambda_1(x_2 - \dot{x}_1) + \lambda_2(u - \dot{x}_2)\,dt$$

The modified Lagrangian is

$$L' = u^2(t) + \lambda_1(x_2 - \dot{x}_1) + \lambda_2(u - \dot{x}_2)$$

Utilizing the Euler-Lagrange equations, we get

$$\frac{\delta L'}{\delta x_1} - \frac{d}{dt}\frac{\delta L'}{\delta \dot{x}_1} = 0 \tag{1}$$

$$\frac{\delta L'}{\delta x_2} - \frac{d}{dt}\frac{\delta L'}{\delta \dot{x}_2} = 0 \tag{2}$$

$$\frac{\delta L'}{\delta u} - \frac{d}{dt}\frac{\delta L'}{\delta \dot{u}} = 0 \tag{3}$$

Equation (1) provides the following outcome

$$-\frac{d}{dt}(-\lambda_1) = 0 \qquad \text{or} \qquad \dot{\lambda}_1 = 0$$

Equation (2) provides the following outcome.

$$\lambda_1 - \frac{d}{dt}(-\lambda_2) = 0 \qquad \text{or} \qquad \lambda_1 + \dot{\lambda}_2 = 0 \qquad \text{or} \qquad \lambda_1 = -\dot{\lambda}_2$$

Equation (3) provides the following outcome.

$$2u + \lambda_2 = 0 \qquad \text{or} \qquad 2u = -\lambda_2$$

Since $\dot{\lambda}_1 = 0$, we get

$$\lambda_1 = K_1 \text{ (where } K_1 \text{ is a constant)}$$

Since $\dot{\lambda}_2 = -\lambda_1 = = -K_1$, we get

$$\lambda_2 = -K_1 t + K_2$$

Since $2u = -\lambda_2$, we get

$$2u = K_1 t - K_2$$

or

$$u = \frac{1}{2}(K_1 t - K_2)$$

Since $u(t) = \ddot{\phi}(t)$ and $\ddot{\phi}(t) = \dot{x}_2(t)$, we get

$$\dot{x}_2(t) = \frac{1}{2}K_1 t - \frac{1}{2}K_2$$

or

$$x_2(t) = \frac{1}{4}K_1 t^2 - \frac{1}{2}K_2 t + K_3 \tag{4}$$

Since $x_2(t) = \dot{x}_1(t)$, we get

$$\dot{x}_1(t) = \frac{1}{4}K_1 t^2 - \frac{1}{2}K_2 t + K_3$$

If the performance index = $\int_0^3 (u_1^2 + u_2^2)dt$, find the optimum value of the same when $X^T(0) = [1, 1]$ and $X_1(3) = 0$.

Solution Since

$$\dot{X} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} X + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u$$

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

we have

$$\dot{X}_1 = X_2 + u_1 \qquad \text{or} \qquad X_2 + u_1 - \dot{X}_1 = 0$$

and

$$\dot{X}_2 = u_2 \qquad \text{or} \qquad u_2 - \dot{X}_2 = 0$$

When the performance index is, $J = \int_0^3 (u_1^2 + u_2^2)dt$, then the modified performance index will be

$$J' = \int_0^3 \left[(u_1^2 + u_2^2) + \lambda^T (AX + Bu - \dot{X}) \right] dt$$

Now,

$$\lambda^T(AX + Bu - \dot{X}) = [\lambda_1 \quad \lambda_2] \left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - \begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} \right\}$$

$$= [\lambda_1 \quad \lambda_2] \left\{ \begin{bmatrix} X_2 \\ 0 \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - \begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} \right\}$$

$$= [\lambda_1 \quad \lambda_2] \begin{bmatrix} X_2 + u_1 - \dot{X}_1 \\ u_2 - \dot{X}_2 \end{bmatrix}$$

$$= \lambda_1(X_2 + u_1 - \dot{X}_1) + \lambda_2(u_2 - \dot{X}_2)$$

Thus,

$$J' = \int_0^3 \left[(u_1^2 + u_2^2) + \lambda_1(X_2 + u_1 - \dot{X}_1) + \lambda_2(u_2 - \dot{X}_2) \right] dt$$

Applying the Euler-Langrange equation,

$$\frac{\delta L'}{\delta X_1} - \frac{d}{dt}\frac{\delta L'}{\delta \dot{X}_1} = 0 \tag{1}$$

$$\frac{\delta L'}{\delta X_2} - \frac{d}{dt}\frac{\delta L'}{\delta \dot{X}_2} = 0 \quad (2)$$

$$\frac{\delta L'}{\delta u_1} - \frac{d}{dt}\frac{\delta L'}{\delta \dot{u}_1} = 0 \quad (3)$$

$$\frac{\delta L'}{\delta u_2} - \frac{d}{dt}\frac{\delta L'}{\delta \dot{u}_2} = 0 \quad (4)$$

Here

$$L' = u_1^2 + u_2^2 + \lambda_1(X_2 + u_1 - \dot{X}_1) + \lambda_2(u_2 - \dot{X}_2)$$

From (1)

$$\frac{-d}{dt}(-\lambda_1) = 0 \qquad \text{or} \qquad \dot{\lambda}_1 = 0 \quad (5)$$

From (2)

$$\lambda_1 - \frac{d}{dt}(-\lambda_2) = 0 \qquad \text{or} \qquad \lambda_1 + \dot{\lambda}_2 = 0$$

or

$$\dot{\lambda}_2 = -\lambda_1 \quad (6)$$

From (3)

$$2u_1 + \lambda_1 = 0 \qquad \text{or} \qquad u_1 = -\frac{\lambda_1}{2} \quad (7)$$

From (4)

$$2u_2 + \lambda_2 = 0 \qquad \text{or} \qquad u_2 = -\frac{\lambda_2}{2} \quad (8)$$

From (5)

$$\lambda_1 = K_1$$

From (6)

$$\lambda_2 = -\lambda_1 t + K_2 = -K_1 t + K_2 \qquad (\because \lambda_1 = K_1)$$

From (7)

$$u_1 = \frac{-K_1}{2}$$

From (8)

$$u_2 = \frac{-\lambda_2}{2} = -\frac{(-K_1 t + K_2)}{2} = \frac{K_1 t - K_2}{2}$$

$$\dot{X}_2 = u_2 \frac{K_1 t}{2} - \frac{K_2}{2}$$

Therefore,

$$X_2 = \frac{K_1 t^2}{4} - \frac{K_2 t}{2} + K_3$$

$$\dot{X}_1 = X_2 + u_1$$

$$= \frac{K_1 t^2}{4} - \frac{K_2 t}{2} + K_3 + \left(-\frac{K_1}{2}\right)$$

Thus,

$$X_1 = \frac{K_1 t^3}{12} - \frac{K_2 t^2}{4} + K_3 t - \frac{K_1}{2} t + K_4$$

$$= K_1\left(\frac{t^3}{12} - \frac{t}{2}\right) - \frac{K_2 t^2}{4} + K_3 t + 4$$

Now, $X_1(0) = 1$, $X_2(0) = 1$, $X_1(3) = 0$. Hence

$$1 = K_4 \quad (\because X_1(0) = 1)$$

$$1 = K_3 \quad (\because X_2(0) = 1)$$

Since $X_1(3) = 0$, we get

$$0 = K_1\left(\frac{27}{12} - \frac{3}{2}\right) - K_2\frac{9}{4} + 3K_3 + K_4$$

$$= K_1\left(\frac{27-18}{12}\right) - K_2\frac{9}{4} + 3 + 1$$

or

$$-4 = K_1\frac{3}{4} - K_2\frac{9}{4}$$

or

$$-16 = 3K_1 - 9K_2$$

Since at $t = 3$, t is fixed and X is free

$$\frac{\delta L'}{\delta \dot{X}_2} = 0 \quad \text{at } t = 3$$

$$-\lambda_2 = 0 \quad \text{at } t = 3 \qquad \text{or} \qquad \lambda_2(3) = 0$$

Again,

$$\lambda_2 = -K_1 t + K_2$$

$$0 = -K_1 \times 3 + K_2 \qquad \text{or} \qquad K_2 = 3K_1$$

Therefore,

$$-16 = 3K_1 - 9K_2 = K_2 - 9K_2 = -8K_2$$

or

$$K_2 = 2$$

Thus,

$$K_2 = 2 = 3K_1 \quad \text{or} \quad K_1 = \frac{2}{3}$$

$$\left(\because \frac{\delta}{\delta X}\lambda^T \dot{X} = 0,\ \frac{d}{dt}\frac{\delta L}{\delta \dot{X}} = 0,\ \frac{d}{dt}\frac{\delta \lambda^T f}{\delta \dot{X}} = 0, \text{and } f \text{ and } L \text{ are not the function of } \dot{X}\right)$$

$L + \lambda^T f$ is defined by the new function H and is termed the state function of Pontryagin. Thus,

$$H(X, u, \lambda, t) = L(X, u, t) + \lambda^T f(X, u, t)$$

From $\frac{\delta}{\delta X}[L + \lambda^T f] + \dot{\lambda} = 0$, we have

$$\frac{\delta H}{\delta X} + \dot{\lambda} = 0 \qquad \text{or} \qquad \dot{\lambda} = -\frac{\delta H}{\delta X}$$

Again,

$$\frac{\delta}{\delta u}[L + \lambda^T (f - \dot{X})] - \frac{d}{dt}\frac{\delta}{\delta \dot{u}}[L + \lambda^T (f - \dot{X})] = 0$$

or

$$\frac{\delta}{\delta u}[L + \lambda^T f] = 0 \qquad \text{or} \qquad \frac{\delta H}{\delta u} = 0$$

Now,

$$\frac{\delta H}{\delta \lambda} = \frac{\delta}{\delta \lambda}(L + \lambda^T f) = f = \dot{X} \qquad (\because f(X, u, t) = \dot{X})$$

Hence the following three equations have been determined:

1. $\frac{\delta H}{\delta X} + \dot{\lambda} = 0$
2. $\frac{\delta H}{\delta u} = 0$
3. $\dot{X} = \frac{\delta H}{\delta \lambda}$

With the above three equations, the optimal control problem can be solved in the following manner.

1. First of all, solve $\frac{\delta H}{\delta u} = 0$ and determine the solution of u in terms of X, λ, and t. Say the solution is $u^0 = u^0(X, \lambda, t)$.
2. Now apply this result to the Pontryagin's function. The optimal Pontryagin function will be

$$H^0(X, \lambda, t) = H(X, u^0[X, \lambda, t), \lambda, t]$$

3. Now apply

$$\dot{X} = \frac{\delta H^0}{\delta \lambda} \qquad \text{and} \qquad \dot{\lambda} = \frac{-\partial H^0}{\partial X}$$

The optimal $X^0(t)$ and $\lambda(t)$ are found by solving the above equations with the boundary conditions which are given as the data.

EXAMPLE 13.9 The performance index of the system is, $J = \int_0^3 (x_1^2 + u^2)\,dt$. The system is described by $\dot{X}_1 = u - X_1$, $X_1(0) = 1$. Find $u(t)$ to minimize the system considering the above two-point boundary value problem (*TPBVP*).

Solution Let us define a state variable as

$$\dot{X}_0 = X_1^2 + u^2 \text{ and } X_0(0) = 0$$

Therefore,

$$J = \int_0^3 (X_1^2 + u^2)\,dt = \int_0^3 \dot{X}_0\,dt = \left[X_0\right]_0^3$$

$$= X_0(3) - X_0(0)$$

Now, $X_0(0) = 0$ (already defined). Therefore,

$$J = X_0(3)$$

When $J = X_0(3)$ is to be minimized, $J = -X_0(3)$ is to be maximized.

Since $\dfrac{\delta H}{\delta \lambda} = f$, we can write in summation form

$$H(X, \lambda, u, t) = \sum_{n=0}^{1} \lambda_n f_n = \lambda_0 f_0 + \lambda_1 f_1$$

Since there are two state variables, one is $\dot{X}_1$ and the other one is $\dot{X}_0$ which is defined, we have

$$H = \lambda_0 \dot{X}_0 + \lambda_1 \dot{X}_1$$

and

$$\frac{\delta H}{\delta \lambda_0} = \dot{X}_0 = f_0 \qquad [\because \dot{X}_0 = f_0, \dot{X}_1 = f_1 \text{ and } \dot{X} = f(X, u, t)]$$

Thus,

$$H = \lambda_0 (X_1^2 + u^2) + \lambda_1 (u - X_1) \tag{1}$$

Let us consider the terminal conditions:

$$\lambda_0(3) = -C_0$$

$$\lambda_1(3) = -C_1$$

$$J = C_0 X_0(3) + C_1 X_1(3)$$

Here,

$$C_0 = 1,\ C_1 = 0 \qquad (\because J = X_0(3))$$

Again for optimal condition

$$\dot{\lambda} = -\frac{\delta H}{\delta X} \qquad \text{and} \qquad \frac{\delta H}{\delta u} = 0$$

Now,

$$\frac{\delta H}{\delta X_0} = -\dot{\lambda}_0 = 0 \quad \text{(from Eq. (1))}$$

$$\lambda_0(3) = -1 \quad \because \lambda_0(3) = -C_0 \text{ and } C_0 = 1$$

From

$$\frac{\delta H}{\delta X_1} = -\dot{\lambda}_1 \text{ and } \frac{\delta H}{\delta X_1} = 2X_1\lambda_0 - \lambda_1 \quad \text{(from Eq. (1))}$$

we have

$$-\dot{\lambda}_1 = 2X_1\lambda_0 - \lambda_1 \text{ or } \dot{\lambda}_1 = \lambda_1 - 2X_1\lambda_0$$

Now,

$$\lambda_1(3) = -C_1 = 0, \frac{\delta H}{\delta u} = 0$$

and

$$\frac{\delta H}{\delta u} = 0 = 2u\lambda_0 + \lambda_1 \quad \text{(From Eq. (1))}$$

At the end terminal, $\lambda_0(3) = -1$,
Again

$$2u\lambda_0 + \lambda_1 = 0 \quad \text{or} \quad u = -\frac{\lambda_1}{2\lambda_0}$$

Hence $u(t)$ at the terminal when $t = 3$ will be

$$u = \frac{-\lambda_1}{2(-1)} = \frac{\lambda_1}{2}$$

Again

$$\dot{X}_1 = u - X_1 \quad \text{or} \quad \dot{X}_1 = \frac{\lambda_1}{2} - X_1$$

and

$$X_1(0) = 1$$

and

$$\dot{\lambda}_1 = \lambda_1 - 2X_1\lambda_0$$

$$= \lambda_1 + 2X_1 \quad [\because \text{at the end terminal } \lambda_0(3) = -1]$$

$$\lambda_1(3) = 0$$

Taking the Laplace transform of $\dot{X}_1 = \frac{\lambda_1}{2} - X_1$, we get

$$sX_1(s) - X_1(0) = \frac{\lambda_1(s)}{2} - X_1(s)$$

or

$$X_1(t) = \frac{1}{2}\Bigg[\lambda_1(0)\left\{\frac{\sqrt{2}-1}{2\sqrt{2}}\left(-\sqrt{2}\,e^{-\sqrt{2}t}\right)+\frac{\sqrt{2}+1}{2\sqrt{2}}\left(\sqrt{2}\,e^{\sqrt{2}t}\right)\right\} -$$

$$\frac{1}{\sqrt{2}}\left(-\sqrt{2}\,e^{-\sqrt{2}t}\right)+\frac{1}{\sqrt{2}}\left(\sqrt{2}\,e^{\sqrt{2}t}\right)-\lambda_1(0)\left\{\frac{\sqrt{2}-1}{2\sqrt{2}}e^{-\sqrt{2}t}+\frac{\sqrt{2}+1}{2\sqrt{2}}e^{\sqrt{2}t}\right\}+\frac{1}{\sqrt{2}}e^{-\sqrt{2}t}-\frac{1}{\sqrt{2}}e^{\sqrt{2}t}\Bigg]$$

$$= \frac{1}{2}\Bigg[\lambda_1(0)\left\{-\frac{\sqrt{2}-1}{2}e^{-\sqrt{2}t}+\frac{\sqrt{2}+1}{2}e^{\sqrt{2}t}-\frac{\sqrt{2}-1}{2\sqrt{2}}e^{-\sqrt{2}t}-\frac{\sqrt{2}+1}{2\sqrt{2}}e^{\sqrt{2}t}\right\}+e^{-\sqrt{2}t}+e^{\sqrt{2}t}$$

$$+\frac{1}{\sqrt{2}}e^{-\sqrt{2}t}-\frac{1}{\sqrt{2}}e^{\sqrt{2}t}\Bigg]$$

Now putting the value of $\lambda_1(0)$, $X_1(t)$ can be determined.

Since both the $\lambda_1(t)$ and $X_1(t)$ are calculated at the optimum condition, $u(t)$ can be determined from the formula

$$u(t) = \frac{\lambda_1(t)}{2}$$

13.6.3 Bang-Bang Control

Suppose, we have to find out a control input for transferring a dynamical system from a given initial state X_0 at time t_0 to a required final state X_f in minimum time. Over and above, the magnitude of the control inputs are within certain non-equality constraints. For example, the input $u(t)$ which is to be found out may lie between M_1 and M_2, that means $M_1 \le u(t) \le M_2$. If the number of inputs are more, the generalized form is

$$M_{j1} \le u_j(t) \le M_{j2}$$

where $j = 1, 2, \ldots, n$.

Let us consider that the system is linear, time-invariant dynamic system.

The performance index will be

$$J = \int_{t_0}^{t_f} 1 \cdot dt = \int_{t_0}^{t_f} dt$$

Here the end terminal time t_f is free, but the final state $X(t_f)$ is fixed.

The dynamic LTI system is

$$\dot{X} = AX + Bu$$

Now, $\dot{X} = f$. Therefore,

$$H = L + \lambda^T f = L + \lambda^T(AX + Bu)$$

Here $L = 1$, since

$$J = \int_{t_0}^{t_f} 1 \cdot dt$$

$$H = 1 + \lambda^T(AX + Bu)$$

This H is also termed 'Hamiltonian' function. According to Pontryagin's optimal policy,

$$\dot{X}^0 = AX^0 + Bu^0$$

$$\dot{\lambda}^0 = -\frac{\delta H}{\delta X} = -A^T\lambda^0$$

(The suffix 0 indicates optimal value.)

We also know, $\frac{\delta}{\delta X}\lambda^T AX = A^T\lambda$.

Let us consider that the optimum value of the input is u^0 and at that time $H(X^0, u^0, \lambda^0)$ will be minimum. That means mathematically, we can say

$$H(X^0, u^0, \lambda^0) \le H(X^0, u, \lambda^0)$$

Putting the vale of H, we can write

$$\lambda^{0T}AX^0 + \lambda^{0T}Bu^0 \le \lambda^{0T}AX^0 + \lambda^{0T}Bu$$

or

$$\lambda^{0T}Bu^0 \le \lambda^{0T}Bu$$

Again if the matrix B is considered as, $B = b_1b_2, ..., b_n$, that means, B is of n vectors, we can then write

$$\lambda^{0T}Bu = \sum_{i=1}^{n} \lambda^{0T} b_i u_i$$

The input vectors $u_1, u_2, ..., u_n$ are independent of one another. Hence to minimize $\lambda^{0T}Bu$, all values of $\lambda^{0T}b_j u_j$ for $j = 1$ to n are to be minimized. If the coefficient of $u_j(t)$ is positive, we have to select, $u_j^0(t)$, the smallest one, that means, as per the constraint it would be M_{j1}, whereas if the coefficient $u_j(t)$ is negative, we have to select, $u_j^0(t)$ the largest one, that means, according to the constraint it would be M_{j2}. Thus,

$$u_j^0(t) = \begin{array}{ll} M_{j2} & \text{for } \lambda^{0T}b_j < 0 \\ M_{j1} & \text{for } \lambda^{0T}b_j > 0 \\ \text{indeterminable for } \lambda^{0T}b_j = 0 & \end{array}$$

We usually assume that $\lambda^{0T} b_j \neq 0$ for finite interval of time. This time the optimal problem is termed bang-bang in nature. The optimal control input will be either of the constraint limits.

To find optimal trajectories

Suppose a system is described as follows:

$$\dot{X}_1 = X_2$$

$$\dot{X}_2 = u$$

$$|u(t)| \le N$$

We have to find out the optimal control input $u^0(t)$ which will transfer the system state from a specified initial state X_0 to a specified final state X_f in a minimum time.

Since

$$\dot{X}_1 = X_2$$

$$\dot{X}_2 = u$$

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

Therefore,

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The Hamiltonian function,

$$H = L + \lambda^T(AX + Bu)$$

For minimum time problem

$$L = 1$$

Therefore,

$$H = 1 + \lambda^T(AX + Bu)$$

$$= 1 + [\lambda_1 \lambda_2] \left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \right\}$$

$$= 1 + [\lambda_1 \lambda_2] \left\{ \begin{bmatrix} X_2 \\ u \end{bmatrix} \right\}$$

$$= 1 + \lambda_1 X_2 + \lambda_2 u$$

According to Pontryagin's optimal principle, we have

$$\frac{\delta H}{\delta \lambda_1} = \dot{X}_1$$

$$\dot{X}_1 = X_2$$

$$\frac{\delta H}{\delta \lambda_2} = \dot{X}_2$$

or

$$u = \dot{X}_2$$

Again

$$\dot{\lambda}_1 = -\frac{\delta H}{\delta X_1}$$

$$\dot{\lambda}_1 = 0 \qquad (\because H = 1 + \lambda_1 X_2 + \lambda_2 u)$$

$$\dot{\lambda}_2 = -\frac{\delta H}{\delta X_2}$$

or

$$\dot{\lambda}_2 = -\lambda_1$$

If the optimal values are expressed as λ_1^0, λ_2^0, u^0, X^0, then we have already shown

$$\lambda^{0T}AX^0 + \lambda^{0T}Bu^0 \le \lambda^{0T}AX^0 + \lambda^0TBu$$

or

$$\lambda^{0T}Bu^0 \le \lambda^{0T}Bu$$

Now,

$$\lambda^{0T}Bu^0 = [\lambda_1^0\lambda_2^0]\begin{bmatrix}0\\1\end{bmatrix}u^0 = \lambda_2^0u^0$$

Therefore,

$$\lambda_2^0\,u^0 \le \lambda_2^0\,u$$

When $\lambda_2^0\,u^0 \le \lambda_2^0\,u$ and $|u(t)| \le N$, then the case can be described in terms of the Signum function in the following manner.

$$u^0(t) = -N\ \text{Sgn}\ [\lambda_2^0(t)]$$

$$\text{Sgn}\ [\lambda_2^0(t)] = \begin{cases} +1 & \text{for} \ \ \lambda_2^0(t) > 0 \\ -1 & \text{for} \ \ \lambda_2^0(t) < 0 \\ \text{undefind} & \text{for} \ \ \lambda_2^0(t) = 0 \end{cases}$$

At the optimum condition $\dot{\lambda}_1 = 0$; that means, $\dot{\lambda}_1^0 = 0$

or

$$\lambda_1^0(t) = c$$

Similarly for optimum condition, $\dot{\lambda}_2 = -\lambda_1$

That means, $\dot{\lambda}_2^0 = -\lambda_1^0$

or

$$\dot{\lambda}_2^0 = -c$$

or

$$\lambda_2^0(t) = -ct + d$$

where c and d are arbitrary constants.

From $\lambda_2^0(t) = -ct + d$, it is clear that the sign of $\lambda_2^0(t)$ will be changed only once when t varies on the real line.

Now the optimal trajectories can be determined in the following manner. Let us take some value of N. Say, for example, the value of N is one. The final state at which we have to reach in minimum time is considered, say, the origin.

When $u(t) = 1$,

$$\dot{X}_2(t) = 1$$

Integrating

$$X_2(t) = t + K_1 \tag{13.26}$$

Again,

$$\dot{X}_1 = X_2$$

or

$$\dot{X}_1 = t + K_1$$

Integrating

$$X_1(t) = \frac{t^2}{2} + K_1 t + K_2$$

where K_1 and K_2 are constants. Squaring (13.26), we obtain

$$X_2^2 = t^2 + 2tK_1 + K_1^2$$

$$= 2\left(\frac{t^2}{2} + K_1 t + K_2\right) - 2K_2 + K_1^2$$

$$= 2X_1 + K_3 \qquad (\text{where } K_3 = K_1^2 - 2K_2)$$

The above equation is the equation of a series of parabolas for different values of K_3.

Figure 13.10 shows the trajectories indicating the direction of increasing t by arrows. Hence, if the state point is to reach the origin with the application $u(t) = 1$, it can come only through the trajectory PO. Suppose,

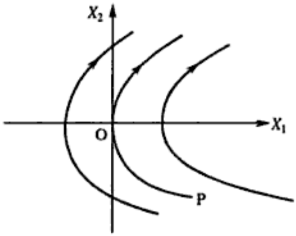


Fig. 13.10 Trajectories.

$$u(t) = -1$$

$$\dot{X}_2 = -1, \quad \text{i.e.} \quad X_2 = -t + K_4$$

Again,

$$\dot{X}_1 = X_2 \quad \text{or} \quad \dot{X}_1 = -t + K_4$$

or

$$X_1 = -\frac{t^2}{2} + K_4 t + K_5$$

Now,

$$X_2^2 = t^2 - 2tK_4 + K_4^2$$

$$= -2\left(-\frac{t^2}{2} + K_4 t + K_5\right) + 2K_5 + K_4^2 = -2X_1 + K_6$$

where K_6 is a constant.

Figure 13.11 shows the curve X_2 versus X_1 for different values of K_6. These are all parabolas. The arrows indicate the direction of the movement of the trajectories with the increasing value of time. QO will be the trajectory for proceeding towards the origin since the optimal control input occurs on values at either of the constraint limits specified, the optimal control input $u(t)$ will provide in this problem either +1 or −1.

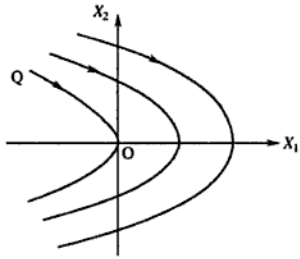


Fig. 13.11

Figure 13.12 shows the series of the time optimal trajectories. Even if the initial state is not on PO or QO, the initial input will be such that it will transfer the trajectory either to PO or to QO curve. Thus the input will switch from +1 to –1 or vice versa and the final state will be at the origin.

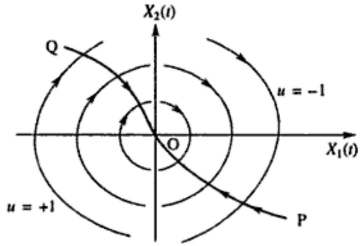


Fig. 13.12 Time optimal trajectories

EXAMPLE 13.10 Apply the calculus of variations and Pontryagin's optimum policy on the following system and show that the optimal input will be the same in both the cases.

$$J = \int_0^2 (X^2 + m^2)\, dt$$

$$\dot{X} = -X + m$$

where m is the input.

Solution The modified performance index will be in calculus of variations,

$$J = \int_0^2 \left[(X^2 + m^2) + \lambda(-X + m - \dot{X})\right] dt$$

Using the Euler-Lagrange equation, we get

$$\frac{\delta L'}{\delta X} - \frac{d}{dt}\frac{\delta L'}{\delta \dot{X}} = 0$$

$$\frac{\delta L'}{\delta m} - \frac{d}{dt}\frac{\delta L'}{\delta \dot{m}} = 0$$

where $L' = X^2 + m^2 + \lambda(-X + m - \dot{X})$.

From

$$\frac{\delta L'}{\delta X} - \frac{d}{dt}\frac{\delta L'}{\delta \dot{X}} = 0$$

we get

$$2X - \lambda - \frac{d}{dt}(-\lambda) = 0$$

or

$$2X - \lambda + \dot{\lambda} = 0$$

From

$$\frac{\delta L'}{\delta m} - \frac{d}{dt}\frac{\delta L'}{\delta \dot{m}} = 0$$

we get

$$2m + \lambda = 0 \quad \text{or} \quad \lambda = -2m$$

Now,

$$2X - \lambda + \dot{\lambda} = 0 \tag{1}$$

$$2m + \lambda = 0 \tag{2}$$

$$\dot{X} = -X + m \tag{3}$$

Thus,

$$2X + 2m + \dot{\lambda} = 0$$

$$\dot{\lambda} = -2\dot{m} \quad (m = \dot{X} + X \quad \text{or} \quad \dot{m} = \ddot{X} + \dot{X})$$

or

$$\dot{\lambda} = -2(\ddot{X} + \dot{X})$$

Therefore, from (1)

$$2X + 2(\dot{X} + X) - 2(\ddot{X} + \dot{X}) = 0 \qquad \text{or} \qquad \ddot{X} - 2X = 0$$

or

$$\frac{d^2X}{dt^2} - 2X = 0$$

Let

$$X = Ae^{mt}$$

Therefore,

$$Am^2e^{mt} - 2Ae^{mt} = 0 \quad \text{or} \quad m^2 - 2 = 0$$

or

$$m = \pm\sqrt{2}$$

Therefore,

$$X = A_1 e^{\sqrt{2}t} + A_2 e^{-\sqrt{2}t}$$

Let

$$\alpha(X, t) = X^T R(t) X = \int_t^N (X^T Q X + U^T P U)\, d\tau$$

where $R(t)$ is a positive-definite matrix.

When $t = N$, $R(t) = 0$ since the right-hand side integration will be zero. Now,

$$\frac{\delta \alpha}{\delta X} = \nabla \alpha = 2R(t)X$$

$$u = -\frac{1}{2} P^{-1} B^T 2R(t) X = -P^{-1} B^T R(t) X$$

$$\frac{\delta \alpha}{\delta t} = X^T \dot{R}(t) X$$

Putting the above values into (13.27), we get

$$\{2R(t)X\}^T AX - \frac{1}{4}\{2R(t)X\}^T BP^{-1} B^T 2R(t)X + X^T QX + \frac{\delta}{\delta t} X^T R(t) X = 0$$

or

$$2X^T R(t) AX - X^T R(t) BP^{-1} B^T R(t) X + X^T QX + X^T \dot{R}(t) X = 0$$

$$\left(\because \frac{\delta \alpha}{\delta t} = X^T \dot{R}(t) X, \text{ where } \nabla \alpha = 2R(t) X\right)$$

or

$$X^T [2R(t)A - R(t) BP^{-1} B^T R(t) + Q + \dot{R}(t)] X = 0$$

If $2R(t)A$ is symmetric, then

$$2R(t)A = 2\frac{R(t)A + A^T R(t)}{2}$$

$$= R(t)A + A^T R(t)$$

or

$$X^T [R(t)A + A^T R(t) - R(t) BP^{-1} B^T R(t) + Q + \dot{R}(t)] X = 0$$

The above equation is termed Matrix–Riccati equation.

When the upper limit N tends to ∞, $R(t)$ will be constant (Say R). Therefore,

$$\dot{R}(t) = 0$$

Hence, the Matrix–Riccati equation becomes

$$A^T R + RA - RBP^{-1} B^T R + Q = 0$$

EXAMPLE 13.11 Determine the optimum input when the performance index is

$$\int_0^\infty X^T \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} X + M^T \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} M\, dt$$

where M is the input and the system equation is

Thus, we get

$$-(a^2 + b^2) + 1 = 0 \quad \text{or} \quad a^2 + b^2 = 1 \tag{1}$$

$$a - (ab + bc) = 0 \quad \text{or} \quad a = b(a + c) \tag{2}$$

$$-(b^2 + c^2) + 1 + 2b = 0 \quad \text{or} \quad b^2 + c^2 = 1 + 2b \tag{3}$$

Now, (1) – (3)

$$a^2 - c^2 = -2b \quad \text{or} \quad (a + c)(a - c) = -2b$$

or

$$\frac{a}{b}(a - c) = -2b \quad [\because \text{ from (2), } a + c = \frac{a}{b}]$$

or

$$a - c = -\frac{2b^2}{a} \tag{4}$$

Again

$$a + c = \frac{a}{b} \tag{5}$$

Adding (4) and (5), we get

$$2a = -\frac{2b^2}{a} + \frac{a}{b} \quad \text{or} \quad a^2 = \frac{2b^3}{1 - 2b} \tag{6}$$

Since

$$a^2 + b^2 = 1$$

we have

$$\frac{2b^3}{1 - 2b} + b^2 = 1 \quad \text{or} \quad b^2 + 2b - 1 = 0$$

or

$$b = -1 \pm \sqrt{2} = 0.414, -2.414$$

Since $b = 0.414$ (a positive number) and $a^2 + b^2 = 1$, we have

$$a^2 = 1 - (0.414)^2 \quad \text{or} \quad a = 0.91$$

Also

$$a + c = \frac{a}{b} \quad \text{or} \quad 0.91 + c = \frac{0.91}{0.414}$$

or

$$c = 1.29$$

Now the optimal input will be

$$M = -P^{-1} B^T R X$$

$$= -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0.91 & 0.414 \\ 0.414 & 1.29 \end{bmatrix} X(t)$$

$$= -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0.91 & 0.414 \\ 0.414 & 1.29 \end{bmatrix} X(t)$$

$$= -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0.91 & 0.414 \\ 0.414 & 1.29 \end{bmatrix} X(t) = -\begin{bmatrix} 0.91 & 0.414 \\ 0.414 & 1.29 \end{bmatrix} X(t)$$

Table 13.1

From state	*To state*	u_i*(required)*	X_i*(average)*	$\Delta J = (X_i + 2u_i)\Delta t$
a	*a*	0	5	5
a	*b*	–1	4.5	2.5
a	*c*	–2	4.0	0
b	*a*	+1	4.5	6.5
b	*b*	0	4.0	4.0
b	*c*	–1	3.5	1.5
b	*d*	–2	3.0	–1.0
c	*a*	+2	4.0	8.0
c	*b*	+1	3.5	5.5
c	*c*	0	3.0	3.0
c	*d*	–1	2.5	0.5
c	*e*	–2	2.0	–2.0
d	*b*	+2	3.0	7.0
d	*c*	+1	2.5	4.5
d	*d*	0	2.0	2.0
d	*e*	–1	1.5	–0.5
e	*c*	+2	2.0	6.0
e	*d*	+1	1.5	3.5
e	*e*	0	1.0	1.0

Hence the minimum value of the objective function

$$= 0 - 2 + 1 + 1 + 3.5 = 3.5$$

Basic equations of dynamic programming can be derived in the following manner. First of all, we have to take the performance index leaving the final decision.
That means

$$J = \sum_{K=0}^{N-1} f_K(X_K, u_K, S_K)$$

where X is the present state, u is the input and S is the possible state. Since one stage is left, the performance index available for optimization will be $J_1 = f_{N-1}(X_{N-1}, u_{N-1}, S_{N-1})$

Let us assume U^0_{N-1} which optimizes the performance index. That is,

$$U^0_{N-1} = \psi_{N-1}(X_{N-1}, S_{N-1})$$

According to the imbedding principle, U^0_{N-1} is found out by all the possible values of X_{N-1}. The value of X_{N-1} is unknown because the optimal policy over the interval 0 to $(N-2)$ has not yet been found out. Therefore,

$$J^0_1 = J^0_1(X_{N-1})$$

Now let us consider the case where two stages remain before the end of the process. That is,

$$J_2 = f_{N-2}(X_{N-2}, U_{N-2}, S_{N-2}) + f_{N-1}(X_{N-1}, U_{N-1}, S_{N-1})$$
$$= f_{N-2} + f_{N-1}$$

According to the theory of optimality, the control policy U_{N-1} will be optimal. Hence the above relation may be written as

$$J_2 = f_{N-2} + J_1^0(X_{N-1})$$

Again X_{N-1} is the function of X_{N-2} and U_{N-2}, therefore the value of optimal input U_{N-2}^0 will be

$$U_{N-2}^0 = \psi_{N-2}(X_{N-2}, S_{N-2})$$

In this way, the process is continued till the initial time is reached.

The result of this optimal closed-loop control is

$$U_K^0 = \psi_K(X_K, S_K).$$

where the stage K lies between 0 to $N - 1$.

EXAMPLE 13.12 The given system is described by

$$\dot{X}_1(t) = 0.4X_1(t) + X_2(t) + u(t)$$

$$\dot{X}_2(t) = 0.6X_2(t) + 0.5u(t)$$

The performance index is as follows:

$$J = \frac{1}{2}\int_0^5 \left[X_1^2(t) + X_2^2(t) + u^2(t)\right] dt$$

How do you make the system optimal in the form of summation of discretization values?

Solution The above system is to be transferred to difference equation.

Let the initial time be 0 and final time t_f. Divide the time interval $0 \le t \le t_f$ into N equal increments of time duration T.

$$\frac{X_1(t+T) - X_1(t)}{T} = -\,0.4X_1(t) + X_2(t) + u(t)$$

or

$$X_1(t + T) - X_1(T) = -\,0.4TX_1(t) + Tx_2(t) + Tu(t)$$

Let $t = nT$, then

$$X_1(nT + T) - X_1(nT) = -\,0.4\,TX_1(nT) + TX_2(nT) + Tu(nT)$$

or

$$X_1(nT + T) = X_1(nT) - 0.4TX_1(nT) + TX_2(nT) + Tu(nT)$$

Let us assume $T = 1$, then

$$X_1(n + 1) = X_1(n)[1 - 0.4] + X_2(n) + u(n)$$

or

$$X_1(n + 1) = 0.6\,X_1(n) + X_2(n) + u(n)$$

Similarly, from

$$\dot{X}_2(t) = -\,0.6X_2(t) + 0.5u(t)$$

we get

$$X_2(n + 1) = 0.4X_2(n) + 0.5u(n)$$

If $T = 1$, then $n = 5$ since $t = nT$ and the upper limit is 5 seconds.

If any value of $J_2^*(X(2))$ is required for a value of $X(2)$ which is in between 0 and 0.5, then the same can be found by interpolation.

Now, the next step is to choose an allowable value of $X(1)$ and then $X(2)$ for each allowable control value of $u(1)$. For example,

$$X(1) = 0, \quad X(1) = 0.5 \quad X(1) = 1$$

Take $X(1) = 0$, the allowable values of $u(1)$ are –1, –0.5, 0, 0.5, 1
Applying the equation

$$X(n + 1) = X(n) + u(n)$$

we have

$$X(2) = X(1) + u(1)$$

$$X(2) = 0 - 1 = -1$$

$$X(2) = 0 - 0.5 = -0.5$$

$$X(2) = 0 + 0 = 0$$

$$X(2) = 0 + 0.5 = 0.5$$

$$X(2) = 0 + 1 = 1.0$$

Now,

$$J_1(X(1), u(1)) = 3u^2(1) + J_2^*[X(2)]$$

When,

$$X(1) = 0, u(1) = 0$$

Then,

$$X(2) = 0$$

$$\text{At} \quad X(2) = 0, \quad J_2^*(X(2)) = 0$$

Therefore,

$$J_1(X(1), u(1)) = 0$$

In this way Table 13.2 can be developed.

Table 13.2

$X(1)$	$u(1)$	$X(2)$	$J_2^*(X(2))$	$J_2(X(1), X(2))$
0	–1	–1	—	—
	–0.5	–0.5	—	—
	0	0	0	0
	0.5	0.5	0.25	1.0
	1.0	1.0	1	4.0
0.5	–1	–0.5	—	—
	–0.5	0.0	0.0	0.75
	0	0.5	0.25	0.25
	0.5	1.0	1.0	1.75
	1	1.5	—	—
1.0	–1	0	0	3
	–0.5	0.5	0.25	1.0
	0	1.0	1.0	1.0
	0.5	1.5	—	—
	1.0	2.0	—	—

Thus, it can be concluded, that if $X(0) = 0.5$, $u^*(0) = 0$, $u^*(1) = 0$, the performance index is then 0.25. In this way for different values of $X(0)$, the optimal input, i.e. control values can be calculated.

13.7 STATE REGULATOR PROBLEM

From the dynamic programming analysis, we came to know that the control problem is a multistage decision problem. Here the calculations of optimal decisions proceed from the last decision back to the first decision.

Let us consider a state regulator problem and apply dynamic programming for obtaining the optimal solution.

We will consider discrete time systems and formulate the state regulator problem in the following manner. The performance index is assumed as follows.

$$J = \frac{1}{2}X^T(N)H\,X(N) + \frac{1}{2}\sum_{K=0}^{N-1}\left[X^T(K)QX(K) + u^T(K)Ru(K)\right]$$

where the initial state $X(0) = X_0$.
According to dynamic programming principle, the first step is

$$J_{N-1}(X(N-1), u(N-1))$$

$$= \frac{1}{2}X^T(N)H\,X(N) + \frac{1}{2}[X^T(N-1)QX(N-1) + u^T(N-1)Ru(N-1)]$$

$$= \frac{1}{2}[AX(N-1) + Bu(N-1)]^T M(0)\,[AX(N-1) + Bu(N-1)]$$

$$+ \frac{1}{2}[X^T(N-1)\,QX(N-1) + u^T(N-1)\,Ru(N-1)]$$

where $H = M(0)$, and $\dot{X} = AX(N) + Bu(N)$ in a discrete form is generally represented by $X(N) = AX(N-1) + Bu(N-1)$.

We know that the necessary condition for minimization is

$$\frac{\partial J_{N-1}}{\partial u_{N-1}} = 0$$

$$Ru(N-1) + B^T M(0)\,[AX(N-1) + Bu(N-1)] = 0$$

$$\left[\because \frac{d(X^T AX)}{dX} = 2AX\right]$$

or

$$Ru(N-1) + B^T M(0)\,AX(N-1) + B^T M(0)\,Bu(N-1) = 0$$

or

$$u(N-1)[R + B^T M(0)\,B] = -B^T M(0)\,AX(N-1)$$

or

$$u(N-1) = -[R + B^T M(0)\,B]^{-1}\,B^T M(0)\,AX(N-1)$$

Now, the minimum value of J_{N-2} will be given by

$$\frac{\partial J_{N-2}}{\partial u_{N-2}} = 0$$

That is,

$$0 = Ru(N-2) + B^T M(1)[AX(N-2) + Bu(N-2)]$$

or

$$0 = Ru(N-2) + B^T M(1)AX(N-2) + B^T M(1)Bu(N-2)$$

or

$$[R + B^T M(1)B]u(N-2) = -B^T M(1)AX(N-2)$$

or

$$u(N-2) = -[R + B^T M(1)B]^{-1} B^T M(1)AX(N-2) = E(N-2)X(N-2)$$

Therefore the minimum value of J_{N-2} is

$J_{N-2}\,(X(N-2))$

$$= \frac{1}{2} X^T(N-2)\,QX(N-2) + \frac{1}{2} X^T(N-2)\,E^T(N-2)\,RE(N-2)\,X(N-2)$$

$$+ \frac{1}{2}\left[AX(N-2) + BE(N-2)\,X(N-2)\right]^T M(1)\left[AX(N-2) + BE(N-2)\,X(N-2)\right]$$

$$= \frac{1}{2} X^T(N-2)\left[Q + E^T(N-2)\,RE(N-2) + \{A + BE(N-2)\}^T M(1)\,\{A + BE(N-2)\}\right]X(N-2)$$

$$= \frac{1}{2} X^T(N-2)\,M(2)\,X(N-2)$$

Similarly,

$$J_{N-i} = \frac{1}{2} X^T(N-i)\,M(i)\,X(N-i)$$

$$u(N-i) = E(N-i)\,X(N-i)$$

$$E(N-i) = -\,[R + B^T M(i-1)B]^{-1}\,B^T M(i-1)A$$

$$M(i) = [A + BE\,(N-i)]^T\,M(i-1)\,[A + BE(N-i)] + E^T(N-i)\,RE\,(N-i) + Q$$

EXAMPLE 13.14 A second-order linear system is described by

$$\dot{X}_1(t) = -\,0.3X_1(t) + X_2(t) + u(t)$$

$$\dot{X}_2(t) = -\,0.4X_2(t) + 0.5u(t)$$

The performance index to be minimized is $J = \frac{1}{2}\int_0^5 \left[X_1^2(t) + X_2^2(t) + u^2(t)\right]dt$. Determine the optimal control policy.

Solution From the above system equations, the difference equation will be

$$X_1(n+1) = 0.7X_1(n) + X_2(n) + u(n)$$

$$X_2(n+1) = 0.6X_2(n) + 0.5u(n)$$

The performance index will be transferred to the approximate summation form

$$= -\left[1+\begin{bmatrix}1 & 0.5\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}1\\0.5\end{bmatrix}\right]^{-1}\begin{bmatrix}1 & 0.5\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}0.7 & 1\\0 & 0.6\end{bmatrix}$$

$$= -\left[1+1.25\right]^{-1}\begin{bmatrix}1 & 0.5\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}0.7 & 1\\0 & 0.6\end{bmatrix} = -\left[2.25\right]^{-1}\begin{bmatrix}1 & 0.5\end{bmatrix}\begin{bmatrix}0.7 & 1\\0 & 0.6\end{bmatrix}$$

$$= -\frac{1}{2.25}\begin{bmatrix}0.7 & 1.3\end{bmatrix}$$

$$= -\begin{bmatrix}0.311 & 0.578\end{bmatrix}$$

In the same manner, $E(2)$, $E(1)$, $E(0)$ can be calculated.
Hence the optimum policy is decided, since

$$u(N-i) = E(N-i)\,X(N-i)$$

State regulator problem for continuous time system

In the case of continuous time problem,

$$J = \frac{1}{2}X^T(t_1)HX(t_1) + \frac{1}{2}\int_{t_0}^{t_1}\left[X^TQX + u^TRu\right]dt$$

Let $u(\tau)$ be the optimum value.

$$J(X(t_0),\ \underset{t_0 \le \tau \le t_1}{t_0},\ u(\tau)) = \frac{1}{2}X^T(t_1)HX(t_1) + \frac{1}{2}\int_{t_0}^{t_1}[X^T(t)QX(t) + u^T(t)Ru(t)]\,dt$$

To find out optimum value $u(\tau)$

$$J(X(t_0), t_0) = \frac{1}{2}\int_{t_0}^{t_0+\Delta t_0}\left[X^T(t)QX(t) + u^T(t)Ru(t)\right]dt$$

$$+\frac{1}{2}\int_{t_0+\Delta t_0}^{t_1}\left[X^T(t)QX(t) + u^T(t)Ru(t)\right]dt + \frac{1}{2}X^T(t_1)HX(t_1)$$

Therefore

$$J(X(t_0), t_0) = \frac{1}{2}\int_{t_0}^{t_0+\Delta t_0}[X^T(t)QX(t) + u^T(t)Ru(t)]\,dt + J(X(t_0+\Delta t_0), t_0+\Delta t_0)$$

Now,

$$J(X(t_0), t_0) = \frac{1}{2}\int_{t_0}^{t_0+\Delta t_0}\left(X^T(t)QX(t) + uT(t)Ru(t)\right)dt + J(X(t_0), t_0) + \frac{\delta J(X(t_0), t_0)}{\delta t_0}\Delta t_0$$

$$+\left[\frac{\delta J(X(t_0), t_0)}{\delta X}\right]^T\left[X(t_0+\Delta t_0) - X(t_0)\right] \qquad \text{(This is as per Taylor's theorem)}$$

or

$$0 = \frac{1}{2}\int_{t_0}^{t_0+\Delta t_0} (X^T(t)\,Q\,X(t) + u^T(t)\,Ru(t))dt + \frac{\delta J(X(t_0),t_0)}{\delta t_0}\Delta t_0 + \left[\frac{\delta J(X(t_0),t_0)}{\delta X}\right]^T \dot{X}(t_0)\,\Delta t_0$$

$$\left(\because\ \dot{X}(t_0) = \frac{X(t_0+\Delta t_0) - X(t_0)}{\Delta T_0}\right)$$

or

$$-\frac{\delta J}{\delta t_0}(X(t_0),t_0)\,\Delta t_0 = \frac{1}{2}\int_{t_0}^{t_0+\Delta t_0}\left(X^T(t)\,Q\,X(t) + u^T(t)\,Ru(t)\right)dt$$
$$+\left[\frac{\delta J(X(t_0),t_0)}{\delta X}\right]^T\left[AX(t_0) + Bu(t_0)\right]\Delta t_0$$

Since Δt_0 is a very small value, we can write

$$-\frac{\delta J}{\delta t_0}(X(t_0),t_0)\Delta t_0 = \frac{1}{2}\left[X^T(t_0)\,Q\,X(t_0) + u^T(t_0)\,Ru(t_0)\right]\Delta t_0$$
$$+\left[\frac{\delta J}{\delta X}(X(t_0),t_0)\right]^T\left[AX(t_0) + Bu(t_0)\right]\Delta t_0$$

or

$$-\frac{\delta J}{\delta t_0}(X(t_0),t_0) = \frac{1}{2}\left[X^T(t_0)\,Q\,X(t_0) + u^T(t_0)\,Ru(t_0)\right] + \left[\frac{\delta J(X(t_0),t_0)}{\delta X}\right]^T\left[AX(t_0) + Bu(t_0)\right]$$

Now, differentiating the above w.r.t. u, we can determine the condition for minimization. That is,

$$0 = Ru(t) + B^T\frac{\delta J}{\delta X}\left(X(t_0),t_0\right)$$

or

$$u(t_0) = -R^{-1}B^T\frac{\delta J}{\delta X}\left(X(t_0),t_0\right)$$

Hence, at the minimum value of J, $u(t_0) = -R^{-1}B^T\frac{\delta J}{\delta X}\left(X(t_0),t_0\right)$

Therefore,

$$-\frac{\delta J\left(X(t_0),t_0\right)}{\delta t_0} = \frac{1}{2}\left\{X^T(t_0)\,Q\,X(t_0) + \left[R^{-1}B^T\frac{\delta J\left(X(t_0),t_0\right)}{\delta X}\right]^T RR^{-1}B^T\frac{\delta J\left(X(t_0),t_0\right)}{\delta X}\right\}$$
$$+\left[\frac{\delta J\left(X(t_0),t_0\right)}{\delta X}\right]^T\left[AX(t_0) - BR^{-1}B^T\frac{\delta J\left(X(t_0),t_0\right)}{\delta X}\right]$$
$$= \frac{1}{2}X^T(t_0)\,Q\,X(t_0) + \frac{1}{2}\left(\frac{\delta J\left(X(t_0),t_0\right)}{\delta X}\right)^T BR^{-1}B^T\frac{\delta J\left(X(t_0),t_0\right)}{\delta X}$$
$$+\left[\frac{\delta J\left(X(t_0),t_0\right)}{\delta X}\right]^T AX(t_0) - \left(\frac{\delta J\left(X(t_0),t_0\right)}{\delta X}\right)^T BR^{-1}B^T\frac{\delta J\left(X(t_0),t_0\right)}{\delta X} \qquad (13.28)$$

EXAMPLE 13.15 A system is defined as follows:

$$\dot{X}(t) = 3X(t) + u(t)$$

If the following performance index is to be minimized, determine the control law.

$$J = \frac{1}{2}\int_{t}^{t_1}\left(4X^2 + \frac{1}{5}u^2\right)dt$$

where $t_1 = 2$ second.

Solution

$$\dot{X} = 3X(t) + u(t)$$

$$A = 3,\ B = 1$$

$$J = \frac{1}{2}\int_{t}^{t_1}\left(4X^2 + \frac{1}{5}u^2\right)dt$$

$$Q = 4,\quad R = \frac{1}{5} \qquad \text{and} \qquad H = 0 = M(t_1)$$

Applying the Matrix–Riccati equation,

$$\dot{M}(t) + Q - M(t)\,BR^{-1}B^TM(t) + M(t)A + A^TM(t) = 0$$

we get

$$\dot{M}(t) + 4 - 5M^2(t) + 3M(t) + 3M(t) = 0$$

or

$$\dot{M}(t) + 4 - 5M^2(t) + 6M(t) = 0$$

or

$$\dot{M}(t) = 5M^2(t) - 6M(t) - 4$$

or

$$\frac{dM(t)}{dt} = 5M^2(t) - 6M(t) - 4$$

or

$$\int_{t}^{t_1}\frac{dM(t)}{5M^2(t) - 6M(t) - 4} = \int_{t}^{t_1}dt$$

or

$$\int_{t}^{t_1}\frac{dM(t)}{(M(t) - 1.677)\,(M(t) + 0.477)} = \int_{t}^{t_1}dt$$

or

$$\int_{t}^{t_1}\left[\frac{A_1}{M(t) - 1.677} + \frac{B_1}{M(t) + 0.477}\right]dM(t) = \int_{t}^{t_1}dt$$

or

$$\int_{t_1}^{t}\left[\frac{A_1}{M(t) - 1.677} + \frac{B_1}{M(t) + 0.477}\right]dM(t) = \int_{t_1}^{t}dt$$

or

$$\left[A_1 \log(M(t) - 1.677) + B_1 \log(M(t) + 0.477)\right]_{t_1}^{t} = t - t_1$$

or

$$\left[\frac{1}{2.154} \log(M(t) - 1.677) - \frac{1}{2.154} \log(M(t) + 0.477)\right]_{t_1}^{t} = t - t_1$$

or

$$\frac{1}{2.154} \log \frac{M(t) - 1.677}{M(t) + 0.477} = t - t_1$$

or

$$\log \frac{M(t) - 1.677}{M(t) + 0.477} = 2.154(t - t_1)$$

or

$$\frac{M(t) - 1.677}{M(t) + 0.477} = e^{2.154(t - t_1)}$$

or

$$M(t) - 1.677 = e^{2.154(t - t_1)} \, [M(t) + 0.477]$$

or

$$M(t)\left[1 - e^{2.154(t - t_1)}\right] = 1.677 + 0.477\, e^{2.154(t - t_1)}$$

or

$$M(t) = \frac{0.477\, e^{2.154(t - t_1)} + 1.677}{1 - e^{2.154(t - t_1)}}$$

or

$$M(t) = \frac{0.477\, e^{2.154(t - 2)} + 1.677}{1 - e^{2.154(t - 2)}}$$

The control policy will be

$$\begin{aligned} u(t) &= -R^{-1}B^T M(t) \\ &= -5M(t) \\ &= \frac{-5\left[0.477\, e^{2.154(t - 2)} + 1.677\right]}{1 - e^{2.154(t - 2)}} \end{aligned}$$

13.8 PARAMETER OPTIMIZATION

The method of the solution of the optimal regulator problem has already been studied. Now in the parameter optimization, we will solve the control problem where the elements of the feedback matrix of K are considered. Suppose the system is represented by

$$\dot{X} = AX + Bu$$

The performance index is

$$J = \frac{1}{2} \int_0^\infty \left(X^T Q X + u^T R u\right) dt$$

where Q is a positive semi-definite, real, symmetric constant matrix and R is a positive definite, real symmetric constant matrix. Now,

$$u = KX(t)$$

where K is the constant matrix. Also,

X is $n \times 1$ state vector matrix

u is $m \times 1$ control vector matrix

A is $n \times n$ constant matrix

B is $n \times m$ constant matrix

K is $m \times n$ constant matrix

We know

$$\dot{X} = AX + Bu = AX + BKX = (A + BK)X$$

Therefore, the performance index

$$J = \frac{1}{2}\int_0^{\infty}(X^TQX + X^TK^TRKX)\,dt$$

$$= \frac{1}{2}\int_0^{\infty}X^T(Q + K^TRK)X\,dt$$

Let us assume

$$X^T(Q + K^TRK)X = -\frac{d}{dt}(X^TMX) = -\dot{X}^TMX - X^TM\dot{X}$$

when M is a real symmetric positive-definite matrix.
Thus,

$$X^T(Q + K^TRK)X = -X^T[(A + BK)^TM + M(A + BK)]X$$

or

$$X^T[Q + K^TRK + (A + BK)^TM + M(A + BK)]\,X = 0$$

Therefore,

$$Q + K^TRK + (A + BK)^TM + M(A + BK) = 0$$

Now,

$$J = \frac{1}{2}\int_0^{\infty} -\frac{d}{dt}(X^TMX)\,dt = -\frac{1}{2}X^TMX\Big|_0^{\infty}$$

$$= -\frac{1}{2}X^T(\infty)MX(\infty) + \frac{1}{2}X^T(0)MX(0)$$

At the final time $\to \infty$, $X(\infty) \to 0$, for optimal system to be stable, therefore,

$$J = \frac{1}{2}X^T(0)MX(0)$$

EXAMPLE 13.16 Determine the optimum damping coefficient which minimizes the following performance index

$$J = \int_0^\infty E^2(t)\, dt$$

where E is the error.

The initial conditions are

$$\theta(0) = 2, \qquad \dot{\theta}(0) = 0$$

where θ is the output.

Solution Let us consider a second-order system as shown in Fig. 13.15, where ζ is the damping coefficient.

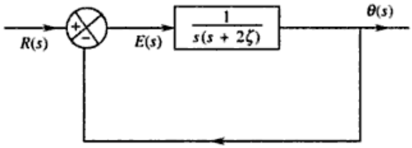


Fig. 13.15

Now,

$$[R(s) - \theta(s)] \frac{1}{s(s+2\zeta)} = \theta(s)$$

Simplifying, we get

$$\theta(s) = \frac{R(s)}{s^2 + 2\zeta s + 1}$$

or

$$s^2\theta(s) + 2\zeta s\theta(s) + \theta(s) = R(s)$$

If the input is taken 0, then

$$s^2\, \theta(s) + 2\zeta s\theta(s) + \theta(s) = 0$$

Taking the inverse Laplace transform,

$$\ddot{\theta} + 2\zeta\dot{\theta} + \theta = 0$$

Let

$$\theta = X_1,\ \dot{\theta} = X_2$$

Then,

$$\dot{X}_1 = \dot{\theta} = X_2$$

Again,

$$\ddot{\theta} + 2\zeta\dot{\theta} + \theta = 0$$

or

$$\dot{X}_2 + 2\zeta X_2 + X_1 = 0$$

or

$$\dot{X}_2 = -2\zeta X_2 - X_1$$

The signal flow graph of the above system is shown in Fig. 13.16

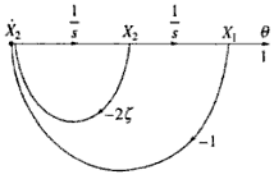


***Fig. 13.16** Signal flow graph of Fig. 13.15.*

Since there is no external input, the optimum policy u will be

$$u = -2\zeta X_2 - X_1. \quad \text{Hence } u = \dot{X}_2.$$

Therefore,

$$\dot{X}_1 = X_2$$

$$\dot{X}_2 = u$$

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

Since

$$\theta(0) = 2, \qquad \dot{\theta}(0) = 0$$

$$X_1(0) = 2, \qquad X_2(0) = 0$$

$$u = -\begin{bmatrix} 1 & 2\zeta \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

The performance index

$$J = \int_0^\infty E^2(t)\, dt = \int_0^\infty (0 - \theta)^2 dt$$

$$= \int_0^\infty \theta^2 dt = \int_0^\infty X_1^2\, dt = \frac{1}{2} \int_0^\infty 2X_1^2\, dt$$

Therefore,

$$2X_1^2 = X^T Q X$$

$$= \begin{bmatrix} X_1 & X_2 \end{bmatrix} Q \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

Hence,

$$Q = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

We already know that

$$(A + BK)^T M + M(A + BK) + Q = 0$$

we get

$$M_{11} = 2\zeta + \frac{1}{2\zeta}$$

Therefore,

$$M = \begin{bmatrix} 2\zeta + \frac{1}{2\zeta} & 1 \\ 1 & \frac{1}{2\zeta} \end{bmatrix}$$

Now the optimal value of the performance index is

$$J = \frac{1}{2} X^T(0)\, MX(0)$$

$$= \frac{1}{2} X^T(0) \begin{bmatrix} 2\zeta + \frac{1}{2\zeta} & 1 \\ 1 & \frac{1}{2\zeta} \end{bmatrix} \begin{bmatrix} X_1(0) \\ X_2(0) \end{bmatrix}$$

$$= \frac{1}{2} [X_1(0)\; X_2(0)] \begin{bmatrix} 2\zeta + \frac{1}{2\zeta} & 1 \\ 1 & \frac{1}{2\zeta} \end{bmatrix} \begin{bmatrix} X_1(0) \\ X_2(0) \end{bmatrix}$$

$$= \frac{1}{2} \left[\left(2\zeta + \frac{1}{2\zeta}\right) X_1(0) + X_2(0) \quad X_1(0) + \frac{1}{2\zeta} X_2(0) \right] \begin{bmatrix} X_1(0) \\ X_2(0) \end{bmatrix}$$

$$= \frac{1}{2} \left[\left\{ \left(2\zeta + \frac{1}{2\zeta}\right) X_1(0) + X_2(0) \right\} X_1(0) + \left\{ X_1(0) + \frac{1}{2\zeta} X_2(0) \right\} X_2(0) \right]$$

Now, $X_1(0) = 2$, $X_2(0) = 0$. Thus,

$$J = \frac{1}{2} \left(2\zeta + \frac{1}{2\zeta}\right) 2.2 = \frac{4\zeta^2 + 1}{\zeta}$$

Now the value of ζ can be found out by making

$$\frac{\delta J}{\delta \zeta} = 0$$

or

$$\frac{\left\{ \frac{\partial}{\partial \zeta} (4\zeta^2 + 1) \right\} \zeta - (4\zeta^2 + 1)}{\zeta^2} = 0$$

or

$$\frac{8\zeta \cdot \zeta - 4\zeta^2 - 1}{\zeta^2} = 0$$

or

$$4\zeta^2 = 1 \quad \text{or} \quad \zeta = \frac{1}{2} = 0.5$$

Again

$$\frac{\partial^2 J}{\partial \zeta^2} = \frac{\partial}{\partial \zeta}\left(\frac{4\zeta^2 - 1}{\zeta^2}\right)$$

$$= \frac{8\zeta \cdot \zeta^2 - 2\zeta(4\zeta^2 - 1)}{\zeta^4}$$

$$= \frac{8\zeta^3 - 8\zeta^3 + 2\zeta}{\zeta^4} = \frac{2\zeta}{\zeta^4} > 1 \text{ for } \zeta = 0.5$$

The damping coefficient will therefore be 0.5 and at this value the performance index is minimum.

SUMMARY

Optimization of objective functions by the steepest descent method is explained. Optimization with constraint by the gradient method is covered next. Jordan's elimination technique is described. Minimization of functions by a numerical method is also described. The Fletcher–Powell method is explained. The Newton–Raphson method for error minimization is also shown. Performance indices such as ISE, ITAE, IAE, and ITSE are defined. Characteristics of the plant, requirements of the plant and the data of the plant received by the controller are described. Minimum time problem, minimum energy problem, minimum fuel problem, state regulator problem, output regulator problem, servomechanism or tracking problem, are all fully dealt with.

Mathematical procedures for optimal control design are described one by one. Calculus of variations is utilized for solving two-point boundary value problems (TPBVPs). Both fixed-end problems and variable-end point problem are discussed.

The method of determination of optimal value of state and Lagrangian multiplier is shown with an example.

An example related to Euler's equation is solved. An example related to Euler–Lagrange equation is also solved. Pontryagin's optimum policy is described with an example. Bang-bang control is explained. The procedure of determining optimal trajectories is explained. Hamilton–Jacobi principle for solving optimization problems is explained. Matrix–Ricatti equation is also established. On dynamic programming, both the imbedding principle and the optimality principle are presented.

Examples of dynamic programming are solved. State regulator problem is solved with the principle of dynamic programming. Parameter optimization is discussed with the help of an example.

Find the optimal control law using the Hamilton–Jacobi equation that minimizes the performance index

$$J = \int_0^{t_f} (X^2 + u^2)\, dt$$

where t_f is specified.

11. Explain the gradient technique to solve optimal control problems.
12. What is maximum principle? How do you implement control and state constraints?
13. Consider a simple open-loop system that consists of an integration and an amplifier having a gain of 2. It is desired to control the state of this system between $X(0)$ and $X(T)$ in order to minimize,

$$S = \int_0^{T} (4X^2(t) + 2u^2(t))\, dt$$

Using the calculus of variations, find the optimal control $u^0(t)$ that can achieve this.

14. Given the system

$$\dot{X}_1 = X_2$$

$$\dot{X}_2 = -2X_2 - 4X_1 + u$$

and

$$|u(t)| \le U$$

find the control $u(t)$ such that the system is taken from the initial state $c(t_0)$ to the equilibrium state $c(t_f) = 0$ in the shortest period of time.

15. Determine the optimal control policy $u^0(t)$ using dynamic programming for the system

$$\ddot{c} + 4\dot{c} + 3c = 6u$$

to minimize

$$S = \int_0^{T} u^2(t)\, dt$$

such that

$$-1 \le u(t) \le 1.$$

16. Determine the optimal control law for the system

$$\dot{X} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} X + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$Y = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} X$$

such that the following performance index is minimized

$$J = \int_0^{\infty} (Y_1^2 + Y_2^2 + u^2) dt$$

Chapter 14

Introduction to Neural Fuzzy Systems and Adaptive Learning Systems

14.1 INTRODUCTION

Fuzzy logic is based on the way the brain deals with inexact information, while neural networks are modelled after the physical architecture of the brain. To a certain extent, both these systems and their techniques have been successfully applied to a variety of control systems and devices in order to improve their intelligence. Fuzzy systems combine fuzzy sets with fuzzy rules to produce overall complex nonlinear behaviour. Neural networks, on the other hand, are trainable dynamical systems whose learning, noise, and generalization abilities grow out of their connectionist structures, their dynamics and their distributed data representation. Neural networks provide fuzzy systems with learning abilities and fuzzy systems provide neural networks with a structural framework. Fuzzy logic and neural networks are constituents of an emerging research area, called *soft computing*, a term coined by Lotfi Zadeh (the father of fuzzy logic). In the partnership of fuzzy logic neural networks and probabilistic reasoning, fuzzy logic is concerned in the main with imprecision and approximate reasoning; neural networks, on the other hand, are concerned with learning and probabilistic reasoning with uncertainty. Since fuzzy logic, neural networks and probabilistic reasoning are complementary rather than competitive, it is frequently advantageous to employ them in combination.

14.1.1 Fuzzy Systems

Fuzzy sets are introduced as a mathematical way to represent vagueness in linguistics. In a nonfuzzy set, an element of the universe either belongs to or does not belong to the set. It is either yes (in the set) or no (not in the set). A fuzzy set is a generalization of an ordinary set in that it allows the degree of membership for each element to range over the interval [0, 1]. A fuzzy set has an infinite number of membership functions that may represent it. Fuzziness is often confused with probability. The fundamental difference between these two phenomena is that fuzziness deals with deterministic plausibility, while probability concerns the likelihood of nondeterministic, stochastic events. Fuzziness is one aspect of uncertainty. It is the vagueness found in the definition of a concept or the meaning of a term such as 'young person' or 'large room'. However, the uncertainty of probability generally relates to the occurrence of phenomena, as symbolized by the concept of randomness. For example, the statements such as: "There is a 50-50 chance that he will be there", "it will rain tomorrow", "Roll the dice and get a four", have the uncertainty of randomness. The fuzzy logic is appropriate to use in the following cases.

(a) When the process is concerned with continuous phenomena that are not easily broken down into discrete segments.

(b) When a mathematical model of the process does not exist or if it exists, it is too difficult to encode or is too complex to be evaluated fast enough for real time operation or involves too much memory on the designated chip architecture.

(c) When high ambient noise levels must be dealt with, or when it is important to use inexpensive sensors and/or low-precision microcontrollers.

(d) When the process involves human interaction.

(e) When an expert is available who can specify the rules underlying the system behaviour and the fuzzy sets that represent the characteristic of each variable.

Fuzzy logic techniques find their application in the following areas:

(a) Control

(b) Pattern recognition (Image, audio signal processing)

(c) Quantitative analysis (operations research, management)

(d) Inference (expert systems for diagnosis, planning, and prediction, natural language processing, intelligent interface, intelligent robots, software engineering).

(e) Information retrieval (databases)

14.2 NEURAL NETWORKS

Neural networks are a new generation of information processing systems that are deliberately constructed to make use of some of the organizational principles that characterize the human brain. The neural networks have a large number of highly interconnected processing elements (nodes) that usually operate in parallel. Models of networks are based on three basic entities.

(a) Models of the processing elements themselves

(b) Models of interconnections and structures

(c) Learning rules

Each node in a neural collects the values from all its input connections, performs a predefined mathematical operation and produces a single output value.

In a neural network, each node output is connected through weights to other nodes or to itself. Hence the structure that organizes these nodes and the connection geometry among them should be specified for a neural network.

There are two kinds of learning in neural networks—parameter learning and structure learning. Parameter learning concerns the updating of the connection weights and structure learning focuses on the change in the network structure. Each kind of learning can be further classified into three categories—supervised learning, reinforcement learning, and unsupervised learning. Neural networks offer the following salient characteristics and properties.

1. Neural networks are able to learn arbitrary nonlinear input-output mapping directly from the training data.

2. Neural networks can sensibly interpolate input patterns that are new to the network. From a statistical point of view, neural networks can fit the desired function in such a way that they have the ability to generalize situations that are different from the collected training data.
3. Neural networks can automatically adjust their connection weights or even network structures to optimize their behaviour as controllers, predictors, pattern recognizers, decision makers, and so on.
4. The performance of a neural network is degraded gracefully under faulty conditions such as damaged nodes or connections.

The following are some of the uses of the neural networks.

(a) For nonlinear mapping, such as robot control and noise removal.

(b) When only a few decisions are required from a massive amount of data, such as speech recognition and fault prediction.

(c) When a near-optimal solution to a combinatorial optimization problem is required in a short time, such as airline scheduling and network routing.

14.3 FUZZY-NEURAL INTEGRATED SYSTEM

This is a promising approach for reaping the benefits of both the fuzzy systems and the neural networks by merging or fusing them into an integrated system. The fusion of these two different technologies can be realized in three directions, resulting in systems with different characteristics.

(a) Neural-fuzzy systems—use of neural networks as a tool in fuzzy models

(b) Fuzzy-neural networks—fuzzification of conventional neural network models

(c) Fuzzy-neural-hybrid systems—Incorporation of fuzzy logic technology and neural networks into hybrid systems

14.3.1 Comparison of Fuzzy Systems, Neural Networks, and Conventional Control Theory

	Fuzzy system	*Neural network*	*Control theory*
Mathematical model	Slightly good	Bad	Good
Learning ability	Bad	Good	Bad
Knowledge representation	Good	Bad	Slightly bad
Expert knowledge	Good	Bad	Slightly bad
Nonlinearity	Good	Good	Bad
Optimization ability	Bad	Slightly Good	Slightly bad
Fault tolerance	Good	Good	Bad
Uncertainty tolerance	Good	Good	Bad
Real time operation	Good	Slightly Good	Good

14.4 APPLICATION OF FUZZY CONTROLLERS

Over the past decade, we have witnessed a very significant increase in the number of application of fuzzy logic-based techniques to various commercial and industrial products and system, especially in controlling nonlinear, time-varying, ill-defined systems and in managing complex systems with multiple independent decision making processes. Fuzzy logic control-based systems have proved to be superior in performance when compared to conventional control systems.

The cement kiln control system was the first successful industrial application of a fuzzy logic control. In contrast to previous analog fuzzy logic controllers which were designed based on a continuous state space model, a discrete-event fuzzy controller was designed for airport control. Fuzzy control has also been successfully applied to automatic train operation systems. Fuzzy logic control systems have also found application in household appliances such as air-conditioners, washing machines, video recorders, television auto contrast and brightness control cameras, auto focusing and jitter control, vacuum cleaners, microwave ovens, palmtop computers, and so forth.

14.5 DIFFERENCES BETWEEN CLASSICAL SET AND FUZZY SET

The differences between classical set and fuzzy set are explained with an example.

Suppose, there exists the real line and classical set that represents 'real numbers' greater than and equal to 5. Then the characteristic function of the classical set will be as shown in Fig. 14.1.

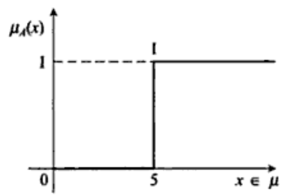


Fig. 14.1

Let fuzzy set $\bar{A}$ represent 'real numbers close to 5'; it would then be as shown in Fig. 14.2. The classical set will be expressed mathematically as

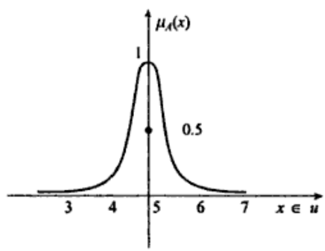


Fig. 14.2

Error sum

$$\Sigma r(K) = \sum_{i=0}^{K} r(i-1)$$

Control change

$$\Delta c(K) = c(K) - c(K-1)$$

Hence the control law

$$\Delta c(K) = f(r(K), \Delta r(K))$$

$$c(K) = c(K-1) + \Delta c(K)$$

Example of the rules base of fuzzy logic control

1. IF the error $r(K)$ is positive AND the change of the error $\Delta r(K)$ is approximately zero, THEN the change of the control $\Delta c(K)$ is positive.

ALSO

2. IF the error $r(K)$ is negative AND the change of the error $\Delta r(K)$ is approximately zero, THEN the change of the control $\Delta c(K)$ is negative.

ALSO

3. IF the error $r(K)$ is approximately zero AND the change of the error $\Delta r(K)$ is approximately zero, THEN the change of the control $\Delta c(K)$ is approximately zero.

ALSO

4. IF the error $r(K)$ is approximately zero AND the change of the error $\Delta r(K)$ is positive, THEN the change of the control $\Delta c(K)$ is positive.

ALSO

5. IF the error $r(K)$ is approximately zero AND the change of the error $\Delta r(K)$ is negative, THEN the change of the control $\Delta c(K)$ is negative.

Relationship between PI and fuzzy control

The fuzzy logic control narrates with the help of fuzzy IF-THEN rules the relationship between the change of control $\Delta c(K) = c(K) - c(K-1)$ and at the same time the error $r(K)$ and its change $\Delta r(K) = r(K) - r(K-1)$. That is,

$$\Delta c(K) = f(r(K), \Delta r(K))$$

If we just compare this with the *PI* controller, it is observed that in the case of the *PI* controller,

$$\Delta c(K) = K_P\, \Delta r(K) + K_I\, r(K)$$

where K_P and K_I are the parameters of the *PI*-controller. Hence, it can be concluded that both the control laws provide a relationship between the variables $r(K)$ and $\Delta r(K)$ with $\Delta c(K)$. The main difference is that the case of the *PI* controller, the relationship is linear whereas in the case of fuzzy logic control it is nonlinear.

control system also accommodates moderate engineering design errors or uncertainties. It compensates the failure of the minor system components and increases system reliability.

Adaption is a basic characteristic of living organisms because they try to maintain physiological equilibrium in the midst of changing environmental conditions. Similarly, an adaptive control system is one that continuously and automatically measures the dynamic characteristics of the plant, compares them with the required dynamic characteristics and utilizes the difference to vary the adjustable parameters or to generate an actuating signal. As a result, the optimal performance can be maintained even with the environmental change. Otherwise, the system may continuously measure its own performance as per the given performance index and develop its own parameters for maintaining optimal performance even with the environmental changes. In other words, there must be some self-organizing scheme.

An adaptive controller consists of the following three functions:

(a) Identification of the dynamic characteristics of the plant

(b) Decision making based on the identification of the plant

(c) Modification or actuation based on the decision made

Figure 14.4 shows the block diagram representation of the adaptive control system.

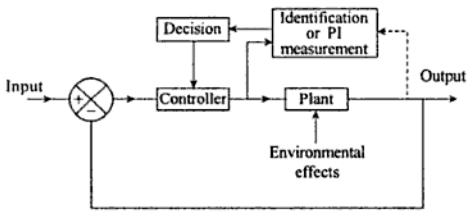


Fig. 14.4 Block diagram of the adaptive control system.

14.7.1 Identification of the Dynamic Characteristics of the Plant

The dynamic characteristics of the plant are measured and identified continuously, without affecting the normal operation of the system. Identification is made from normal operating data of the plant or by utilizing the test signals, for example, sinusoidal signals of small amplitude or various stochastic signals of small amplitude. Stochastic signals are quite impressive in some cases. The output is analyzed as a function of the stochastic input for determining the response characteristics. The identification time should be sufficiently short compared with the rate of change of environment. But during a short time of identification, it may be difficult to identify the plant completely. That is why, partial identification is appropriate. Where the plant identification is very different, the performance index is measured directly and an adaptive controller based on it is built.

SUMMARY

The fuzzy system is defined. The cases in which fuzzy logic can be used are explained. The areas of application of fuzzy logic are described.

The neural network is defined. The salient characteristics and properties of neural networks are described. The uses of the neural network are also stated.

The fuzzy-neural integrated system is also explained.

A comparison of fuzzy systems, neural networks and conventional control theory is tabulated.

The applications of fuzzy controllers are also described.

The difference between the classical set and the fuzzy set is discussed. The fuzzy logic control system is defined. An example of the rules base of the fuzzy logic control is also shown. Relationships of PI, PD, and PID controllers with fuzzy logic control are dealt with.

An idea of the adaptive control system is given. Three main functions of an adaptive system, i.e. identification of the dynamic characteristics of the plant, decision making and modification are explained.

The principle of learning systems is explained.

QUESTIONS

1. Explain the concept of fuzzy systems. What are the areas of their application?
2. What are neural networks? What are the salient characteristics and properties of neural networks? What are the uses of neural networks?
3. What is a fuzzy-neural integrated system?
4. Compare fuzzy systems, neural networks, and conventional control theory.
5. What are the applications of fuzzy controllers?
6. Describe the differences between the classical set and the fuzzy set.
7. Explain the principle of the fuzzy logic controller. Give an example of the rules base of the fuzzy logic control.
8. Compare *PI*, *PD*, and *PID* Controllers with the fuzzy logic controller.
9. What is an adaptive control system? Explain with the help of a block diagram.
10. Give an idea about 'Learning System'.

Chapter 15

Miscellaneous Solved Problems

PROBLEM 15.1 Draw the block diagram for the speed control system shown in Fig. 15.1.

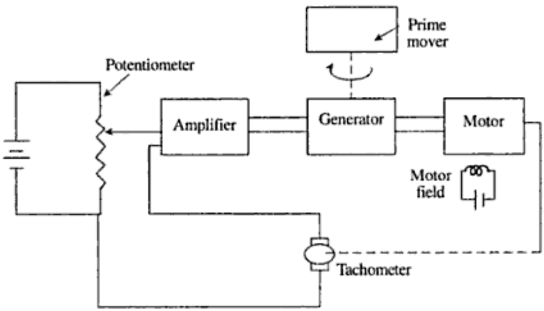


Fig. 15.1 Speed control system.

Solution Figure 15.2 depicts the block diagram of the speed control system shown in Fig. 15.1. Here the potentiometer is the reference. The amplifier and the generator are the controller. The motor is the plant. The tachometer is the feedback element.

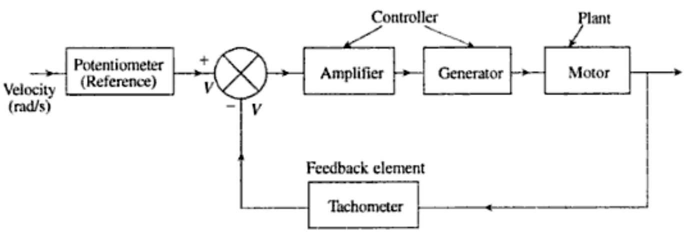


Fig. 15.2 Block diagram of the system in Fig. 15.1.

PROBLEM 15.2 Determine the damping ratio, the undamped natural frequency, the damped natural frequency, the damping coefficient, and the time constant for the following second-order system.

$$\frac{d^2y}{dt} + 2\frac{dy}{dt} + 2y = 2x$$

Solution We know that the second-order system is represented by $\dfrac{d^2y}{dt^2} + 2\zeta\omega_n\dfrac{dy}{dt} + \omega_n^2 y = \omega_n^2 x$

where

ζ = damping ratio

ω_n = undamped natural frequency.

Therefore,

$$2\zeta\omega_n = 2, \qquad \omega_n^2 = 2$$

or

$$\zeta\omega_n = 1, \qquad \omega_n = \sqrt{2}$$

or

$$\zeta = \frac{1}{\sqrt{2}}$$

Damped natural frequency

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

$$= \sqrt{2}\sqrt{1-\frac{1}{2}} = 1$$

Damping coefficient, $\zeta\omega_n = 1$

Time constant of the system, $\dfrac{1}{\zeta\omega_n} = 1$

PROBLEM 15.3 Prove that the Laplace transform of the unit-impulse function is one.

Solution

$$\int_{0^-}^{\infty} \delta(t)\, e^{-st} dt = \int_{0^-}^{\infty} \lim_{\Delta t \to 0}\left[\frac{u(t) - u(t-\Delta t)}{\Delta t}\right] e^{-st} dt$$

$$= \lim_{\Delta t \to 0}\left[\int_{0^-}^{\infty} \frac{u(t)}{\Delta t} e^{-st} dt - \int_{0^-}^{\infty} \frac{u(t-\Delta t)}{\Delta t} e^{-st} dt\right]$$

$$= \lim_{\Delta t \to 0}\left[\frac{1}{s} - \frac{e^{-\Delta t s}}{s}\right]$$

($\because$ the Laplace transform of $u(t) = 1$ and the Laplace transform of $f(t - T) = e^{-sT}F(s)$)

Now

$$e^{-\Delta ts} = 1 - \Delta ts + \frac{(\Delta t \cdot s)^2}{2!} - \frac{(\Delta t \cdot s)^3}{3!} + \cdots$$

Laplace transform of $\delta(t)$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left[\frac{1}{s} - \frac{1}{s} + \Delta t - \frac{(\Delta t)^2 s}{2!} + \frac{(\Delta t)^3 s^2}{3!} - \cdots\right]$$

$$= \lim_{\Delta t \to 0}\left[1 - \frac{(\Delta t)}{2!}s + \frac{(\Delta t)^2 s^2}{3!} - \cdots\right]$$

$$= 1$$

PROBLEM 15.4 Find the transfer function of the gyroscopic system shown in Fig. 15.3 considering θ as the output and ω as the input.

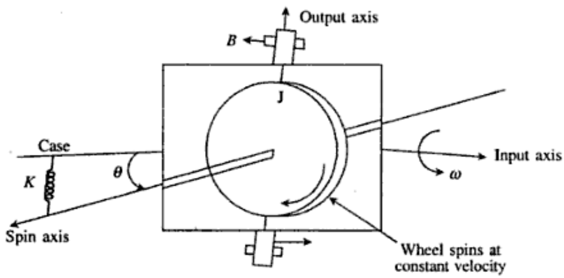


Fig. 15.3 *Gyroscope.*

Solution In the case of gyroscope, the differential equation is

$$\frac{Jd^2\theta}{dt^2} + B\frac{d\theta}{dt} + K\theta = H\omega$$

where J is the moment of inertia, B is the viscous friction coefficient, and H is the angular momentum stored in the spinning wheel.

Taking the Laplace transform of the above equation, we get

$$(Js^2 + Bs + K)\theta(s) = H\omega(s)$$

where the initial condition is assumed zero.

Therefore,

$$\frac{\theta(s)}{\omega(s)} = \frac{H}{Js^2 + Bs + K}$$

PROBLEM 15.5 Determine the type of the closed-loop system shown in Fig. 15.4.

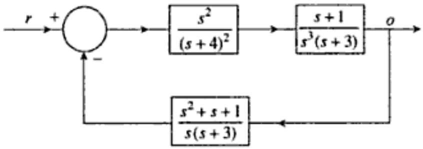


Fig. 15.4 Closed-loop system.

Solution The open-loop transfer function of the system is

$$GH = \frac{s^2(s+1)(s^2+s+1)}{s^4(s+4)^2(s+3)^2} = \frac{(s+1)(s^2+s+1)}{s^2(s+4)^2(s+3)^2}$$

From the open-loop transfer function, it is clear that the given system is a type-2 system.

PROBLEM 15.6 Compare the sensitivities of the two systems, shown in Fig. 15.5 and Fig. 15.6 with respect to the parameter K' for nominal values $K' = K'' = 100$ and also compare the transfer functions of the two systems.

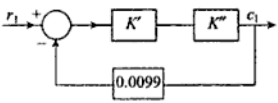


Fig. 15.5

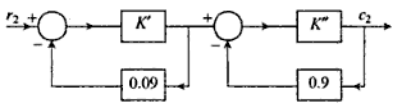


Fig. 15.6

Solution The transfer function of Fig. 15.5 is

$$T_1 = \frac{c_1}{r_1} = \frac{K'K''}{1 + 0.0099\,K'K''}$$

$$= \frac{(100)(100)}{1 + (0.0099)(100)(100)} = 100$$

The transfer function of Fig. 15.6 is

$$T_2 = \frac{c_2}{r_2} = \left(\frac{K'}{1 + 0.09\,K'}\right)\left(\frac{K''}{1 + 0.09\,K''}\right)$$

$$= \left(\frac{100}{1+0.09\times 100}\right)\left(\frac{100}{1+0.09\times 100}\right) = 100$$

Hence, both the transfer functions are equal. The sensitivity of the transfer function T_1 with respect to K' is

$$= \frac{\dfrac{\delta T_1}{T_1}}{\dfrac{\delta K'}{K'}} = \frac{\delta T_1}{\delta K'}\cdot\frac{K'}{T_1}$$

$$= \frac{K''(1+0.0099\,K'K'') - K'K''(0.0099\,K'')}{(1+0.0099\,K'K'')^2}\cdot\frac{K'}{\dfrac{K'K''}{1+0.0099\,K'K''}}$$

$$= \frac{1}{(1+0.0099\,K'K'')} = \frac{1}{1+(0.0099)(100)(100)} = 0.01$$

Again for the second system, the sensitivity of the transfer function T_2 with respect to K' is

$$= \frac{\dfrac{\delta T_2}{T_2}}{\dfrac{\delta K'}{K'}} = \frac{\delta T_2}{\delta K'}\cdot\frac{K'}{T_2}$$

$$= \left\{\frac{\delta}{\delta K'}\left[\left(\frac{K'}{1+0.09\,K'}\right)\left(\frac{K''}{1+0.09\,K''}\right)\right]\right\}\frac{K'}{\dfrac{K'K''}{(1+0.09K')(1+0.09K'')}}$$

$$= \frac{K''(1+0.09K')(1+0.09\,K'') - K'K''[0.09\,(1+0.09\,K'')]}{(1+0.09K')^2(1+0.09K'')^2}\times\frac{K'(1+0.09K')(1+0.09K'')}{K'K''}$$

$$= \frac{1}{1+0.09K'} = \frac{1}{1+0.09\times 100} = 0.1$$

Hence, it is observed that the sensitivity of T_1 with respect to K' is 0.01 and the sensitivity of T_2 with respect to K' is 0.1.

Therefore the second system shown in Fig. 15.6 is 10 times more sensitive than the first system shown in Fig. 15.5 with respect to variations in K'.

PROBLEM 15.7 Determine the bandwidth of the system whose transfer function is $\dfrac{1}{s+1}$.

Solution In Fig. 15.7, $o(t)$ is the output and $r(t)$ is the input and $\left|\dfrac{O}{R}(j\omega)\right|$ is the magnitude of the transfer function in the frequency domain.

When $\dfrac{O(s)}{R(s)} = \dfrac{1}{s+1}$ (in Laplace transform), then

$$M = \frac{\dfrac{\omega_n^2}{j\omega(j\omega + 2\zeta\omega_n)}}{1 + \dfrac{\omega_n^2}{j\omega(j\omega + 2\zeta\omega_n)}}$$

$$= \frac{\omega_n^2}{(\omega_n^2 - \omega^2) + 2j\zeta\omega_n\,\omega_n} \qquad \text{(after simplification)}$$

or

$$M^2 = \frac{\omega_n^4}{(\omega_n^2 - \omega^2)^2 + 4\zeta^2\omega_n^2\,\omega^2}$$

Now,

$$\frac{d}{d\omega} M^2 = 0 \qquad \text{(for maximum value of the frequency response)}$$

or

$$-\omega_n^4\left[2(\omega_n^2 - \omega^2)(-2\omega) + 4\zeta^2\omega_n^2\,2\omega\right] = 0$$

or

$$\omega = \pm\,\omega_n\sqrt{1 - 2\zeta^2} \quad \text{(after simplification)}$$

At

$$\omega = \pm\,\omega_n\sqrt{1 - 2\zeta^2}$$

$$M^2 = \left[\frac{\omega_n^4}{(\omega_n^2 - \omega_n^2(1 - 2\zeta^2))^2 + 4\zeta^2\omega_n^4(1 - 2\zeta^2)}\right]$$

$$= \frac{1}{4\zeta^2(1 - \zeta^2)}$$

or

$$M = \frac{1}{2\zeta\sqrt{1 - \zeta^2}}$$

Hence the maximum value of frequency response is $\dfrac{1}{2\zeta\sqrt{1 - \zeta^2}}$.

PROBLEM 15.11 If $\dfrac{dX_1}{dt} = aX_1 + bX_2, \qquad \dfrac{dX_2}{dt} = cX_1 + dX_2$

Determine the sufficient conditions on a, b, c, and d so that the asymptotically stable condition can be achieved.

Let us choose a function $W = X_1^2 + X_2^2$ to apply the Liapunov theory. Now,

$$\frac{dW}{dt} = 2X_1\frac{dX_1}{dt} + 2X_2\frac{dX_2}{dt}$$

Damping ratio, $$\zeta = \frac{f}{2}\sqrt{\frac{1}{JK}}$$

$$= \frac{3\times10^{-4}}{2}\sqrt{\frac{1}{5.5\times10^{-2}\times(10^{-2})^2 K}}$$

$$= \frac{3\times10^{-4}}{2}\sqrt{\frac{1}{5.5\times10^{-6} K}}$$

$$= \sqrt{\frac{4.09\times10^{-3}}{K}}$$ (Damping ratio as a function of *K*)

(c) At critically damped condition, $\zeta = 1$
Therefore,

$$\sqrt{\frac{4.09\times10^{-3}}{K}} = 1 \quad \text{or} \quad K = 4.09 \times 10^{-3}\ \text{N}\cdot\text{m/rad}$$

(d) and (e):

Damping ratio, $$\zeta = \sqrt{\frac{4.09\times10^{-3}}{1.5\times10^{-2}}} = 0.522$$

Natural frequency, $$\omega_n = 10^3\sqrt{\frac{K}{5.5}}$$

$$= 10^3\sqrt{\frac{1.5\times10^{-2}}{5.5}} = 52.2\ \text{rad/s}$$

$$\text{Frequency of damped oscillation} = \frac{\omega_n}{2\pi}\sqrt{1-\zeta^2}$$

$$= \frac{52.2}{6.28}\times\sqrt{1-(0.522)^2} = 7\ \text{rad/s}$$

$$\text{Period of damped oscillation} = \frac{1}{\text{frequency of damped oscillation}} = 0.143\ \text{s}$$

$$= \frac{1}{\frac{52.2}{6.28}\sqrt{1-(0.522)^2}}\ \text{s}$$

PROBLEM 15.14 The servomotor of a position servo (Fig. 15.10) drives the load with the help of a 20 : 1 reduction gear and a tachogenerator with the help of a 2 : 1 step-up gear. The load inertia is 20×10^{-6} kg · m^2 and the inertia of the servomotor and the tachogenerator armature are 0.45×10^{-6} kg · m^2 and 0.35×10^{-6} kg · m^2, respectively. Determine the inertia referred to the motor shaft. Also calculate the inertia referred to the load.

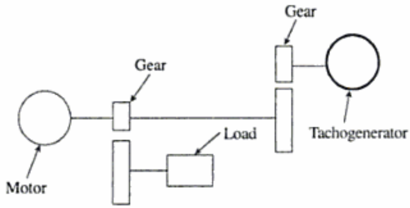


Fig. 15.10 Position servo

Motor inertia referred to the load side

$$= (20)^2 \times 0.45 \times 10^{-6} = 1.8 \times 10^{-4} \text{ kg} \cdot \text{m}^2$$

The transformation ratio of gear trains between the load shaft and the tachogenerator = 20 × 2 = 40. Hence the tachogenerator inertia referred to the load side will be

$$= (40)^2 \times 0.35 \times 10^{-6} = 5.6 \times 10^{-4} \text{ kg} \cdot \text{m}^2$$

Hence, the total inertia referred to the load end

$$= 20 \times 10^{-6} + 1.8 \times 10^{-4} + 5.6 \times 10^{-4}$$
$$= 760 \times 10^{-6} \text{ kg} \cdot \text{m}^2$$

Therefore, the inertia referred to the motor side $= \dfrac{760 \times 10^{-6}}{400}$ kg · m^2

PROBLEM 15.15 A dc motor drive has fixed excitation. Its moment of inertia is 6.5×10^{-2} kg · m^2 and the friction of the motor is 3.5×10^{-2} N · m/rad/s. The motor drives a load which has an inertia of 420 kg · m^2 and friction of 220 N · m/rad/s through a 100 : 1 reduction gear. If the armature current of 0.5 amp is required to produce a torque of 1 N · m at the motor shaft, find the response of the output speed at the load end when a step input of 1 ampere is fed to the armature from a constant current generator. Also, determine the steady-state output speed in rpm. See Fig. 15.11.

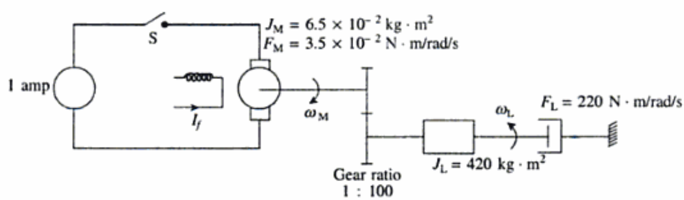


Fig. 15.11 DC motor drive.

Solution Transferring the load moment of inertia and the friction on the motor side, the total moment of inertia J and friction F will be as follows.

$$J = J_M + a^2 J_L$$

$$= \frac{2}{5.7 \times 10^{-2}} \left[1 - e^{\frac{-5.7}{10.7}t} \right] \text{ rad/s}$$

Now,

$$\omega_M(t) = 2\pi N_M$$

where N_M is the rps (revolutions per second) of the motor.

Therefore,

$$N_M = \frac{\omega_M(t)}{2\pi}$$

Thus, motor speed in rpm

$$= \frac{\omega_M(t)}{2\pi} \times 60$$

Load speed will be

$$= \frac{1}{100} \frac{\omega_M(t)}{2\pi} \times 60 \text{ rpm}$$

$$= \frac{1}{100} \frac{60}{2\pi} \frac{2}{5.7 \times 10^{-2}} \left[1 - e^{\frac{-5.7}{10.7}t} \right]$$

Steady output speed

$$= \lim_{t \to \infty} \frac{1}{100} \frac{60}{2\pi} \frac{2}{5.7 \times 10^{-2}} \left[1 - e^{\frac{-5.7}{10.7}t} \right]$$

$$= \frac{1}{100} \frac{60}{2\pi} \cdot \frac{2}{5.7 \times 10^{-2}}$$

$$= 3.35 \text{ rpm}$$

PROBLEM 15.16 Figure 15.12 is a device by virtue of which the liquid level in a tank is controlled. The input valve closes completely when the level of water in the tank attains the height H_r. The fluid level in the tank is H at a particular time. The valve gap opening x is controlled with the help of float F and level L. The value of discharge Q_i is proportional to the gate opening x. The output Q_0 is proportional to head H. Determine the signal flow graph.

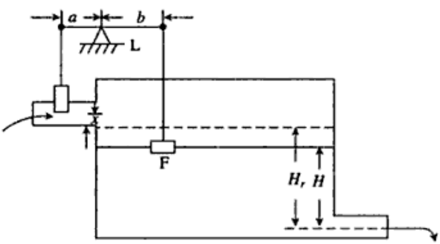


Fig. 15.12 Control of liquid level in a tank.

Solution If the cross-sectional area of the tank is A, then

$$Q_i = A\frac{dH}{dt} + K_1 H \qquad (\because Q_0 \propto H)$$

From Fig. 15.13, we get $F_1 a = F_2 b$ or $\frac{F_2}{F_1} = \frac{a}{b}$. Now the work done in the right-hand side will be equal in magnitude to the work done in the left-hand side. Therefore,

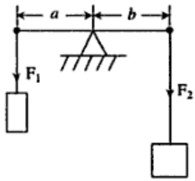


Fig. 15.13

$$F_1 x = F_2(H_r - H)$$

or

$$x = \frac{F_2}{F_1}(H_r - H)$$

or

$$x = \frac{a}{b}(H_r - H)$$

Again it is mentioned

$$Q_i = Kx \text{ when } K \text{ is constant}$$

$$Q_i = A\frac{dH}{dt} + K_1 H$$

Taking Laplace transform

$$Q_i(s) = AsH(s) + K_1 H(s)$$

or

$$H(s) = \frac{Q_i(s)}{As + K_1}$$

$$Q_i(s) = KX(s)$$

$$X(s) = \frac{a}{b}(H_r(s) - H(s))$$

Hence the signal flow graph will be as shown in Fig. 15.14.

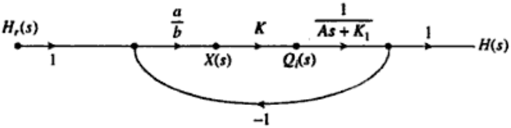


Fig. 15.14

PROBLEM 15.17 Determine the signal flow graph and the transfer function of the cathode follower circuit shown in Fig. 15.15.

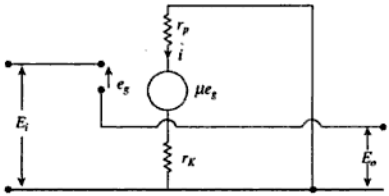


Fig. 15.15 *Cathode follower circuit.*

Solution From Fig. 15.15, we have the relations:

$$E_i = e_g + E_o$$

$$E_o = ir_K$$

$$\mu e_g = ir_p + ir_K = ir_p + E_o$$

Taking Laplace transform of the above equations, we get

$$E_i(s) = e_g(s) + E_o(s)$$

$$E_o(s) = i(s)r_K$$

$$\mu e_g(s) = i(s)r_p + E_o(s)$$

The signal flow graph is shown in Fig. 15.16.

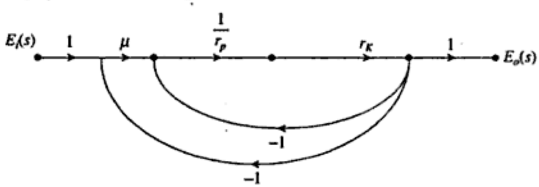


Fig. 15.16 *Signal flow graph of Fig. 15.15.*

According to Mason's gain formula,

$$\frac{E_o(s)}{E_i(s)} = \frac{\mu r_K / r_p}{1 - (-r_K/r_p) - \left(-\mu r_K/r_p\right)} = \frac{\mu r_K}{r_p + r_K(1+\mu)}$$

PROBLEM 15.18 Determine the signal flow graph and the transfer function of the speed control system shown in Fig. 15.17. Assume that the resistance and inductance of both the exciter and the generator are negligibly small.

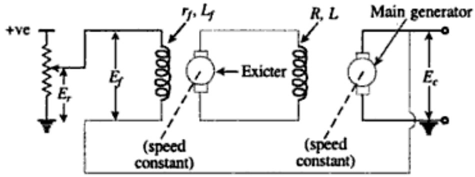


Fig. 15.17 Speed control system.

Solution The following mathematical relations are to obtained after taking the Laplace transforms.

$$E_f(s) = E_r(s) - E_c(s)$$

$$I_f(s) = \frac{E_f(s)}{r_f + sL_f}$$

$$E(s)_{\text{Exciter}} = KI_f(s)$$

$$I(s)_{\text{Exciter}} = \frac{E(s)_{\text{Exciter}}}{R + sL}$$

$$E_c(s) = K_1 I(s)_{\text{Exciter}}$$

Figure 15.18 describes the signal flow graph applying the Mason's gain formula; the transfer function can be found as follows.

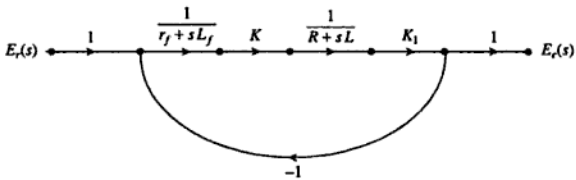


Fig. 15.18 Signal flow graph of Fig. 15.17.

$$\frac{E_c(s)}{E_r(s)} = \frac{\left(\frac{1}{r_f + sL_f}\right)(K)\left(\frac{1}{R + sL}\right)(K_1)}{1 - \left\{\left(\frac{1}{r_f + sL_f}\right)(K)\left(\frac{1}{R + sL}\right)(K_1)(-1)\right\}}$$

$$= \frac{KK_1}{(r_f + sL_f)(R + sL) + KK_1}$$

PROBLEM 15.19 Determine the transfer function of the control system shown in Fig. 15.19 with the help of the signal flow graph.

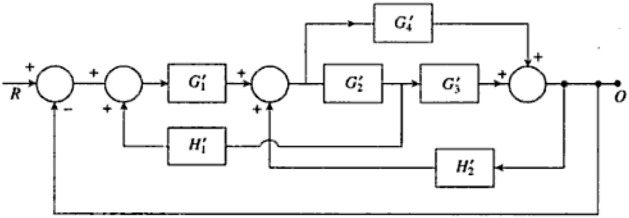


Fig. 15.19 Control system.

Solution Figure 15.20 describes the signal flow graph of Fig. 15.19.

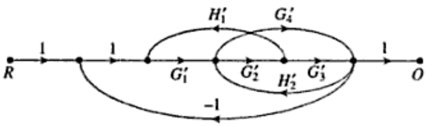


Fig. 15.20 Signal flow graph of Fig. 15.19.

As per Mason's gain formula, the transfer function will be

$$\frac{O}{R} = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta}$$

$$= \frac{G_1'G_2'G_3' + G_1'G_4'}{1-(-G_1'G_2'G_3' + G_1'G_2'H_1' + G_2'G_3'H_2' + G_4'H_2' - G_1'G_4')}$$

Therefore,

$$\frac{O}{R} = \frac{G_1'G_2'G_3' + G_1'G_4'}{1 + G_1'G_2'G_3' - G_1'G_2'H_1' - G_2'G_3'H_2' - G_4'H_2' + G_1'G_4'}$$

PROBLEM 15.20 Find the transfer function of the control system shown in Fig. 15.21.

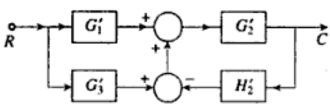


Fig. 15.21 Control system.

Solution Figure 15.21 can be converted to Fig. 15.22 as follows:

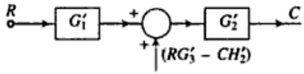


Fig. 15.22 Simplified block diagram of Fig. 15.21.

Therefore,

$$[R \cdot G_1' + (RG_3' - CH_2')]\,G_2' = C$$

or

$$RG_1'G_2' + RG_3'G_2' - CH_2'G_2' = C$$

or

$$C(1 + H_2'G_2') = R(G_1'G_2' + G_3'G_2')$$

or

$$\frac{C}{R} = \frac{G_2'(G_1' + G_3')}{1 + G_2'H_2'}$$

PROBLEM 15.21 What do you mean by the principle of causality in dynamic programming?

Solution If in dynamic programming the future state is determined by the present state, then the principle followed is termed the ***principle of causality***. With the help of this principle, the initial state $x(0)$ and the control sequence $u[0, N - 1]$ uniquely determine the trajectory

$$x[1, N] = \{x(1), x(2), \ldots, x(N)\}$$

Figure 15.23 describes the schematic diagram of the same.

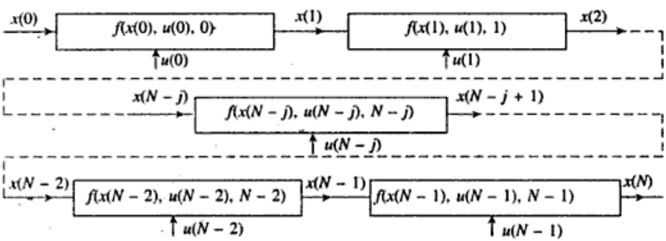


Fig. 15.23 Schematic diagram of the principle of causality.

PROBLEM 15.22 What are the rules for simplifying the complex block diagram configurations of control systems?

Solution The rules for simplifying the complex block diagram configurations are the following:

PROBLEM 15.23 How do you find the closed-loop frequency response from the inverse polar plot?

Solution Let us consider a unity feedback system. The transfer function of the unity feedback system is

$$T(j\omega) = \frac{G(j\omega)}{1+G(j\omega)} = Me^{j\alpha}$$

The inverse transfer function of the unity feedback system is

$$\frac{1}{T(j\omega)} = \frac{1}{M}e^{-j\alpha} = \frac{1+G(j\omega)}{G(j\omega)}$$

$$= 1 + \frac{1}{G(j\omega)}$$

Suppose the polar plot of $1/G(j\omega)$ is as shown in Fig. 15.30.

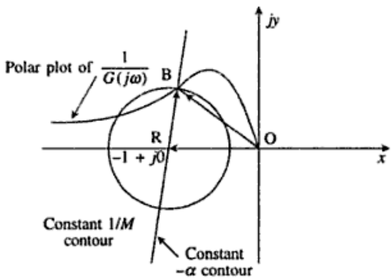


Fig. 15.30 Polar plot $1/G(j\omega)$.

Let us consider the point B on the polar plot, where the frequency is ω_1. The vector **OB** will represent $1/G(j\omega_1)$.

Again,

$$\mathbf{RB} = \mathbf{OB} - \mathbf{OR}$$

where R is at $-1 + j0$. Therefore,

$$\mathbf{RB} = \frac{1}{G(j\omega_1)} - (-1)$$

$$= \frac{1}{G(j\omega_1)} + 1$$

Since

$$\frac{1}{M}e^{-j\alpha} = 1 + \frac{1}{G(j\omega)} \quad \text{(already defined)}$$

$$\frac{1}{G(j\omega_1)} + 1 = \frac{1}{M_1} e^{-j\alpha_1}$$

or

$$\text{RB} = \frac{1}{M_1} e^{-j\alpha_1}$$

Hence, the circle with radius RB and centre R will be the constant $1/M$ circle. The contours of constant values of $-\alpha$ are radial lines passing through $-1 + j0$. This is the method of finding out the closed-loop frequency response from the inverse polar plot.

PROBLEM 15.24 Develop the state equation and the output equation of the circuit shown in Fig. 15.31.

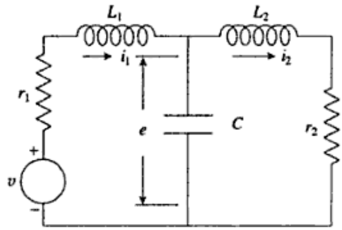


***Fig. 15.31** Circuit diagram.*

Solution From Fig. 15.31, the following equation can be developed.

$$v = i_1 r_1 + L_1 \frac{di_1}{dt} + e$$

$$e = i_2 r_2 + L_2 \frac{di_2}{dt}$$

$$i_1 = C \frac{de}{dt} + i_2$$

Let

$$e = x_1, \quad i_1 = x_2; \quad i_2 = x_3$$

Then,

$$v = x_2 r_1 + L_1 \dot{x}_2 + x_1$$

$$x_1 = x_3 r_2 + L_2 \dot{x}_3$$

$$x_2 = C\dot{x}_1 + x_3$$

Therefore,

$$\dot{x}_1 = \frac{1}{C} x_2 - \frac{1}{C} x_3$$

$$\dot{x}_2 = -\frac{1}{L_1} x_1 - \frac{r_1}{L_1} x_2 + \frac{v}{L_1}$$

$$\dot{x}_3 = -\frac{1}{L_2}x_1 - \frac{r_2}{L_2}x_3$$

Writing in matrix form, we have

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{C} & -\frac{1}{C} \\ -\frac{1}{L_1} & -\frac{r_1}{L_1} & 0 \\ \frac{1}{L_2} & 0 & -\frac{r_2}{L_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L_1} \\ 0 \end{bmatrix} v \quad (1)$$

Let us consider that the outputs are i_2r_2 and i_2. Therefore,

$$y_1 = i_2r_2 \quad \text{and} \quad y_2 = i_2$$

or

$$y_1 = x_3r_2 \quad \text{and} \quad y_2 = x_3$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & r_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad (2)$$

Here equation (1) is the state equation and equation (2) is the output equation.

PROBLEM 15.25 Find out the state equation and the output equation of the electric drive shown in Fig. 15.32.

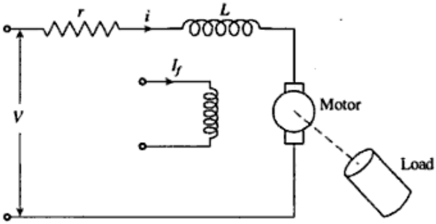


Fig. 15.32 Electric drive.

Solution In the drive shown, the field current I_f is constant. Here

$$V = ri + L\frac{di}{dt} + K\frac{d\theta}{dt}$$

$$\text{Torque} = T = K_1 i$$

$$T = J\frac{d^2\theta}{dt_2} + F\frac{d\theta}{dt}$$

where

J = moment of inertia

F = viscous friction coefficient.

Let

$$\theta = x_1, \quad \dot{\theta} = x_2, \quad i = x_3$$

Thus,

$$\dot{x}_1 = x_2$$

$$V = rx_3 + L\dot{x}_3 + Kx_2$$

$$T = K_1x_3 = J\dot{x}_2 + Fx_2$$

Therefore,

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{F}{J}x_2 + \frac{K_1}{J}x_3$$

$$\dot{x}_3 = -\frac{K}{L}x_2 - \frac{r}{L}x_3 + \frac{V}{L}$$

In matrix form, therefore, the state equation is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\frac{F}{J} & \frac{K_1}{J} \\ 0 & \frac{-K}{L} & -\frac{r}{L} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{L} \end{bmatrix} V$$

With $y = \theta = x_1$ as the output, the output equation is

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

PROBLEM 15.26 Find the stability or the system whose system matrix is given by

$$\begin{bmatrix} 5 & -6 & -12 \\ -1 & 1 & 2 \\ 5 & -6 & -11 \end{bmatrix}$$

Solution The characteristic equation for the system will be

$$|[sI - A]| = 0$$

The matrix with two rows and two columns has the minor = 1 × 1 − 1 × 1 = 0. Therefore, the matrix does not have rank 2.

A system will be reachable if and only if the matrix W_C has rank n of nth order system. Therefore, the given system is not termed reachable.

PROBLEM 15.28 A discrete system has the characteristic equation

$$z^3 + 6z^2 + 8z - 0.4 = 0$$

Determine whether the system is stable or not.

Solution Apply the bilinear transformation over the system characteristic equation by putting

$$z = \frac{r+1}{r-1}$$

Therefore,

$$\left(\frac{r+1}{r-1}\right)^3 + 6\left(\frac{r+1}{r-1}\right)^2 + 8\left(\frac{r+1}{r-1}\right) - 0.4 = 0$$

Simplifying, we obtain

$$14.6r^3 + 2.2r^2 - 12.2r + 3.4 = 0$$

Applying the Routh's criterion,

r^3	14.6	−12.2
r^2	2.2	3.4
r	−34.76	
r^0	3.4	

As there are two sign changes in the first column, the system is unstable.

PROBLEM 15.29 Find the transfer function of the circuit shown in Fig. 15.33 with the help of the signal flow graph.

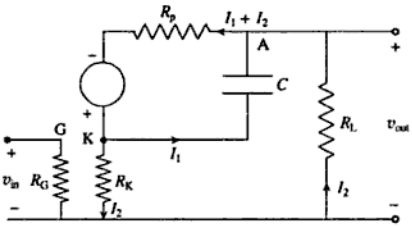


Fig. 15.33 *Circuit diagram.*

Solution The signal flow graph of the circuit shown in Fig. 15.33 is described in Fig. 15.34.

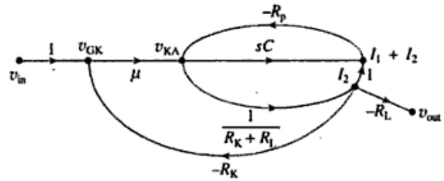


Fig. 15.34 Signal flow graph of Fig. 15.33.

Applying the Mason's gain formula, we get the transfer function as

$$\frac{v_{out}}{v_{in}} = \frac{\mu \frac{1}{R_K + R_L}(-R_L)}{1 - \left[sC(-R_p) + \frac{\mu(R_K)}{R_K + R_L} + \frac{1}{R_K + R_L}(-R_p) \right]}$$

$$= \frac{-\mu R_L}{(R_K + R_L) R_p Cs + (\mu + 1) R_K + R_p + R_L}$$

Note. The mathematical modelling of the above is

$$v_{in} = v_{GK} + I_2 R_K$$

$$v_{KA} = \mu v_{GK} + (I_1 + I_2) R_p$$

$$I_1 = \frac{v_{KA}}{\frac{1}{sC}}$$

$$I_2 = \frac{v_{KA}}{R_K + R_L}$$

$$v_{out} = -I_2 R_L$$

PROBLEM 15.30 How do you convert the following nonlinear differential system

$$\frac{d^2y}{dt^2} + y \cos y = x$$

to a linear system when $x = 0$, $y = 0$?

Solution

$$\frac{d^2y}{dt^2} + y \cos y = x$$

or

$$\frac{d^2y}{dt^2} + y\left(1 - \frac{y^2}{2!} + \cdots\right) - x = 0$$

The damped natural frequency is therefore $\sqrt{3}/2$ rad/second.

$$\text{The time constant} = \frac{1}{\frac{5}{2}} = \frac{2}{5} = 0.4\text{ s}$$

$$\text{Damping ratio} = \frac{5}{2\sqrt{7}}$$

$$\text{Undamped natural frequency} = \sqrt{7}\text{ rad/s}$$

PROBLEM 15.32 Find the free response of the system

$$t^2\frac{d^2y}{dt^2} - 2t\frac{dy}{dt} + 2y = 0$$

with the initial conditions

$$y(0) = 0,\ \left.\frac{dy}{dt}\right|_{t=0} = 1,\ \left.\frac{d^2y}{dt^2}\right|_{t=0} = 0$$

Solution Putting $t = e^x$, i.e. $x = \log t$, we have

$$\frac{dy}{dt} = \frac{dy}{dx}\cdot\frac{dx}{dt}$$

$$= \frac{dy}{dx}\cdot\frac{1}{t}$$

or

$$t\frac{dy}{dt} = \frac{dy}{dx}$$

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\cdot\left(\frac{dy}{dt}\right)$$

$$= \frac{d}{dt}\cdot\left(\frac{1}{t}\frac{dy}{dx}\right)$$

$$= -\frac{1}{t^2}\frac{dy}{dx} + \frac{1}{t}\frac{d}{dt}\left(\frac{dy}{dx}\right)$$

$$= -\frac{1}{t^2}\frac{dy}{dx} + \frac{1}{t}\frac{d}{dx}\left(\frac{dy}{dx}\right)\frac{dx}{dt}$$

$$= -\frac{1}{t^2}\frac{dy}{dx} + \frac{1}{t}\frac{d^2y}{dx^2}\cdot\frac{1}{t}$$

$$= \frac{1}{t^2}\left(-\frac{dy}{dx}+\frac{d^2y}{dx^2}\right)$$

or

$$t^2\frac{d^2y}{dt^2} = -\frac{dy}{dx}+\frac{d^2y}{dx^2}$$

Therefore, the differential equation

$$t^2\frac{d^2y}{dt^2}-2t\frac{dy}{dt}+2y = 0$$

can be written as

$$\frac{-dy}{dx}+\frac{d^2y}{dx^2}-2\frac{dy}{dx}+2y = 0$$

or

$$\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y = 0$$

With $y = Ae^{mx}$, the differential equation becomes

$$Am^2e^{mx} - 3Ame^{mx} + 2Ae^{mx} = 0$$

or

$$m^2 - 3m + 2 = 0$$

or

$$m = 2 \quad \text{or} \quad 1$$

Therefore,

$$\begin{aligned} y &= A_1e^{2x} + A_2e^{x} \\ &= A_1e^{2\log t} + A_2e^{\log t} \\ &= A_1e^{\log t^2} + A_2e^{\log t} \\ &= A_1t^2 + A_2t \end{aligned}$$

$$y(0) = 0, \quad \left.\frac{dy}{dt}\right|_{t=0} = 1$$

$$\frac{dy}{dt} = 2tA_1 + A_2$$

When

$$t = 0,$$

$$\left.\frac{dy}{dt}\right|_{t=0} = 1 = A_2$$

Again,

$$\left.\frac{d^2y}{dt^2}\right|_{t=0} = 0 = 2A_1 \text{ or } A_1 = 0$$

Therefore, $y = t$ is the free response of the system.

PROBLEM 15.33 What is singular point? What is Liapunov's stability criterion? Determine whether the following system is stable or not:

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} + \left(\frac{dx}{dt}\right)^3 + x = 0$$

Solution If a system is described by

$$\frac{dx_1}{dt} = f_1(x_1, x_2); \quad \frac{dx_2}{dt} = f_2(x_1, x_2)$$

then a single equation is developed by eliminating the independent variable time t, such as:

$$\frac{dx_1}{dx_2} = \frac{f_1(x_1, x_2)}{f_2(x_1, x_2)}$$

The solution of the above equation will describe a trajectory in the phase plane. The point (x_1, x_2) in the phase plane, which will satisfy the two equations $f_1(x_1, x_2) = 0$ and $f_2(x_1, x_2) = 0$, is termed the singular point.

Liapunov's stability criterion can be described as follows:

If the origin is a singular point, then it is stable if a function $V(x_1, x_2)$ can be found such that:

(a) $V(x_1, x_2)$ is positive for all values of x_1 and x_2 except that it may be zero for $x_1 = x_2 = 0$.

(b) dV/dt is never positive. If dV/dt is never zero except at the origin, then the origin is asymptotically stable.

The given system is

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} + \left(\frac{dx}{dt}\right)^3 + x = 0$$

Let

$$x_1 = x, \; x_2 = \frac{dx}{dt}, \quad \text{then}$$

$$\dot{x}_1 = \dot{x} = x_2$$

Therefore,

$$\dot{x}_2 + x_2 + (x_2)^3 + x_1 = 0$$

$$\dot{x}_1 = x_2 = f_1(x_1, x_2)$$

$$\dot{x}_2 = -x_1 - x_2 - (x_2)^3 = f_2(x_1, x_2)$$

Here, (x_1, x_2) has the singular point at (0, 0) because $f_1(x_1, x_2) = 0$ and $f_2(x_1, x_2) = 0$ at (0, 0). Let us consider $V = x_1^2 + x_2^2$ which is positive for all x_1 and x_2 except $x_1 = x_2 = 0$.

Again,

$$\frac{dV}{dt} = 2x_1\frac{dx_1}{dt} + 2x_2\frac{dx_2}{dt}$$

$$= 2x_1x_2 + 2x_2\,[-x_1 - x_2 - (x_2)^3]$$

$$= \frac{B}{A}(e^{Ah} - 1)$$

Therefore,

$$x(nh + h) = e^{Ah}x(nh) + \frac{B}{A}(e^{Ah} - 1)u(nh)$$

where h is the period of periodic sampling.

PROBLEM 15.36 Find the impulse response of

$$G(s) = \frac{25}{(s^2 + 4s + 25)}$$

with the help of MATLAB.

Solution

```
num= [0 0 25];
den= [1 4 25];
Impulse (num, den)
```

The plot of the impulse response generated by MATLAB is shown in Fig. 15.36.

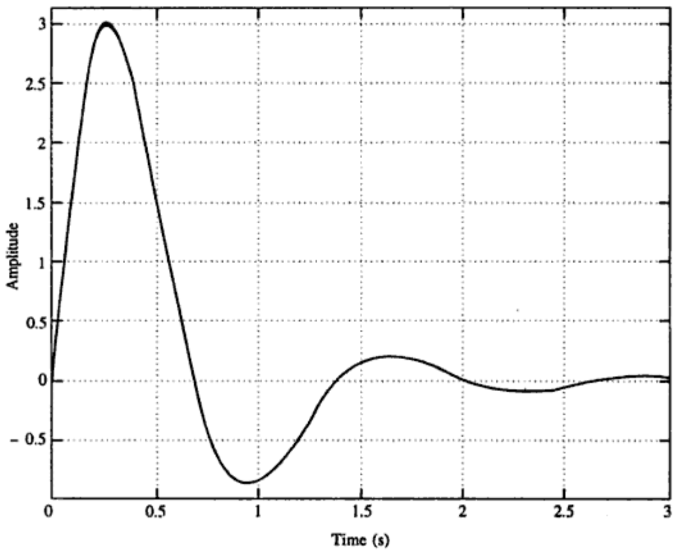


Fig. 15.36 Impulse response of $G(s) = 25/(s^2 + 4s + 25)$.

PROBLEM 15.37 Find the step response of

$$G(s) = \frac{25}{(s^2 + 4s + 25)}$$

with the help of MATLAB.

Solution

```
num = [0 0 25];
den = [1 4 25];
step (num, den)
```

The plot of the step response generated by MATLAB is shown in Fig. 15.37.

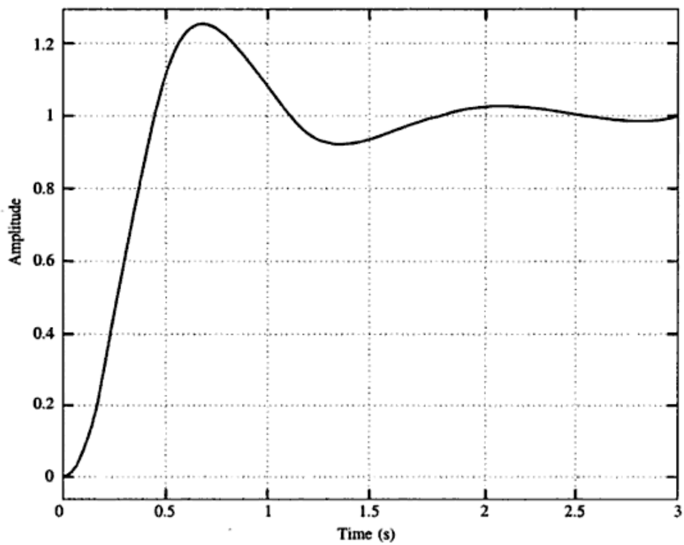


Fig. 15.37 Unit step response of $G(s) = 25/(s^2 + 4s + 25)$.

PROBLEM 15.38 How do you find the roots of the characteristic equation

$$s^4 + 2s^3 + s^2 - 2s - 1 = 0$$

with the help of MATLAB?

Solution

```
p = [1 2 1 -2 -1]

roots (p)

>> pwd

ans =

C:\MATLAB\bin

>> cd ..

>> ban7

p =

     1     2     1    -2    -1

ans =

    -1.2071  +  0.9783i
    -1.2071  -  0.9783i
     0.8832
    -0.4690

>>
```

PROBLEM 15.40 Develop the Nyquist diagram of the following unity feedback system

$$G(s) = \frac{1}{s^2 + 0.8s + 1}$$

with the help of MATLAB.

Solution

```
num = [0 0 1];
den = [1 .8 1];
nyquist (num, den)
```

The Nyquist plot generated by MATLAB is shown in Fig. 15.39.

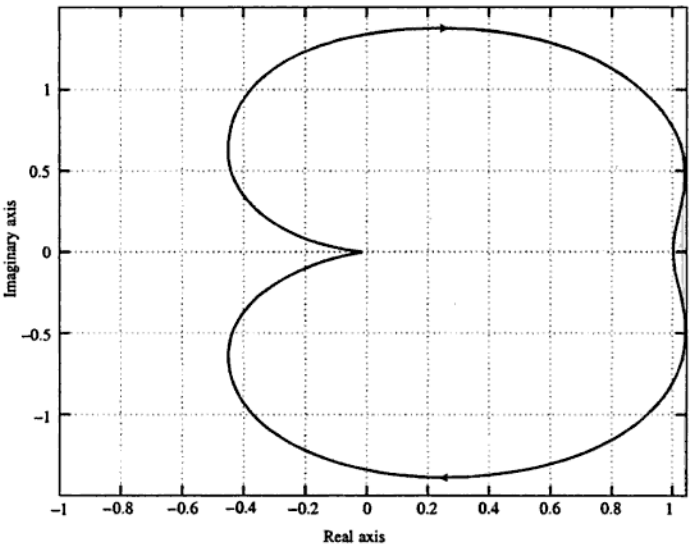


Fig. 15.39 Nyquist plot of $G(s) = 1/(s^2 + 0.8s + 1)$.

PROBLEM 15.41 Develop the Nyquist diagram of the unity feedback system

$$G(s) = \frac{s+2}{(s+1)(s-1)}$$

with the help of MATLAB.

Solution

```
num = [0 1 2];
den = [1 0 -1];
nyquist (num, den)
```

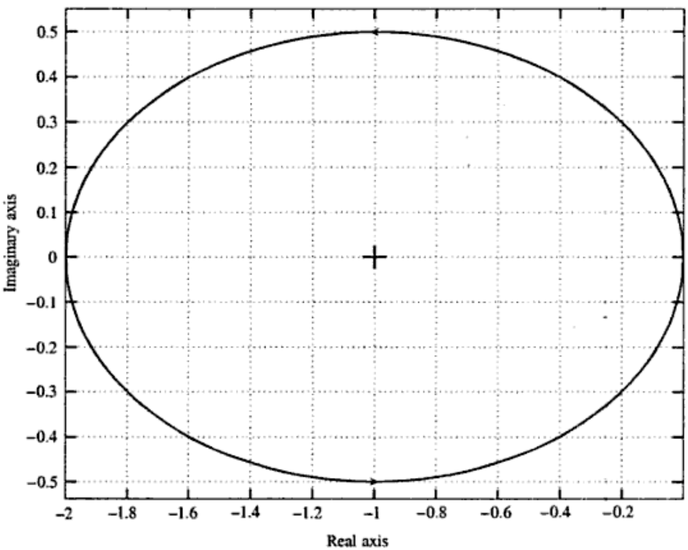


Fig. 15.40 Nyquist plot of G(s) = (s + 2)/(s + 1)(s − 1).

PROBLEM 15.43 Determine the eigenvalues of $A = \begin{bmatrix} 0 & 6 & -5 \\ 1 & 0 & 2 \\ 3 & 2 & 4 \end{bmatrix}$ by MATLAB.

Solution

```
A = [0 6 -5; 1 0 2; 3 2 4]
B = eig(A)

To get started, type one of these: helpwin, helpdesk, or demo.
For product information, type tour or visit www.mathworks.com.

>> pwd

ans =

C:\MATLAB\bin

>> cd ..
>> ban7

A =

     0     6    -5
     1     0     2
     3     2     4

B =

    2.0000
    1.0000 + 0.0000i
    1.0000 - 0.0000i

>>
```

PROBLEM 15.44 Using MATLAB, find the state space realization of an LTI system, considering an SISO system, where

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix};\quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix};\quad C = \begin{bmatrix} 1 & 1 \end{bmatrix};\quad D = 0;\quad E = 0.$$

Solution The state space realization of the LTI system is expressed by the system matrix as follows:

$$\begin{bmatrix} A + j(E - I) & B & na \\ C & D & 0 \\ 0 & 0 & -\text{Inf} \end{bmatrix}$$

PROBLEM 15.45 Calculate the covariant matrix of $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ with the help of MATLAB.

Solution

```
A = [1 0 0; 0 2 0; 0 0 3]
cov(A)

To get started, type one of these: helpwin, helpdesk, or demo.
For product information, type tour or visit www.mathworks.com.

>> pwd

ans =

C:\MATLAB\bin

>> cd ..

>> ban7

A =

     1     0     0
     0     2     0
     0     0     3

ans =

     0.3333    -0.3333    -0.5000
    -0.3333     1.3333    -1.0000
    -0.5000    -1.0000     3.0000

>>
```

PROBLEM 15.46 Find the transfer function (TF) of the system whose signal flow graph is shown in Fig. 15.42.

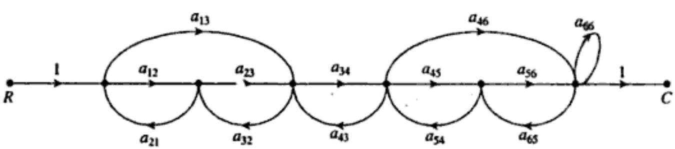


Fig. 15.42 Signal flow graph.

Solution By applying the well-known principles of physics, we find that the variables are related as follows:

$$m_1\ddot{x}_1 + b_1\dot{x}_1 + K_1x_1 + b_2(\dot{x}_1 - \dot{x}_2) + K_2(x_1 - x_2) = 0$$

and

$$m_2\ddot{x}_2 + b_2(\dot{x}_2 - \dot{x}_1) + K_2(x_2 - x_1) = 0$$

which gives the second-order differential equation model of the system.

PROBLEM 15.48 Develop the mathematical model of the system shown in Fig. 15.44.

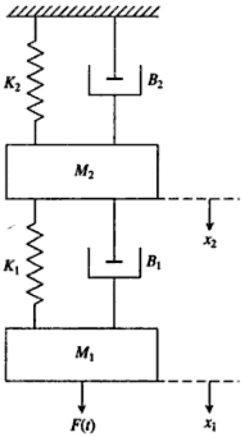


Fig. 15.44 Mathematical model.

Solution

At node x_1:

$$F(t) = M_1\ddot{x}_1 + B_1(\dot{x}_1 - \dot{x}_2) + K_1(x_1 - x_2)$$

At node x_2:

$$M_2\ddot{x}_2 + B_2\dot{x}_2 + K_2x_2 = B_1(\dot{x}_1 - \dot{x}_2) + K_1(x_1 - x_2)$$

The above two equations give the second-order differential equation model of the system.

Problem 15.49 Sketch the root locus of the system

$$1+\frac{K}{s(s+3)(s^2+2s+2)}$$

with the help of MATLAB.

Solution

```
num = [0 0 0 0 1];
den = [1 5 8 6 0];
rlocus (num, den)
```

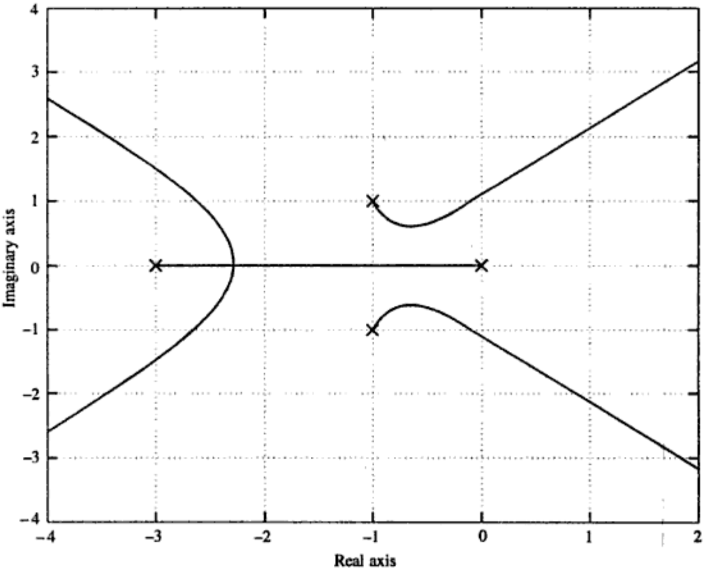


Fig. 15.45 Root locus of $1 + [K/s(s + 3)(s^2 + 2s + 2)]$.

PROBLEM 15.50 Construct the Bode plot for

$$G(s) = \frac{16(s+2)}{s(s+0.5)(s^2+3.2s+64)}$$

with the help of MATLAB.

Solution

```
num = [0 0 0 16 32];
den = [2 7.4 131.2 64 0];
bode (num, den)
```

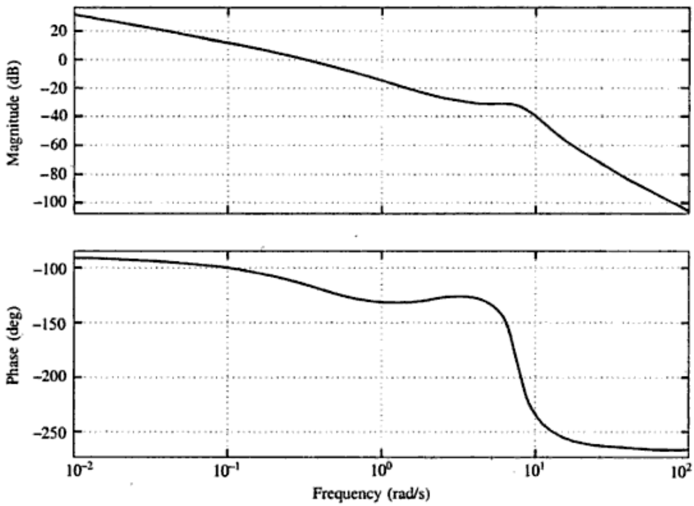


Fig. 15.46 Bode Plot for $G(s) = 16(s+2)/s(s+0.5)(s^2+3.2s+64)$.

PROBLEM 15.51 A first-order system is described by the differential equation

$$\dot{x}(t) = u(t), \quad x(0) = x_0$$

Find the optimal law using the Hamilton–Jacobi equation that minimizes the performance index

$$J = \int_0^{t_f} (x^2 + u^2)dt$$

where t_f is specified.

$$L = t\dot{x}(t) + \dot{x}^2(t)$$

we have

$$\frac{\delta L}{\delta x} - \frac{d}{dt}\frac{\delta L}{\delta \dot{x}} = 0 \qquad \text{(As per Euler's equation)}$$

or

$$0 - \frac{d}{dt}\{t + 2\dot{x}(t)\} = 0$$

or

$$1 + 2\ddot{x}(t) = 0$$

or

$$2\ddot{x} = -1$$

or

$$\frac{d^2x}{dt^2} = -\frac{1}{2}$$

or

$$\frac{dx}{dt} = -\frac{1}{2}t + K$$

or

$$x = -\frac{1}{2}\frac{t^2}{2} + Kt + K_1$$

Now, $x(0) = 1$, therefore, $K_1 = 1$. Since $x(2)$ is free,

$$\left.\frac{\delta L}{\delta \dot{x}}\right|_{t=2} = 0$$

or

$$(t + 2\dot{x}(t))|_{t=2} = 0$$

or

$$\left.\left(t + 2\left(-\frac{1}{2}t + K\right)\right)\right|_{t=2} = 0$$

or

$$2K = 0$$

Therefore,

$$x = -\frac{1}{4}t^2 + 1$$

Hence the equation of the curve is $x = -\frac{1}{4}t^2 + 1$.

PROBLEM 15.53 Find the state model for a system characterized by the differential equation

$$\frac{d^3y}{dt^2} + \frac{6d^2y}{dt^2} + \frac{11\,dy}{dt} + 6y = u$$

Solution The transfer function is

$$T(s) = \frac{Y(s)}{U(s)} = \frac{1}{s^3 + 6s^2 + 11s + 6}$$

$$= \frac{1}{(s+1)(s+2)(s+3)}$$

$$= \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s+3}$$

$$A = \lim_{s \to -1} \frac{1}{(s+2)(s+3)} = \frac{1}{2}$$

$$B = \lim_{s \to -2} \frac{1}{(s+1)(s+3)} = -1$$

$$C = \lim_{s \to -3} \frac{1}{(s+1)(s+2)} = \frac{1}{2}$$

Therefore,

$$Y(s) = \frac{1/2}{s+1} U(s) - \frac{1}{s+2} U(s) + \frac{1/2}{s+3} U(s)$$

Let

$$X_1(s) = \frac{1/2}{s+1} U(s)$$

$$X_2(s) = -\frac{1}{s+2} U(s)$$

$$X_3(s) = \frac{1/2}{s+3} U(s)$$

Therefore,

$$sX_1(s) + X_1(s) = \frac{1}{2} U(s)$$

$$sX_2(s) + 2X_2(s) = -U(s)$$

$$sX_3(s) + 3X_3(s) = \frac{1}{2} U(s)$$

Taking Laplace inverse of the above equation, we have

$$\dot{x}_1 = -x_1 + \frac{1}{2} u$$

$$\dot{x}_2 = -2x_2 - u$$

$$\dot{x}_3 = -3x_3 + \frac{1}{2} u$$

Since

$$Y(s) = X_1(s) + X_2(s) + X_3(s)$$

we get

$$y = x_1 + x_2 + x_3$$

For stability, $\frac{2K}{4} < 1$ or $K < 2$

Matlab plotting of the problem is obtained from the program:

```
num = [0 0 0 2];
den = [1 2 2 0];
nyquist (num, den)
```

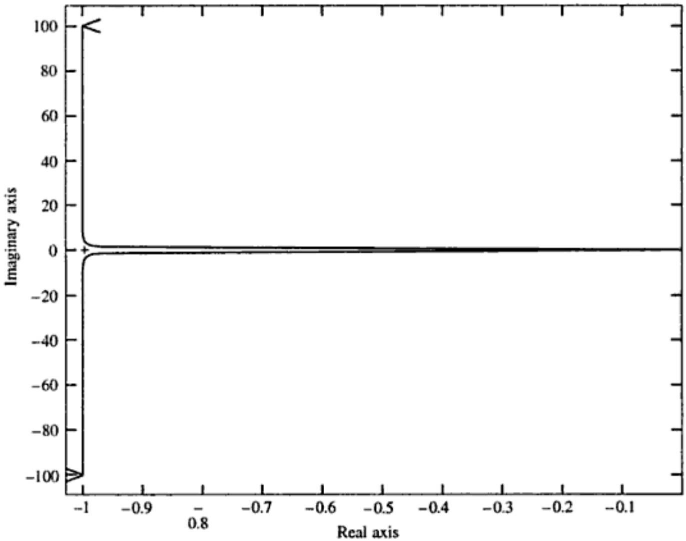


Fig. 15.49 Nyquist diagrams.

PROBLEM 15.55 In the mechanical system shown in Fig. 15.50, determine the value of $X_2(s)/F(s)$ where $x_2(t)$ and $f(t)$ are displacement and force respectively.

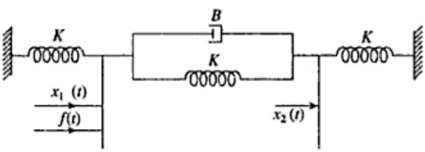


Fig. 15.50 Mechanical system.

Solution The equation of the mechanical system will be:

At node x_1:

$$f = B(\dot{x}_1 - \dot{x}_2) + K(x_1 - x_2) + Kx_1$$

At node x_2:

$$0 = B(\dot{x}_2 - \dot{x}_1) + K(x_2 - x_1) + Kx_2$$

Taking the Laplace transform of the above relations, we have

$$F(s) = BsX_1(s) - BsX_2(s) + KX_1(s) - KX_2(s) + KX_1(s) \quad \text{(i)}$$

and

$$0 = BsX_2(s) - BsX_1(s) + KX_2(s) - KX_1(s) + KX_2(s) \quad \text{(ii)}$$

From Eq. (ii), we have

$$X_1(s) = \frac{(Bs + 2K)}{(Bs + K)} X_2(s)$$

From Eq. (i), we have

$$F(s) = (Bs + 2K)X_1(s) - (Bs + K)X_2(s)$$

or

$$F(s) = (Bs + 2K)\frac{(Bs + 2K)}{(Bs + K)} X_2(s) - (Bs + K)X_2(s)$$

On simplification, we obtain

$$\frac{X_2(s)}{F(s)} = \frac{Bs + K}{2BsK + 3K^2}$$

PROBLEM 15.56 Write the differential equations for the mechanical system shown in Fig. 15.51. Obtain an analogus electrical circuit based on force current analogy.

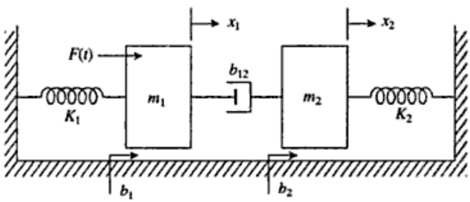


Fig. 15.51 Mechanical system.

Solution The differential equations for the mechanical system will be as follows:

$$m_1\ddot{x}_1 + b_1\dot{x}_1 + K_1x_1 + b_{12}(\dot{x}_1 - \dot{x}_2) = F(t)$$

$$m_2\ddot{x}_2 + b_2\dot{x}_2 + b_{12}(\dot{x}_2 - \dot{x}_1) + K_2x_2 = 0$$

The mechanical network will be as shown in Fig. 15.52.

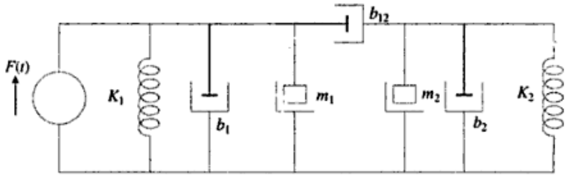


Fig. 15.52 Mechanical network.

The electrical analogus circuit is shown in Fig. 15.53.

At the voltage node v_1:

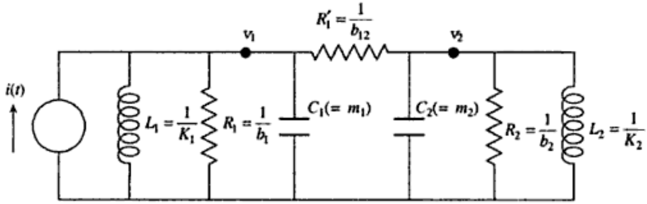


Fig. 15.53 Electrical analogus for the mechanical system of Fig. 15.52.

$$i(t) = \left(\frac{1}{L_1}\int_{-\infty}^{t} v_1\,dt\right) + \frac{v_1}{R_1} + C_1\frac{dv_1}{dt} + \frac{v_1 - v_2}{R_1'}$$

At the voltage node v_2:

$$0 = \frac{v_2 - v_1}{R_1'} + C_2\frac{dv_2}{dt} + \frac{v_2}{R_2} + \frac{1}{L_2}\int_{-\infty}^{t} v_2\,dt$$

Hence

$$i(t) = K_1\int_{-\infty}^{t} v_1\,dt + b_1 v_1 + m_1\frac{dv_1}{dt} + b_{12}(v_1 - v_2)$$

$$0 = b_{12}(v_2 - v_1) + m_2\frac{dv_2}{dt} + b_2 v_2 + K_2\int_{-\infty}^{t} v_2\,dt$$

From the above equations, it is clear that an analogus circuit based on force and current analogy has been developed.

PROBLEM 15.57 A feedback system has derivative feedback as shown in Fig. 15.54.

(a) If $K_0 = 0$, find the damping ratio, undamped natural frequency ω_n of the system. What is the steady-state error to a unit step input?

$$= \frac{1}{\left(-\zeta + j\sqrt{1-\zeta^2}\right)^2 2j\omega_n\sqrt{1-\zeta^2}}$$

$$= \frac{1}{\left\{\zeta^2 - (1-\zeta^2) - 2j\zeta\sqrt{1-\zeta^2}\right\}2j\omega_n\sqrt{1-\zeta^2}}$$

$$= \frac{1}{\left\{2\zeta^2 - 1 - 2j\zeta\sqrt{1-\zeta^2}\right\}2j\omega_n\sqrt{1-\zeta^2}}$$

$$= \frac{1}{2\left[j(2\zeta^2-1)\,\omega_n\sqrt{1-\zeta^2} + 2\zeta(1-\zeta^2)\,\omega_n\right]}$$

$$= \frac{1}{2\omega_n}\,\frac{1}{\left[2\zeta(1-\zeta^2) + j(2\zeta^2-1)\sqrt{1-\zeta^2}\right]}$$

$$= \frac{1}{2\omega_n\sqrt{1-\zeta^2}}\,\frac{1}{\left[2\zeta\sqrt{1-\zeta^2} + j(2\zeta^2-1)\right]}$$

$$D = C^* = \frac{1}{2\omega_n\sqrt{1-\zeta^2}}\,\frac{1}{\left[2\zeta\sqrt{1-\zeta^2} - j(2\zeta^2-1)\right]}$$

Therefore,

$$\frac{\omega_n^2}{s^2(s^2+2\zeta\omega_n s+\omega_n^2)} = \frac{1}{s^2} - \frac{2\zeta}{\omega_n s}$$

$$+ \frac{1}{2\omega_n\sqrt{1-\zeta^2}\left[2\zeta\sqrt{1-\zeta^2} + j(2\zeta^2-1)\right]\left[s - \left(-\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}\right)\right]}$$

$$+ \frac{1}{2\omega_n\sqrt{1-\zeta^2}\left[2\zeta\sqrt{1-\zeta^2} - j(2\zeta^2-1)\right]\left[s - \left(-\zeta\omega_n - j\omega_n\sqrt{1-\zeta^2}\right)\right]}$$

Taking Laplace inverse, the output

$$o(t) = t - \frac{2\zeta}{\omega_n} + \frac{e^{-\zeta\omega_n t + j\omega_n\sqrt{1-\zeta^2}\,t}}{2\omega_n\sqrt{1-\zeta^2}\left[2\zeta\sqrt{1-\zeta^2} + j(2\zeta^2-1)\right]}$$

$$+ \frac{e^{-\zeta\omega_n t - j\omega_n\sqrt{1-\zeta^2}\,t}}{2\omega_n\sqrt{1-\zeta^2}\left[2\zeta\sqrt{1-\zeta^2} - j(2\zeta^2-1)\right]}$$

or

$$\theta = \tan^{-1} \frac{2\zeta\sqrt{1-\zeta^2}}{2\zeta^2 - 1}$$

Therefore,

$$o(t) = t - \frac{2\zeta}{\omega_n} + \frac{e^{-\zeta\omega_n t}}{2\omega_n\sqrt{1-\zeta^2}}\left[2\sin\theta\cos\omega_n\sqrt{1-\zeta^2}\,t + 2\cos\theta\sin\omega_n\sqrt{1-\zeta^2}\,t\right]$$

$$= t - \frac{2\zeta}{\omega_n} + \frac{e^{-\zeta\omega_n t}}{\omega_n\sqrt{1-\zeta^2}}\sin\left[\omega_n\sqrt{1-\zeta^2}\,t + \theta\right]$$

where $\theta = \tan^{-1} \frac{2\zeta\sqrt{1-\zeta^2}}{2\zeta^2 - 1}$.

PROBLEM 15.59 A servo system is represented by the signal flow graph shown in Fig. 15.56. The variable T is the torque and E is the error. Determine:

(a) The overall transfer function if $K_1 = 1$, $K_2 = 5$, and $K_3 = 5$
(b) The sensitivity of the system to changes in K_1 for $\omega = 0$.

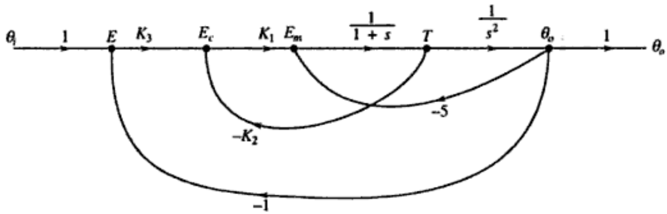


Fig. 15.56 Signal flow graph.

Solution (a) Overall transfer function

$$T = \frac{K_3K_1\frac{1}{1+s}\cdot\frac{1}{s^2}}{1-\left[K_3K_1\frac{1}{1+s}\cdot\frac{1}{s^2}(-1) + \frac{K_1}{1+s}(-K_2) + \frac{1}{(1+s)s^2}(-5)\right]}$$

$$= \frac{K_3K_1}{s^3 + s^2 + K_3K_1 + K_1K_2s^2 + 5}$$

$$= \frac{5}{s^3 + s^2 + 5 + 5s^2 + 5} \qquad (\because K_1 = 1, K_2 = 5, K_3 = 5)$$

$$= \frac{5}{s^3 + 6s^2 + 10}$$

(b) Sensitivity of the system to changes in $K_1 = \frac{\delta T}{\delta K_1} \cdot \frac{K_1}{T}$

Substituting, $s = j\omega$

$$T = \frac{5K_1}{(j\omega)^3 + (j\omega)^2 + 5K_1 + 5K_1(j\omega)^2 + 5} = \frac{5K_1}{5K_1 + 5} \quad (\because \omega = 0)$$

Thus,

$$\frac{\delta T}{\delta K_1} \cdot \frac{K_1}{T} = \frac{\delta}{\delta K_1}\left(\frac{5K_1}{5K_1 + 5}\right) \cdot \frac{K_1}{\dfrac{5K_1}{5K_1 + 5}}$$

$$= \frac{1}{K_1 + 1} \text{ (after differentiation and simplification)}$$

$$= \frac{1}{1+1} = 0.5 \qquad (\because K_1 = 1)$$

PROBLEM 15.60 The open-loop transfer function of a unity feedback system is given by

$$G(s) = \frac{2(s + \alpha)}{s(s + 2)(s + 10)}$$

Sketch the root locus with α varying from 0 to ∞. Find the angle and real axis intercept of the asymptotes, breakaway point and the points crossing imaginary axis, if any.

Solution Here,

$$G(s) = \frac{2(s + \alpha)}{s(s + 2)(s + 10)}$$

The characteristic equation is

$$1 + \frac{2(s + \alpha)}{s(s + 2)(s + 10)} = 0$$

or

$$s^3 + 12s^2 + 22s + 2\alpha = 0$$

or

$$1 + \frac{2\alpha}{s^3 + 12s^2 + 22s} = 0$$

or

$$1 + \frac{2\alpha}{s(s^2 + 12s + 22)} = 0$$

The roots of $s^2 + 12s + 22 = 0$ are -2.26, -9.74.

$$\text{Asymptotic centroid} = \frac{0 - 2.26 - 9.74}{3} = -\frac{12}{3} = -4$$

$$\text{Asymptotic angle} = \frac{(2q+1)\pi}{3}, \; q = 0, 1, 2$$

$$= \frac{\pi}{3}, \pi, \frac{5\pi}{3}$$

For breakaway point,

$$\frac{2\alpha}{s(s+2.26)(s+9.74)} = -1$$

or

$$2\alpha = -s(s + 2.26)(s + 9.74)$$

or

$$\frac{d\alpha}{ds} = -\frac{1}{2}\left[\frac{d}{ds}(s^3 + 12s^2 + 22s)\right] = 0$$

or

$$3s^2 + 24s + 22 = 0$$

or

$$s = \frac{-24 \pm \sqrt{576 - 264}}{6} = -1.06, -6.94$$

Hence the breakaway point is -1.06.

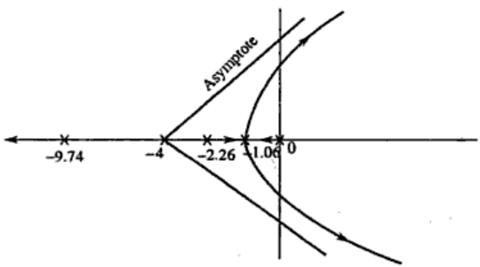


Fig. 15.57 Root locus diagram.

The characteristic equation is

$$s(s + \alpha) + 1 = 0$$

or

$$s^2 + s\alpha + 1 = 0$$

or

$$1 + \frac{s\alpha}{s^2 + 1} = 0$$

The roots are $s^2 + 1 = 0$ or $s = \pm j$.

Breakaway point:

$$\alpha = -\frac{s^2 + 1}{s}$$

or

$$\frac{d\alpha}{ds} = -\frac{d}{ds}\left(\frac{s^2 + 1}{s}\right) = 0$$

or

$$2s \cdot s - (s^2 + 1) = 0$$

or

$$s = \pm 1$$

Hence the breakaway point is at −1. The root locus will be as shown in Fig. 15.59.

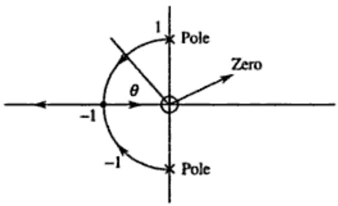


Fig. 15.59 Root locus plot of Fig. 15.58.

Therefore the root locus will be a circle, Now,

$$1 + \frac{s\alpha}{s^2 + 1} = 0$$

or

$$\alpha = \left|\frac{s^2 + 1}{s}\right|$$

When the damping ratio is 0.5

$$\cos\theta = 0.5,\ \cos 60° = 0.5$$

The coordinate will be

$$0.5 + j\sin 60° = 0.5 + j\frac{\sqrt{3}}{2}$$

Therefore,

$$\alpha = \left|\frac{\left(0.5 + j\frac{\sqrt{3}}{2}\right)^2 + 1}{0.5 + j\frac{\sqrt{3}}{2}}\right| = \left|\frac{\frac{1}{4} + 2j\frac{1}{2}\frac{\sqrt{3}}{2} - \frac{3}{4} + 1}{0.5 + j\frac{\sqrt{3}}{2}}\right| = 1$$

The characteristic equation is

$$s^2 + 1 + s = 0 \qquad \text{or} \qquad s = -\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$$

The overall transfer function in the factored form is

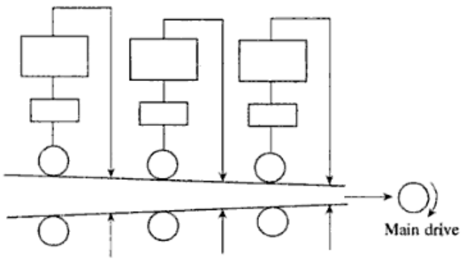
$$\frac{\frac{1}{s(s+1)}}{1 + \frac{1}{s(s+1)}} = \frac{1}{s^2 + s + 1} = \frac{1}{\left[s - \left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)\right]\left[s - \left(-\frac{1}{2} - j\frac{\sqrt{3}}{2}\right)\right]}$$

Chapter

16 *Objective Type Questions*

1. A thermostat-controlled room heater is an example of
   (a) a continuous system (b) a discontinuous system
   (c) a continuous dynamic system (d) none of the above

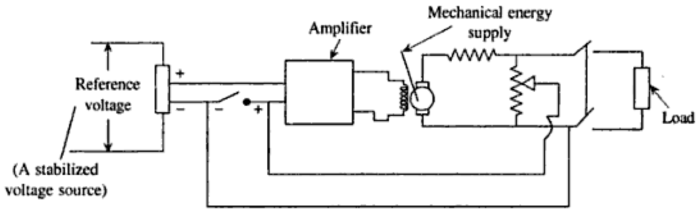
2. The strip gauge control in a continuous rolling mill as shown in the figure below is an example of



   (a) a single input single output system (b) a multivariable control
   (c) a multilevel stochastic control (d) none of the above

3. A nonlinear system is that system
   (a) which obeys the principle of superposition
   (b) which does not obey the principle of superposition
   (c) which is dynamic in nature
   (d) which is none of the above

4. The voltage regulating system shown in the following figure is
   (a) an open-loop system (b) a closed-loop positive feedback system
   (c) a closed-loop negative feedback system (d) none of the above



5. The system described by $\dot{x} = f(x)$ is
   (a) an autonomous system (b) a static system
   (c) a time dependent system (d) none of the above
6. When a system satisfies the properties of homogeneity and additivity, it is termed:
   (a) nonlinear (b) linear
   (c) nonlinear dynamic (d) none of the above
7. When a function is described as follows,

$$u(t) = 0 \quad \text{for } t < 0$$
$$u(t) = 1 \quad \text{for } t > 0$$

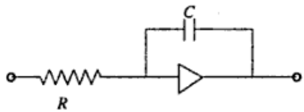
   it is termed
   (a) ramp function (b) impulse function
   (c) step function (d) none of the above
8. The Laplace transform of the impulse function is
   (a) zero (b) one
   (c) greater than one (d) none of the above
9. The stochastic inputs are based on
   (a) probability distribution (b) deterministic approach
   (c) sinusoidal inputs (d) none of the above
10. For the critically damped condition, the damping ratio is
   (a) zero (b) equal to one
   (c) any value greater than zero (d) none of the above
11. In the pulse width modulation
   (a) the amplitudes of the pulses are kept fixed
   (b) the times of the pulses are kept fixed
   (c) both the time and amplitudes of the pulses are kept fixed
   (d) none of the above

12. In the pulse amplitude modulation system
    (a) the amplitudes of the pulses are kept varying
    (b) the times of the pulses are kept varying
    (c) both the amplitudes and the time of pulses are kept fixed
    (d) none of the above

13. In the pulse frequency modulation
    (a) the amplitudes of the pulses are kept varying
    (b) the pulses of constant magnitude are produced at the rate which is a function of the magnitude of the input signal
    (c) pulses of variable magnitude are produced at the rate which is a function of the magnitude of the input signal
    (d) none of the above

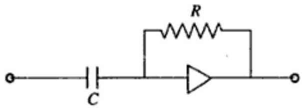
14. The device which converts a continuous signal into a sequence of pulses is termed
    (a) synchro (b) amplifier
    (c) sampler (d) none of the above

15. The circuit shown below is termed



    (a) differentiating unit (b) integrating unit
    (c) adder (d) none of the above

16. The circuit shown below is termed



    (a) integrating unit (b) differentiating unit
    (c) adder (d) none of the above

17. If we have a matrix A of order n having the following characteristic equation

$$f(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_n = 0$$

then

$$f(A) = A^n + a_1A^{n-1} + \cdots + a_nI = 0$$

is according to

(a) the Mason's theorem (b) the Cayley–Hamilton's theorem

(c) the Jordan's canonical theorem (d) none of the above

18. If the eigenvalues $\lambda_i (i = 1, 2, \ldots)$ of an $n \times n$ matrix A are distinct, and if $f(A)$ is any polynomial in A then,

$$f(A) = \sum_{r=1}^{n} \left[f(\lambda_r) \prod_{\substack{r=1 \\ K \neq r}}^{n} \left(\frac{A - \lambda_K I}{\lambda_r - \lambda_K} \right) \right]$$

is according to

(a) the Cayley–Hamilton's theorem (b) the Sylvesters' theorem

(c) the Mason's theorem (d) none of the above

19. If A is any matrix of order n and $X^T BX = X^T AX$, $X^T CX = 0$, then

(a) B is symmetric and C is the skew-symmetric of A

(b) B is skew-symmetric and C is symmetric of A

(c) B and C are both skew symmetric of A

(d) none of the above

20. The matrix shown below is

$$\begin{bmatrix} 4 & -4 & 2 \\ -4 & 5 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

(a) positive definite (b) positive semi-definite

(c) negative definte (d) none of the above

21. If the quadratic form of the matrix A is $10x_1^2 + 4x_2^2 + x_3^2 + 2x_1x_2 - 2x_2x_3 - 4x_1x_3$, the matrix A is given by

(a) $\begin{bmatrix} 10 & 1 & -2 \\ 1 & 4 & -1 \\ -2 & -1 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 10 & 1 & -2 \\ -1 & 4 & -1 \\ -2 & -1 & -1 \end{bmatrix}$

(c) $\begin{bmatrix} 10 & 1 & 2 \\ -1 & 4 & -1 \\ -2 & -1 & 1 \end{bmatrix}$ (d) none of the above.

22. A synchro is very similar to

(a) a rotating amplifier

(b) an induction motor

(c) a miniature three-phase star-connected synchronous motor

(d) none of the above

31. If $f(t)$ and $f'(t)$ have Laplace transforms, and $\lim_{s \to 0} sF(s)$ exists, then

$$\lim_{s \to \infty} sF(s) = \lim_{t \to 0^+} f(t)$$

The above phenomenon is termed

(a) convolution property
(b) final-value theorem
(c) initial-value theorem
(d) none of the above

32. The accuracy of the tachogenerator utilized in the computer application is

(a) a nonlinearity of 2–4 per cent
(b) a nonlinearity of 0.3–0.4 per cent
(c) a nonlinearity of zero per cent
(d) none of the above

33. The synchro is

(a) an electromagnetic transducer
(b) a static transducer
(c) basically an induction machine
(d) none of the above

34. The rotor of a synchro transmitter has

(a) a distributed winding
(b) a concentric winding
(c) a bar and an end ring
(d) none of the above

35. The output of a synchro-error detector is usually

(a) a sinusoidal waveform
(b) a modulated signal
(c) a square waveform
(d) none of the above

36. The type of modulation that occurs in a synchro transmitter is

(a) suppressed carrier modulation
(b) single sideband suppressed carrier modulation
(c) double sideband suppressed carrier modulation
(d) none of the above

37. In the case of a synchro error detector system, the rotor of which synchro is required to be made cylindrical

(a) synchro transmitter
(b) synchro control transformer
(c) synchro generator
(d) none of the above

38. The rotor of the differential synchro is usually

(a) delta connected
(b) star connected
(c) zig-zag connected
(d) none of the above

39. The differential transformer produces an electrical output proportional to the

(a) displacement of the movable core
(b) product of the number of primary and secondary turns
(c) flux density
(d) none of the above

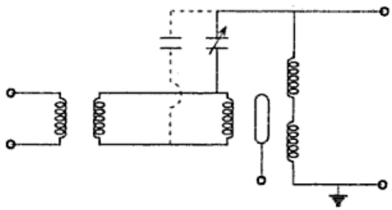
40. The advantage of the differential transformer over the synchro is that
    (a) there is no requirement of brushes and slip rings
    (b) it is entirely a static device
    (c) it can be used over an unlimited range of input positions
    (d) it has none of the above features

41. In the case of the variable reluctance differential transformer, the output voltage will not be shown zero, even though it may be theoretically zero, due to the fact that
    (a) a quadrature voltage is developed
    (b) the two secondary windings are not identical
    (c) there occurs error in the mechanical alignment of the movable part
    (d) none of the above may be true

42. Microsyn is essentially
    (a) a rotary differential transformer (b) an induction motor
    (c) a reluctance motor (d) none of the above

43. In the case of a servomotor
    (a) the speed-torque curve should have a positive slope
    (b) it should be able to withstand frequent starting operations
    (c) the rotor should have a high moment of inertia
    (d) none of the above may be true

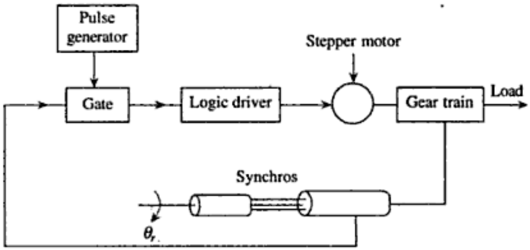
44. The following figure is



    (a) the schematic arrangement to reduce quadrature voltage in a differential transformer
    (b) the schematic arrangement to maintain quadrature voltage in a differential transformer
    (c) the schematic arrangement for tuning the circuit of differential transformer
    (d) none of the above

45. The advantage of the drag-cup rotor in a two-phase control motor is
    (a) to minimize the inertia (b) to minimize arcing
    (c) to minimize noise (d) none of the above

46. The principal disadvantage of the two-phase control motor is
    (a) the inherent inefficiency of a squirrel cage induction motor running at a large slip
    (b) the increase in cost
    (c) the vibrations of the motor
    (d) none of the above
47. The following figure is



    (a) an open-loop control system
    (b) a closed-loop control system
    (c) not a control system
    (d) none of the above
48. The stepper motor is
    (a) an analog device
    (b) a digital device
    (c) a conventional motor
    (d) none of the above
49. The hybrid stepper motor is
    (a) entirely a permanent magnet stepper motor
    (b) entirely a variable reluctance type stepper motor
    (c) a permanent magnet stepper motor having similarity with the variable reluctance motor from the construction point of view
    (d) none of the above
50. The following figure is the schematic view of a two-phase four-pole
    (a) variable reluctance stepper motor (b) permanent magnet stepper motor
    (c) hybrid stepper motor (d) none of the above

58. A moulded-composition potentiometer has the following most important advantage over the wire-wound potentiometer.
   (a) It has finite resolution
   (b) It has infinite resolution
   (c) It is cheap
   (d) None of the above

59. The resolution (per turn voltage) of the wire-wound potentiometer is usually
   (a) infinite
   (b) from 0.06% to 0.4%
   (c) from 0.6% to 4%
   (d) none of the above

60. The ohmic range of ceramic potentiometers usually lies between
   (a) 5 Ω and 50 kΩ
   (b) 1 kΩ and 100 kΩ
   (c) 500 Ω and 2 MΩ
   (d) none of the above

61. The linearity of the conductive plastic potentiometer is
   (a) from ± 0.25% to + 0.5%
   (b) from ± 0.025% to + 0.05%
   (c) from ± 2.5% to + 5%
   (d) none of the above

62. Wire-wound potentiometers are not used as feedback elements because of their
   (a) poor linearity
   (b) high resistance
   (c) noise and finite resolution
   (d) low maximum power rating

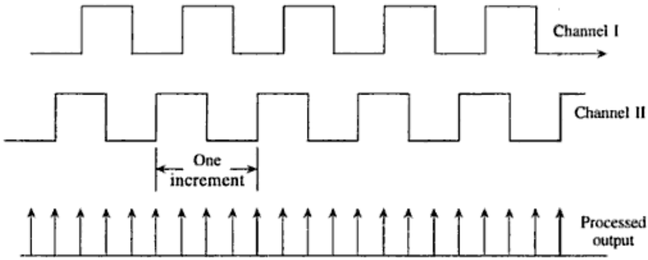
63. If in a potentiometer position feedback system, the reference voltage of the potentiometer is made more than the reference voltage of an analog to digital converter, then
   (a) the potentiometer will be damaged
   (b) the analog-to-digital converter will be damaged
   (c) the buffer will be damaged
   (d) none of the above will be damaged

64. Optical encoders may be used in a control system for
   (a) amplifying the signal
   (b) converting linear or rotary displacement into digitial code or pulse signal
   (c) isolating the low-power system from the high-power system
   (d) none of the above

65. The type of sensor used in the incremental optical encoder is
   (a) LED
   (b) photo diode
   (c) rotating disc
   (d) stationary mask

66. For incremental optical encoder, the value of the maximum resolution is
   (a) thousands of increments/revolution
   (b) hundreds of increments/revolution
   (c) ten thousands of increments/revolution
   (d) none of the above

67. The following figure is



(a) the digitial processing of two-channel incremental optical encoder
(b) the analog simulation of two-channel incremental optical encoder
(c) the analog simulation of two-channel absolute optical encoder
(d) none of the above

68. The servomotors used in position control systems have the ratio of stall torque and no-load speed in the low-speed region that is approximately
(a) one-fourth of that at rated voltage
(b) one-half of that at rated voltage
(c) the same as that at rated voltage
(d) none of the above

69. In a servomotor, the slope of the torque-speed characteristic reduces as the control phase voltage
(a) increases
(b) decreases
(c) remains constant
(d) none of the above

70. The drag-cup rotor of ac tachometers is usually made of
(a) copper
(b) aluminium
(c) iron
(d) none of the above

71. In order to have a highly stable dc coupled amplification in the feedback mode and ease in implementation of analog filtering in the forward path of a control system, the dc servo system needs
(a) a synchro
(b) an OPAMP
(c) a stepper motor
(c) none of the above

72. An ac amplifier compared to a dc amplifier in a control system is
    (a) more stable
    (b) less stable
    (c) same from the stability point of view
    (d) none of the above

73. An ac control system is preferred to a dc control system in an aircraft system due to the advantage gained in
    (a) stability
    (b) weight and size
    (c) sensitivity
    (d) none of the above

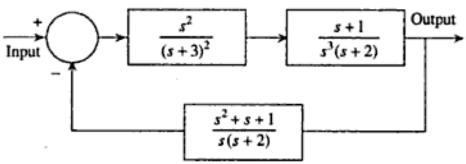
74. For large power applications like heavy drives, the dc system is ideal on account of the advantage gained in
    (a) availability of rugged high power amplification by rotating and static amplifiers
    (b) low cost
    (c) stability
    (d) none of the above

75. A hybrid control system uses
    (a) dc components only
    (b) ac components only
    (c) both dc and ac components
    (d) none of the above components

76. Type 1 system means that the open-loop transfer function has a number of integrations equal to
    (a) zero
    (b) one
    (c) two
    (d) none of the above

77. The standard second-order system indicates
    (a) one forward path integration
    (b) two forward path integrations
    (c) none forward path integration
    (d) none of the above

78. The system shown below is

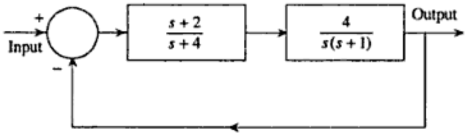


    (a) a Type 0 system
    (b) a Type 1 system
    (c) a Type 2 system
    (d) none of the above

79. The type of a system which has $G = \dfrac{2}{s^2 + 2s + 5}$ and $H = s + 5$ is

(a) a Type 0 system (b) a Type 1 system
(c) a Type 2 system (d) none of the above

80. The acceleration error constant of the block diagram shown in the following figure is



(a) 2 (b) ∞
(c) 0 (d) none of the above

81. The rise time for $c(t) = 1 - e^{-t}$ is

(a) 2.198 s (b) $\dfrac{1}{2.198}$ s
(c) 2.302 s (d) none of the above

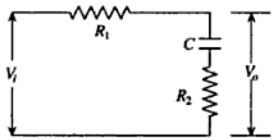
82. The number of octaves between 200 Hz and 800 Hz is

(a) 2 (b) 4
(c) 3 (d) none of the above

83. If the output in response to a unit-step function input for a particular control system is $c(t) = 1 - e^{-t}$, then the delay time will be

(a) $\dfrac{1}{\log_e 2}$ (b) 0.693
(c) 1 (d) none of the above

84. The transfer function of the network shown in the figure below is



(a) $\dfrac{R_2 + \dfrac{1}{Cs}}{R_1 + R_2 + \dfrac{1}{Cs}}$ (b) $\dfrac{R_2}{R_1 + R_2 + \dfrac{1}{Cs}}$

92. The Lagrangian mechanics is defined as
    (a) the difference between the kinetic energy and the potential energy
    (b) the difference between the mechanical rotational system and the mechanical translational system
    (c) the difference between the thermal system and the pneumatic system
    (d) none of the above

93. The sinusoidal transfer function of the transfer function $\dfrac{Ms^2}{Ms^2 + Fs + K}$ is
    (a) $\dfrac{-\omega^2}{\omega^2 + \dfrac{F}{M} j\omega + \dfrac{K}{M}}$
    (b) $\dfrac{-\omega^2}{-\omega^2 + \dfrac{F}{M} j\omega + \dfrac{K}{M}}$
    (c) $\dfrac{\omega^2}{\omega^2 + \dfrac{F}{M} j\omega + \dfrac{K}{M}}$
    (d) none of the above

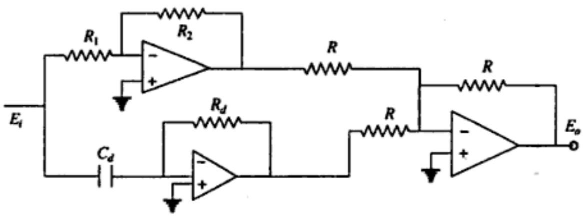
94. For low-speed high-torque applications, the actuators which are preferred are
    (a) chemical
    (b) pneumatic
    (c) hydraulic
    (d) none of the above

95. The output of a sensor is invariably in the form of
    (a) a velocity signal
    (b) an acceleration signal
    (c) an electrical signal
    (d) none of the above

96. The steady-state error of unit-ramp input in the Type 2 system is
    (a) ∞
    (b) 0
    (c) 1
    (d) none of the above

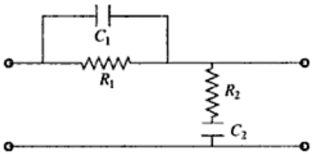
97. The steady-state error of unit-parabolic input in the Type 0 system is
    (a) 0
    (b) ∞
    (c) 1
    (d) none of the above

98. The circuit diagram shown below is



(a) that of a phase-lag network
(b) that of a phase-lead network
(c) that of a lag-lead compensator
(d) none of the above

102. The circuit diagram shown below is



(a) that of a lag-lead compensator
(b) that of a lag compensator
(c) that of a lead compensator
(d) none of the above

103. The digital controller having the $\mathcal{Z}$-transfer function is not realized in one of the following ways:

(a) Pulse data RC-network
(b) Computer program
(c) Digital processor
(d) None of the above

104. The eigenvalues of matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 3 & 0 & 2 \\ -12 & -7 & -6 \end{bmatrix} \quad \text{are}$$

(a) 1, 2, 3
(b) −1, 2, −3
(c) −1, −2, −3
(d) none of the above

105. Hysteresis in mechanical transmission is termed

(a) backlash
(b) dead zone
(c) damping
(d) none of the above

106. Compared to coulomb frictional force, the force of stiction

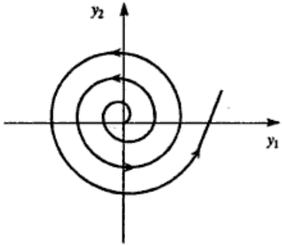
(a) is always greater
(b) is always equal
(c) is always less
(d) is none of the above

107. The relay has definite amount of dead zone in the control system on account of the fact that

(a) the relay coil requires a finite amount of current to actuate the relay
(b) the relay is not much sensitive
(c) the relay is of rotating type
(d) none of the above may be true

(a) has both the eigenvalues real and negative
(b) has both the eigenvalues real and positive
(c) has stable nodal point
(d) has none of the above

114. The phase portrait of the second-order system shown below in the y_1, y_2 plane has



(a) stable focus
(b) unstable focus
(c) stable nodal point
(d) none of the above

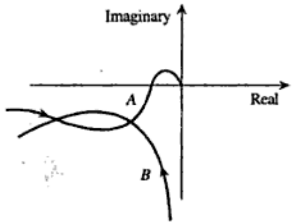
115. If the eigenvalues are on the imaginary axis, the phase portrait has
(a) closed path trajectories
(b) spiral trajectories focusing at the origin
(c) trajectories converging to the origin
(d) none of the above

116. A forced system will be stable
(a) with zero input and arbitrary initial conditions if the resulting trajectory tends towards the equilibrium state
(b) if with bounded input, the system output is bounded
(c) if the trajectory passes through the origin
(d) if none of the above are true

117. The stability of a system which approaches the origin as time tends to infinity is termed
(a) asymptotically stable
(b) limitedly stable
(c) oscillating in nature
(d) none of the above

118. If the trajectories of a nonlinear control system are eventually trapped into the closed curve, then it can be concluded that
(a) a stable steady-state oscillation will never be attained
(b) a stable state oscillation results with the voltage oscillating with fixed amplitude
(c) asymptotic stability has been attained
(d) none of the above are true

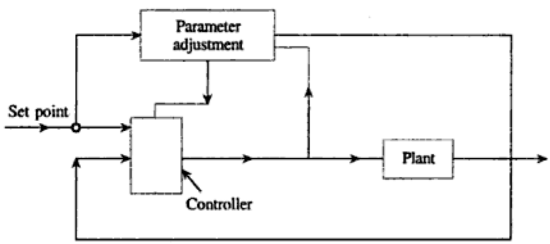
122. In the figure below:



(a) *A* has unstable limit cycle and *B* has stable limit cycle
(b) *A* has stable limit cycle and *B* has unstable limit cycle
(c) both *A* and *B* have unstable limit cycles
(d) none of the above are true

123. Is there any universal method for selecting Liapunov function?
(a) Yes (b) No
(c) Yes, but with certain limitations (d) None of the above

124. The block diagram shown in the figure below is



(a) for a PI controller (b) for a PID controller
(c) for an adaptive controller (d) for none of the above controllers

125. $\frac{dp}{dt} = -KE \cdot \frac{\partial E}{\partial p}$, where p is the adjustable parameter of the controller, E is the error, and K is a constant of proportionality. The above equation is termed
(a) MIT rule (b) model reference adaptive control rule
(c) Lagrangian rule (d) none of the above

126. If a scalar function $W(x, t)$ satisfies the following conditions,

(i) $W(x, t)$ is lower bounded

(ii) $\dot{W}(x, t)$ is negative semi-definite

(iii) $\dot{W}(x, t)$ is uniformly continuous in time

then $\dot{W}(x, t) \to 0$ as $t \to 0$.

The above theory is termed

(a) Liapunov theory (b) Barabalat's lemma

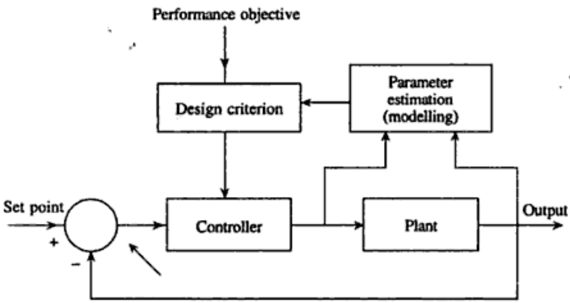
(c) Gradient theory (d) none of the above

127. Self-tuning control is mostly utilized in

(a) continuous time (b) discrete time

(c) non-recursive least squares estimation (d) none of the above

128. The block diagram shown below is



(a) for a self-tuning controller (b) for a model reference adaptive controller

(c) for a fuzzy logic-based control system (d) for none of the above

129. The fuzzy rules are

(a) "If–then" rules (b) MIT rules

(c) "GO TO" rules (d) none of the above

130. The standard formula for the performance index

$$J = \frac{1}{2} Y^T(t_1) H Y(t_1) + \frac{1}{2} \int_{t_0}^{t_1} [Y^T(t) Q Y(t) + u^T(t) R u(t)]\, dt$$

is for the following problem of optimal control system.

(a) State regulator problem (b) Output regulator problem

(c) Servomechanism problem (d) None of the above

131. The standard formula for the performance index

$$J = \frac{1}{2} X^T(t_1) H X(t_1) + \frac{1}{2} \int_{t_0}^{t_1} [X^T(t)\, Q X(t) + u^T(t)\, R u(t)]\, dt$$

is for the following problem of optimal control system.

(a) State regulator problem
(b) Output regulator problem
(c) Tracking problem
(d) None of the above

132. The standard formula for the performance index

$$J = \frac{1}{2} e^T(t_1) H e(t_1) + \frac{1}{2} \int_{t_0}^{t_1} [e^T(t)\, Q e(t) + u^T(t)\, R u(t)]\, dt$$

where $e(t) = [X(t) - r(t)]$ is for the following problem of optimal control system

(a) State regulator problem
(b) Output regulator problem
(c) Tracking problem
(d) None of the above

133. $\int_0^T t\, e^2(t)\, dt$, (where $e(t)$ is the error) is termed

(a) ITAE
(b) ITSE
(c) ISE
(d) none of the above

134. $\int_0^T t\left|e(t)\right| dt$ is termed

(a) ITAE
(b) ITSE
(c) ISE
(d) none of the above

135. $\int_0^{\infty} e^2(t)\, dt$ is termed

(a) ISE
(b) ITSE
(c) ITAE
(d) none of the above

136. The integral square error of the second-order system with unit-step input having damping coefficient ζ and undamped natural frequency ω_n, is

(a) $\frac{1}{2\omega_n}\left(\frac{1}{2\zeta} - 2\zeta\right)$
(b) $\frac{1}{2\omega_n}\left(\frac{1}{2\zeta} + 2\zeta\right)$
(c) $\frac{1}{4\omega_n \zeta}$
(d) none of the above

137. The integral time square error of the second-order system with step input having damping coefficient ζ and undamped natural frequency ω_n, is

(a) $\dfrac{1}{2\omega_n^2}\left(\dfrac{1}{4\zeta^2} - 2\zeta^2\right)$ (b) $\dfrac{1}{2\omega_n^2}\left(\dfrac{1}{4\zeta^2} + 2\zeta^2\right)$

(c) $\dfrac{1}{2\omega_n^2}\zeta^2$ (d) none of the above

138. The following quadratic form

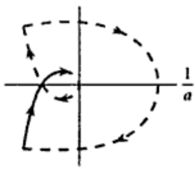
$$W(X) = 10X_1^2 + 4X_2^2 + X_3^2 + 2X_1X_2 - 2X_2X_3 - 4X_1X_3$$

(a) is positive definite (b) is negative definite

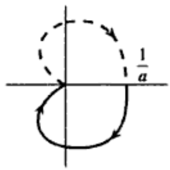
(c) is negative semi-definite (d) is none of the above

139. The Nyquist stability plot for $\dfrac{e^{-Ts}}{s+a}$, $a > 0$ will be:

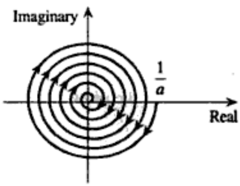
(a)



(b)



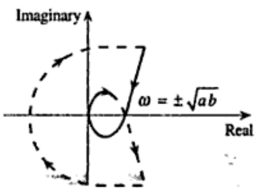
(c)



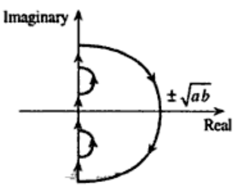
(d) none of the above

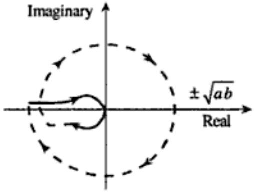
140. The Nyquist stability plot for $GH = \dfrac{s-a}{s(s+b)}$, $a, b > 0$ will be:

(a)



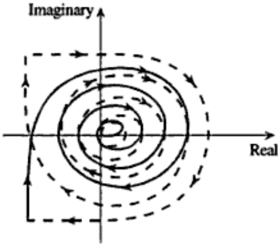
(b)

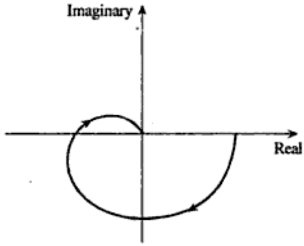


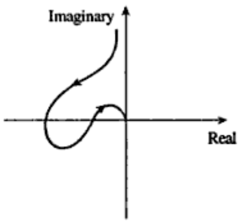
(c) 

(d) none of the above

141. The Nyquist stability plot for $GH = \dfrac{K_1 e^{-Ts}}{s(s+1)}$ is

(a) 

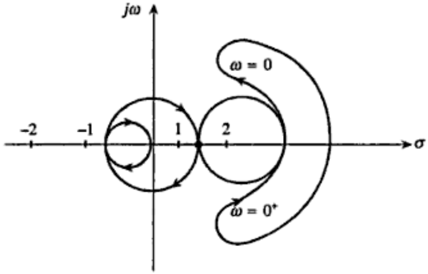
(b) 

(c) 

(d) none of the above

142. Is there any need of compensation to find a maximum overshoot of 20 per cent for the system defined by $GH = \dfrac{4\times10^4}{s^2(s+100)}$

(a) Yes, lag-lead compensator
(b) No compensation
(c) Yes, lag compensator
(d) None of the above

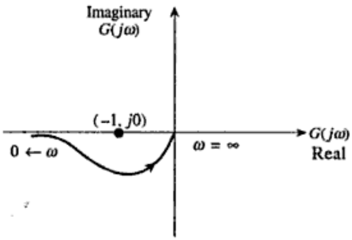


(a) Because of positive feedback, no conclusion about the system stability can be drawn from the given Nyquist plot.
(b) The system is stable.
(c) The system is unstable.
(d) The system has a limited stability.

9. The open-loop transfer function of a control system is given by

$$G(s) = \frac{K(1 + sT_d)}{s^2(1 + sT_1)}$$

The Nyquist plot of the system will be as shown below, if



(a) $T_d = 0$ (b) $T_d < T_1$
(c) $T_d = T_1$ (d) $T_d > T_1$

10. The transfer function of a unity feedback system is given by

$$G(s) = \frac{K}{s^2(s + 2)}$$

where $K = 25$. For this system, the most suitable compensation would be

(a) phase-lag network
(b) phase-lead network
(c) tachometer feedback
(d) tachometer plus high-pass filter feedback.

11. A linear system is given by

$$\dot{x}_1 = x_1$$

$$\dot{x}_2 = x_1 + x_2$$

The state transition matrix is

(a) $\begin{bmatrix} e^{-1} & 0 \\ e^{-1} & te^{-1} \end{bmatrix}$ (b) $\begin{bmatrix} e^{-1} & 0 \\ te^{-1} & e^{-1} \end{bmatrix}$

(c) $\begin{bmatrix} e^{t} & 0 \\ t^{et} & e^{t} \end{bmatrix}$ (d) $\begin{bmatrix} e^{t} & 0 \\ e^{t} & te^{t} \end{bmatrix}$

12. The state equations of a system are given by

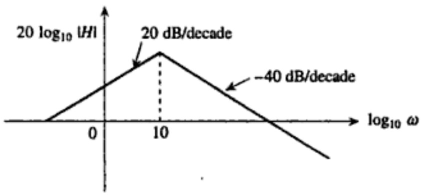
$$\dot{x} = \begin{bmatrix} -3 & 1 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} x$$

The system is

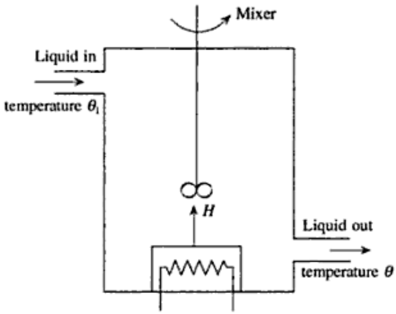
(a) controllable and observable
(b) controllable but not completely observable
(c) neither controllable nor completely observable
(d) not completely controllable but observable.

13. The piece-wise linear approximation for the magnitude response in dB of a rational function is shown below. The rational function has

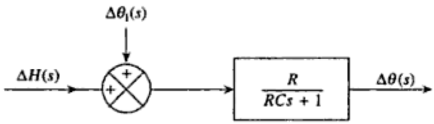


(a) three zeros and one pole (b) two zeros and two poles
(c) one zero and three poles (d) no zero and four poles

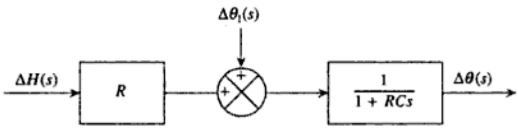
14. The figure below shows a liquid heating system, the liquid flow rate being constant. The block diagram of the system for incremental changes will be (where R is the thermal resistance and C the thermal capacitance)



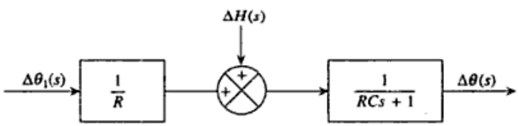
(a)



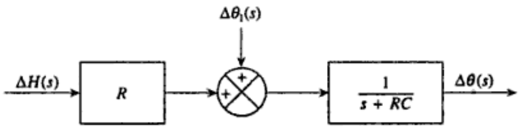
(b)



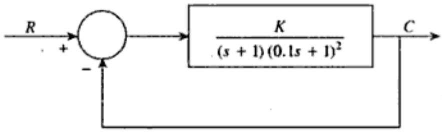
(c)



(d)



19. The system shown below has a unit-step input.



In order that the steady-state error is 0.1, the value of K required should be

(a) 0.1 (b) 0.9
(c) 1.0 (d) 9.0

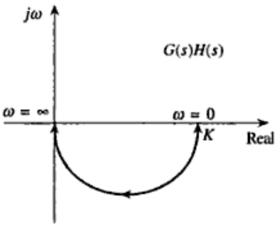
20. Consider the following statements regarding the advantages of the closed-loop negative feedback control system over the open-loop system.

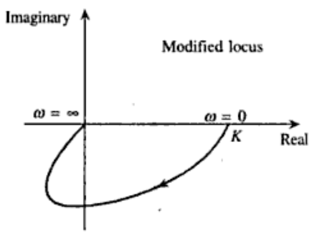
   1. The overall reliability of the closed-loop system is more than that of the open-loop system.
   2. The transient response in the closed-loop system decays more quickly than that in the open-loop system.
   3. In an open-loop system, closing of the loop increases the overall gain of the system.
   4. In the closed-loop system, the effect on the performance of the system due to variations in component parameters is very much reduced.

   Of these statements:

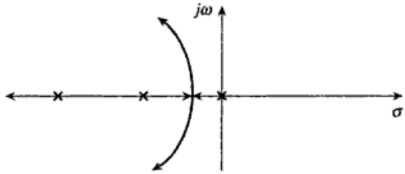
   (a) 1 and 3 are correct (b) 1 and 2 are correct
   (c) 2 and 4 are correct (d) 3 and 4 are correct

21. The Nyquist locus of a transfer function $G(s)H(s) = \dfrac{K}{1 + sT_1}$ is shown below. The locus is modified as shown on addition of a pole or poles to the original function $G(s)H(s)$.

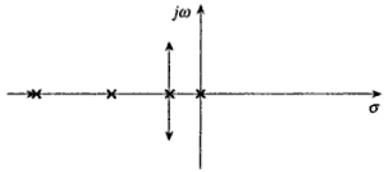




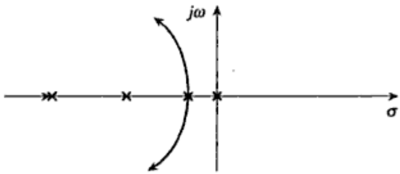
(b)



(c)



(d)



26. The open-loop transfer function of a control system is given by

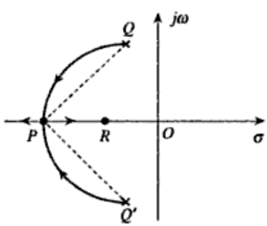
$$\frac{K(s+10)}{s(s+2)(s+a)}$$

The smallest possible value of a for which the system is stable in the closed loop for all positive values of K is

(a) 0 (b) 8

(c) 10 (d) 12

27. The figure below shows the root locus of the open-loop transfer function of a control system.



$PQ = PQ' = 2.6$
$PR = 1.4$
$OR = 2.0$
$OQ = OQ' = 1.4$

The value of the forward path gain K at the point P is

(a) 0.2 (b) 1.4
(c) 3.4 (d) 4.8

28. The open-loop transfer function of a unity negative feedback control system is given by

$$G(s) = \frac{K(s+2)}{(s+1)(s-7)}$$

For $K > 6$, the stability characteristics of the open-loop and closed-loop configurations of the system are, respectively,

(a) stable and stable (b) unstable and stable
(c) stable and unstable (d) unstable and unstable

29. A phase-lag compensation will
    (a) improve the relative stability
    (b) increase the speed of response
    (c) increase the bandwidth
    (d) increase the overshoot

30. The maximum phase shift that can be obtained by using a lead compensator with transfer function

$$G(s) = \frac{4(1 + 0.15s)}{1 + 0.05s}$$

is equal to

(a) 15° (b) 30°
(c) 45° (d) 60°

31. Consider the following statements regarding a first-order system with a proportional (P) controller which exhibits an offset to step input. In order to reduce the offset, it is necessary to
    1. increase the gain of the P-controller
    2. add a derivative mode
    3. add an integral mode

Of these statements

(a) 1, 2, and 3 are correct (b) 1 and 2 are correct
(c) 2 and 3 are correct (d) 1 and 3 are correct

32. The state model of the system shown in the figure below is given by

(a) $$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\beta & -\alpha \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$Y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

(b) $$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\beta & -\alpha \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

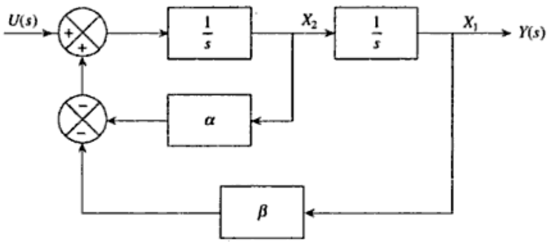
$$Y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

(c) $\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -\beta & -\alpha \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$

$Y = [1 \ \ 0] \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$

(d) $\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\alpha & -\beta \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$

$Y = [1 \ \ 0] \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$



33. A system is described by the state equation

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

The state transition matrix of the system is

(a) $\begin{bmatrix} e^{2t} & 0 \\ 0 & e^{2t} \end{bmatrix}$

(b) $\begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{-2t} \end{bmatrix}$

(c) $\begin{bmatrix} e^{2t} & 1 \\ 1 & e^{2t} \end{bmatrix}$

(d) $\begin{bmatrix} e^{-2t} & 1 \\ 1 & e^{-2t} \end{bmatrix}$

34. An effect of phase-lag compensation on a servo system performance is that
   (a) for a given relative stability, the velocity constant is increased
   (b) the bandwidth of the system is increased
   (c) the time response is made faster
   (d) none of the above is true.

35. Consider the system

$$x(t) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u(t)$$

$$c(t) = \begin{bmatrix} d_1 & d_2 \end{bmatrix} x(t)$$

The condition for complete state controllability and complete observability is

(a) $d_1 > 0$, $b_2 > 0$, b_1 and d_2 can be anything

(b) $d_1 > 0$, $d_2 > 0$, b_1 and b_2 can be anything

(c) $b_1 > 0$, $b_2 > 0$, d_1 and d_2 can be anything

(d) $b_1 > 0$, $d_2 > 0$, b_2 and d_1 can be anything

36. Canonical decomposition of a linear time-invariant system is given by

$$\dot{X} = \begin{bmatrix} -3 & 0 & -8 \\ 0 & -5 & -12 \\ 0 & 0 & -1 \end{bmatrix} X + \begin{bmatrix} 6 & 0 \\ 0 & 6 \\ 0 & 0 \end{bmatrix} u$$

$$Y = \begin{bmatrix} 0 & 0 & -4 \\ 1 & 0 & 4 \end{bmatrix}$$

It is

(a) completely controllable and observable

(b) completely controllable but unobservable

(c) uncontrollable but observable

(d) neither completely controllable nor observable

37. The transfer function of a multi-input multi-output system, with the state space representation of

$$\dot{X} = AX + Bu$$

$$Y = CX + Du$$

where X represents the state, Y the output and u the input vector, will be given by

(a) $C(sI - A)^{-1}B$ (b) $C(sI - A)^{-1}B + D$

(c) $(sI - A)^{-1}B + D$ (d) $(sI - A)^{-1}B$

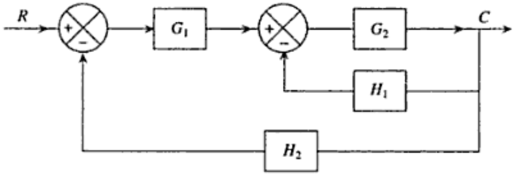
38. What will be the closed-loop transfer function of a unity feedback control system whose step response is given by

$$c(t) = K\,[1 - 1.66\, e^{-8t} \sin\,(st + 37°)]$$

(a) $\dfrac{100K}{s^2 + 16s + 100}$ (b) $\dfrac{K}{s^2 + 16s + 100}$

(c) $\dfrac{10K}{s^2 + 8s + 10}$ (d) None of the above

39. The overall transfer function C/R is given by



$$G_1 = \frac{10}{s}$$

$$G_2 = \frac{10}{s+1}$$

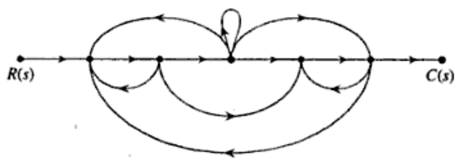
$$H_1 = s + 3$$

$$H_2 = 1$$

(a) $\dfrac{10}{11s^2 + 31s + 10}$ (b) $\dfrac{100}{11s^2 + 31s + 100}$

(c) $\dfrac{100}{11s^2 + 31s}$ (d) none of the above

40. The signal flow graph of a system is shown below.

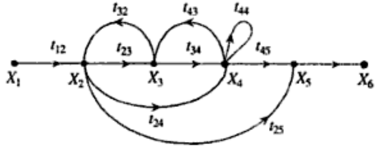


In this graph the number of three non-touching loops is

(a) 0 (b) 1

(c) 2 (d) 3

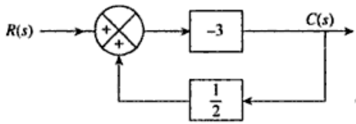
41. The sum of the gain products of all the possible combinations of two non-touching loops in the signal flow graph shown below, is



(a) $t_{23}t_{32} + t_{44}$ (b) $t_{23}t_{32} + t_{34}t_{43}$

(c) $t_{23}t_{32} + t_{34}t_{43} + t_{44}$ (d) $t_{24}t_{43}t_{32} + t_{44}$

42. The closed-loop gain of the system shown below, is



(a) $-\frac{9}{5}$ (b) $-\frac{6}{5}$

(c) $\frac{6}{5}$ (d) $\frac{9}{5}$

43. The response $c(t)$ of a system to an input $r(t)$ is given by the following differential equation:

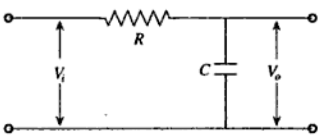
$$\frac{d^2c(t)}{dt^2} + 3\frac{dc(t)}{dt} + 5c(t) = 5r(t)$$

The transfer function of the system is given by

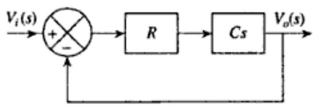
(a) $G(s) = \dfrac{5}{s^2 + 3s + 5}$ (b) $G(s) = \dfrac{1}{s^2 + 3s + 5}$

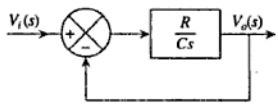
(c) $G(s) = \dfrac{3s}{s^2 + 3s + 5}$ (d) $G(s) = \dfrac{s + 3}{s^2 + 3s + 5}$

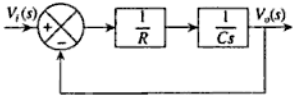
44. For the *RC* circuit shown below, V_i and V_o are the input and the output of the system respectively.

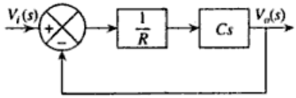


The block diagram of the system is represented by

(a) 

(b) 

(c) 

(d) 

45. The transfer function of a control system is given as

$$T(s) = \frac{K}{s^2 + 4s + K}$$

where K is the gain of the system in rad/amp.
For this system to be critically damped, the value of K should be

(a) 1 (b) 2
(c) 3 (d) 4

46. A linear system initially at rest, is subjected to an input signal

$$r(t) = 1 - e^{-t} \qquad (t \geq 0)$$

The response of the system for $t \geq 0$ is given by

$$c(t) = 1 - e^{-2t}$$

The transfer function of the system is

(a) $\frac{s+2}{s+1}$ (b) $\frac{s+1}{s+2}$

(c) $\frac{2(s+1)}{s+2}$ (d) $\frac{1}{2}\frac{s+1}{s+2}$

47. If the time response of a system is given by the following equation

$$y(t) = 5 + 3\sin(\omega t + \delta_1) + e^{-3t}\sin(\omega t + \delta_2) + e^{-5t}$$

then the steady-state part of the above response is given by

(a) $5 + 3\sin(\omega t + \delta_1)$ (b) $5 + 3\sin(\omega t + \delta_1) + e^{-3t}\sin(\omega t + \delta_2)$
(c) $5 + e^{-5t}$ (d) 5

48. The impulse response of a system is $5e^{-10t}$. Its step response is equal to

(a) $0.5e^{-10t}$ (b) $5(1 - e^{-10t})$
(c) $0.5(1 - e^{-10t})$ (d) $10(1 - e^{-10t})$.

49. The transfer function of a system is $10/(1 + s)$. When operated as a unity feedback system, the steady-state error to a unit-step input will be

(a) zero (b) 1/11
(c) 10 (d) infinity

50. The open-loop transfer function of a system is given by

$$G(s) = \frac{K}{s(s+2)(s+4)}$$

(c) $\frac{40}{81}$ (d) $\frac{28}{81}$

55. Which of the following properties are associated with the state transition matrix $\phi(t)$?

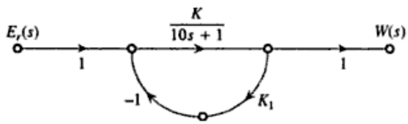
    1. $\phi(-t) = \phi^{-1}(t)$
    2. $\phi(t_1/t_2) = \phi(t_1) \cdot \phi^{-1}(t_2)$
    3. $\phi(t_1 - t_2) = \phi(-t_2)\,\phi(t_1)$

    Select the correct answer using the codes given below.

    (a) 1, 2, and 3 (b) 1 and 2

    (c) 2 and 3 (d) 1 and 3

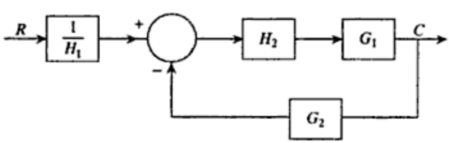
56. Given $KK_1 = 99$; $s = j1$ rad/s, the sensitivity of the closed-loop system, shown below, to variation in parameter K is approximately,



(a) 0.01 (b) 0.1

(c) 1.0 (d) 10

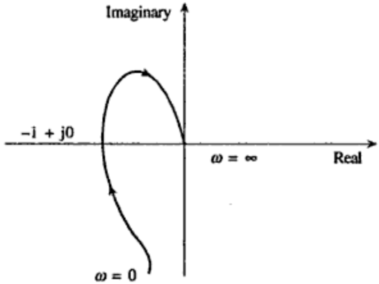
57. The transfer function C/R of the system, shown below, is



(a) $\dfrac{G_1 G_2 H_2}{H_1(1 + G_1 G_2 H_2)}$ (b) $\dfrac{G_1 H_2}{H_1(1 + G_1 G_2 H_2)}$

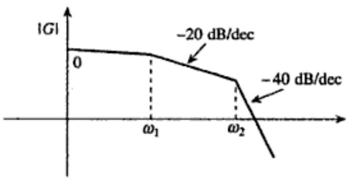
(c) $\dfrac{G_2 H_2}{H_1(1 + G_1 G_2 H_1)}$ (d) $\dfrac{G_2 H_1}{H_2(1 + G_1 G_2 H_2)}$

58. The Nyquist plot for a control system is shown below.

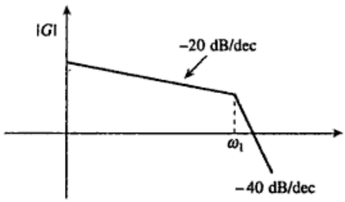


The Bode plot for the same system will be as in

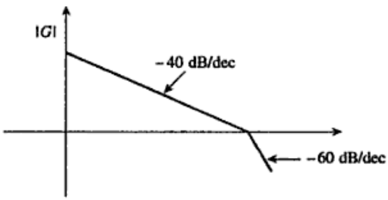
(a)



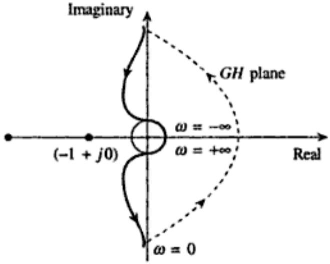
(b)



(c)

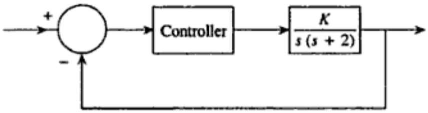


62. For the Nyquist plot of the open-loop transfer function $G(s)H(s)$ shown below:



(a) The open-loop system is unstable but the closed-loop system is stable
(b) Both the open-loop and the closed-loop systems are unstable
(c) The open-loop system is stable but the closed-loop system is unstable
(d) Both the open-loop and the closed-loop systems are stable.

63. In the control system shown below, the controller which can give zero steady-state error to a ramp input, with $K = 9$ is of



(a) proportional type
(b) integral type
(c) derivative type
(d) proportional plus derivative type

64. The gain cross-over frequency and bandwidth of a control system are ω_{cu} and ω_{bu} respectively. A phase-lag network is employed for compensating the system. If the gain cross-over frequency and bandwidth of the compensated system are ω_{cc} and ω_{bc} respectively, then

(a) $\omega_{cc} < \omega_{cu}$; $\omega_{bc} < \omega_{bu}$
(b) $\omega_{cc} > \omega_{cu}$; $\omega_{bc} < \omega_{bu}$
(c) $\omega_{cc} < \omega_{cu}$; $\omega_{bc} > \omega_{bu}$
(d) $\omega_{cc} > \omega_{cu}$; $\omega_{bc} > \omega_{bu}$

65. The transfer function of a certain system is

$$\frac{Y(s)}{U(s)} = \frac{1}{s^4 + 5s^3 + 7s^2 + 6s + 3}$$

The A, B matrix pair of the equivalent state space model will be

(a) $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & -6 & -7 & -5 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & -5 & -6 & -7 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -5 & -7 & -6 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & -6 & -7 & -5 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

66. A linear system is described by the state equations

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} R$$

$$C = X_2$$

where R and C are the input and the output respectively. The transfer function is

(a) $\dfrac{1}{s+1}$ (b) $\dfrac{1}{(s+1)^2}$

(c) $\dfrac{1}{s-1}$ (d) $\dfrac{1}{(s-1)^2}$

67. If the state equation of a dynamic system is given by

$$\dot{X}(t) = AX(t)$$

where $$A = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -3 & 4 \\ 0 & 0 & 0 & -4 & -3 \end{bmatrix}$$

the eigenvalues of the system would be

(a) real non-repeated only (b) real non-repeated and complex

(c) real repeated (d) real repeated and complex

68. A simple electric heater is shown below. The system can be modelled by

(a) a first-order differential equation (b) a second-order differential equation

(c) a third-order differential equation (d) an algebraic equation

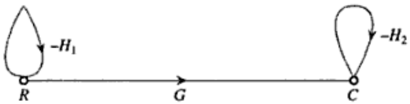
The system given above is

(a) controllable and observable (b) uncontrollable and observable

(c) uncontrollable and unobservable (d) controllable and unobservable

72. Which of the following systems is an open-loop system?
    (a) The respiratory system of a human being
    (b) A system for controlling the movement of the slide of a copying milling machine
    (c) A theromostatic control
    (d) Traffic light control

73. For the signal flow graph shown below, the overall transfer function of the system will be



(a) $\frac{C}{R} = G$ (b) $\frac{C}{R} = \frac{G}{1 + H_2}$

(c) $\frac{C}{R} = \frac{G}{(1 + H_1)(1 + H_2)}$ (d) $\frac{C}{R} = \frac{G}{1 + H_1 + H_2}$

74. A linear second-order system with the transfer function

$$G(s) = \frac{49}{s^2 + 16s + 49}$$

is initially at rest and is subjected to a step-input signal. The response of the system will exhibit a peak overshoot of

(a) 16% (b) 9%

(c) 2% (d) zero

75. A system has the following transfer function

$$G(s) = \frac{100(s + 5)(s + 50)}{s^4(s + 10)(s^2 + 3s + 10)}$$

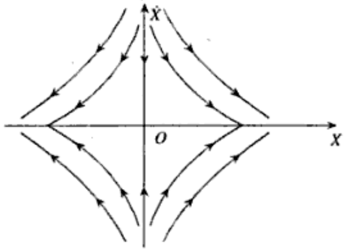
The type and order of the system are respectively

(a) 4 and 9 (b) 4 and 7

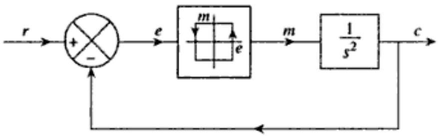
(c) 5 and 7 (d) 7 and 5

76. The open-loop transfer function of a unity feedback control system is

$$G(s) = \frac{K(s + 10)(s + 20)}{s^2(s + 2)}$$

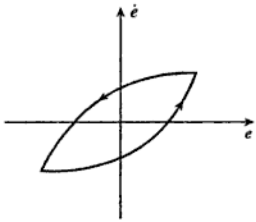


80. A closed-loop nonlinear system is shown below.

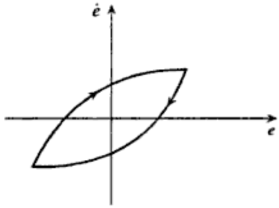


The phase plane plot in the e–$\dot{e}$ plane is

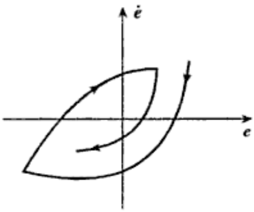
(a)



(b)



(c)



(d) none of the above.

ANSWERS TO OBJECTIVE TYPE QUESTIONS

(1) (b)	(2) (b)	(3) (b)	(4) (c)
(5) (a)	(6) (b)	(7) (c)	(8) (b)
(9) (a)	(10) (b)	(11) (a)	(12) (a)
(13) (b)	(14) (c)	(15) (b)	(16) (b)
(17) (b)	(18) (b)	(19) (a)	(20) (b)
(21) (a)	(22) (c)	(23) (a)	(24) (a)
(25) (a)	(26) (b)	(27) (b)	(28) (b)
(29) (a)	(30) (b)	(31) (c)	(32) (b)
(33) (a)	(34) (b)	(35) (b)	(36) (a)
(37) (b)	(38) (b)	(39) (a)	(40) (a)
(41) (a)	(42) (a)	(43) (b)	(44) (a)
(45) (a)	(46) (a)	(47) (b)	(48) (b)
(49) (c)	(50) (c)	(51) (b)	(52) (b)
(53) (c)	(54) (c)	(55) (a)	(56) (c)
(57) (a)	(58) (b)	(59) (b)	(60) (c)
(61) (a)	(62) (c)	(63) (b)	(64) (b)
(65) (b)	(66) (a)	(67) (a)	(68) (b)
(69) (b)	(70) (b)	(71) (b)	(72) (a)
(73) (b)	(74) (a)	(75) (c)	(76) (b)
(77) (a)	(78) (c)	(79) (a)	(80) (c)
(81) (a)	(82) (a)	(83) (b)	(84) (a)
(85) (c)	(86) (a)	(87) (b)	(88) (b)
(89) (c)	(90) (a)	(91) (a)	(92) (a)
(93) (b)	(94) (c)	(95) (c)	(96) (b)
(97) (b)	(98) (a)	(99) (b)	(100) (c)
(101) (b)	(102) (a)	(103) (d)	(104) (c)
(105) (a)	(106) (a)	(107) (a)	(108) (b)
(109) (a)	(110) (b)	(111) (c)	(112) (b)
(113) (b)	(114) (b)	(115) (a)	(116) (b)
(117) (a)	(118) (b)	(119) (b)	(120) (a)
(121) (b)	(122) (a)	(123) (b)	(124) (c)
(125) (a)	(126) (b)	(127) (b)	(128) (a)
(129) (a)	(130) (b)	(131) (a)	(132) (c)
(133) (b)	(134) (a)	(135) (a)	(136) (b)
(137) (b)	(138) (a)	(139) (c)	(140) (a)
(141) (a)	(142) (a)		

Appendix

A.1 FINITE SUMMATIONS

1. $\sum_{i=0}^{n} a^i = \frac{1-a^{n+1}}{1-a}$ (where $a \neq 1$)

2. $\sum_{i=0}^{n} ia^i = \frac{a}{(1-a)^2}[1-(n+1)a^n + n \cdot a^{n+1}]$ ($a \neq 1$)

3. $\sum_{i=0}^{n} i^2a^i = \frac{a}{(1-a)^3}[(1+a)-(n+1)^2a^n + (2n^2+2n-1)a^{n+1} - n^2a^{n+2}]$ ($a \neq 1$)

4. $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$

5. $\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

6. $\sum_{i=1}^{n} i^3 = n^2\frac{(n+1)^2}{4}$

A.2 LAPLACE TRANSFORMS

Time function, x(t)	*Laplace transform, X(s)*
$x(t)$, where $x(t) = 0$ for $t < 0$	$X(s) = \int_0^\infty x(t)e^{-st}dt$
$\mathcal{L}^{-1}[X(s)] = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}ds = x(t)$	$X(s)$

Time function, x(t)	*Laplace transform, X(s)*
$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$	$X(s) = \begin{bmatrix} X_1(s) \\ X_2(s) \\ \vdots \\ X_n(s) \end{bmatrix}$
$f(t) = \begin{bmatrix} f_{11}(t) & f_{12}(t) & \cdots & f_{1n}(t) \\ f_{21}(t) & f_{22}(t) & \cdots & f_{2n}(t) \\ \vdots & \vdots & & \vdots \\ f_{m1}(t) & f_{m2}(t) & \cdots & f_{mn}(t) \end{bmatrix}$	$F(s) = \begin{bmatrix} f_{11}(s) & f_{12}(s) & \cdots & f_{1n}(s) \\ f_{21}(s) & f_{22}(s) & \cdots & f_{2n}(s) \\ \vdots & \vdots & & \vdots \\ f_{m1}(s) & f_{m2}(s) & \cdots & f_{mn}(s) \end{bmatrix}$
$ax(t) + by(t)$	$aX(s) + bY(s)$
$\frac{d}{dt}x(t) = \dot{x}(t)$	$sX(s) - x(0)$
$\frac{d^n}{dt^n}x(t)$	$s^nX(s) - s^{n-1}x(0) - s^{n-2}\frac{d}{dt}x(0) - \cdots - s\frac{d^{n-2}}{dt^{n-2}}x(0) - \frac{d^{n-1}}{dt^{n-1}}x(0)$
$\int_0^t x(\tau)d\tau$	$\frac{X(s)}{s}$
$\delta(t)$ (unit impulse at $t = 0$)	1
$x(t - \alpha)$	$X(s)e^{-s\alpha}$
$\int_0^t x(\tau)y(t-\tau)d\tau$	$X(s)Y(s)$
$x(t) = 1$ (unit step)	$\frac{1}{s}$
$\frac{t^n}{n!}$	$\frac{1}{s^{n+1}}$
$e^{-\alpha t}$	$\frac{1}{s+\alpha}$
$\frac{t^n}{n!}e^{-\alpha t}$	$\frac{1}{(s+\alpha)^{n+1}}$
$1 - e^{-\alpha t}$	$\frac{a}{s(s+\alpha)}$

Time function, x(t)	*Laplace transform, X(s)*
$\sin \omega t$	$\dfrac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\dfrac{s}{s^2+\omega^2}$
$e^{-\alpha t} \sin \omega t$	$\dfrac{\omega}{(s+\alpha)^2+\omega^2}$
$e^{-\alpha t} \cos \omega t$	$\dfrac{s+\alpha}{(s+\alpha)^2+\omega^2}$

A.3 $\mathcal{Z}$-TRANSFORMS

Time function, x(n)	*$\mathcal{Z}$-transform, X(z)*
$x(n)$, where $x(n) = 0$ for $n < 0$	$X(z) = \sum_{n=0}^{\infty} x(n)z^{-n}$
$\mathcal{J}^{-1}[X(z)] = \dfrac{1}{2\pi j}\int X(z)z^{n-1}\,dz = x(n)$	$X(z)$
$x(n) = \begin{bmatrix} x_1(n) \\ x_2(n) \\ \vdots \\ x_n(n) \end{bmatrix}$	$X(z) = \begin{bmatrix} X_1(z) \\ X_2(z) \\ \vdots \\ X_n(z) \end{bmatrix}$
$f(n) = \begin{bmatrix} f_{11}(n) & f_{12}(n) & \cdots & f_{1k}(n) \\ f_{21}(n) & f_{22}(n) & \cdots & f_{2k}(n) \\ \vdots & \vdots & & \vdots \\ f_{m1}(n) & f_{m2}(n) & \cdots & f_{mk}(n) \end{bmatrix}$	$F(z) = \begin{bmatrix} f_{11}(z) & f_{12}(z) & \cdots & f_{1k}(z) \\ f_{21}(z) & f_{22}(z) & \cdots & f_{2k}(z) \\ \vdots & \vdots & & \vdots \\ f_{m1}(z) & f_{m2}(z) & \cdots & f_{mk}(z) \end{bmatrix}$
$ax(n) + by(n)$	$aX(z) + bY(z)$
$x(n + 1)$	$zX(z) - zx(0)$
$x(n + K)$	$z^K X(z) - \sum_{i=0}^{K-1} X(i)z^{K-i}$
$x(n - K)$	$z^{-K}X(z)$

Time function, $x(t)$	Z-transform, $X(z)$
$\sum_{j=0}^{n} x(n-j)y(j)$	$X(z)Y(z)$
$u(n)$ = unit pulse at $n = 0$	1
$x(n) = 1$	$X(z) = \frac{z}{z-1}$
n	$\frac{z}{(z-1)^2}$
n^2	$\frac{z(z+1)}{(z-1)^3}$
a^n	$\frac{z}{z-a}$
a^{n-1}	$\frac{1}{z-a}$
na^{n-1}	$\frac{z}{(z-a)^2}$
n^2a^{n-1}	$\frac{z(z+a)}{(z-a)^3}$
$\sin \omega n$	$\frac{z \sin \omega}{z^2 - 2z \cos \omega + 1}$
$\cos \omega n$	$\frac{z(z - \cos \omega)}{z^2 - 2z \cos \omega + 1}$

A.4 MATRIX

1. $$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

   [$m \times n$ rectangular matrix]

2. When $m = n$, the matrix A is a square matrix.
3. A^T = Transpose of $A = [a_{ji}]$. It is obtained by interchanging rows and columns.
4. $\bar{A}$ = Conjugate of A. It is obtained by replacing every element in A by its complex conjugate.

42. $\det A^{-1} = \dfrac{1}{\det A}$

43. $\det P^{-1} AP = \det A$

44. $\mathrm{tr}(A^T) = \mathrm{tr}(A)$

45. $\mathrm{tr}(A + B) = \mathrm{tr}(A) + \mathrm{tr}(B)$

46. $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ $[\mathrm{tr}(AB) \neq \mathrm{tr}(A)\,\mathrm{tr}(B)]$

47. $\mathrm{tr}(P^{-1}AP) = \mathrm{tr}(A)$

48. $\mathrm{P}(A^T) = \mathrm{P}(A)$

49. $\mathrm{P}(A) \leq \min(m, n)$: A is an $m \times n$ matrix.

50. $\mathrm{P}(AB) \leq \min(\mathrm{P}(A), \mathrm{P}(B))$

51. $\mathrm{P}(A) = n$ if and only if $\det A^TA \neq 0$ (for a real $m \times n$ matrix A).

52. $\mathrm{P}(A) = m$ if and only if $\det AA^T \neq 0$ (for a real $m \times n$ matrix A).

53. A square real matrix A can be expressed as the sum of a symmetric matrix A_1 and a skew symmetric matrix A_2

$$A = A_1 + A_2$$

$$A_1 = \frac{1}{2}(A + A^T)$$

$$A_2 = \frac{1}{2}(A - A^T)$$

54. Integration and differentiation of vectors and matrices.

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

$$\frac{dx(t)}{dt} = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix}$$

59. Differentiation of scalar/vector with respect to a vector

$$f_1(x) = f(x_1, x_2, \ldots, x_n)$$

$$\frac{\delta f_1(x)}{\delta x} = g_1(x) = \begin{bmatrix} \frac{\delta f_1}{\delta x_1} \\ \frac{\delta f_2}{\delta x_2} \\ \vdots \\ \frac{\delta f}{\delta x_n} \end{bmatrix}$$

where $g_1(x)$ is termed the gradient of $f_1(x)$.

60.

$$f(x) = \begin{bmatrix} f_1(x_1, x_2, \cdots, x_n) \\ f_2(x_1, x_2, \cdots, x_n) \\ \vdots \\ f_m(x_1, x_2, \cdots, x_n) \end{bmatrix}$$

$$\frac{\delta f(x)}{\delta x} = J(x) = \begin{bmatrix} \frac{\delta f_1}{\delta x_1} & \frac{\delta f_1}{\delta x_2} & \cdots & \frac{\delta f_1}{\delta x_n} \\ \frac{\delta f_2}{\delta x_1} & \frac{\delta f_2}{\delta x_2} & \cdots & \frac{\delta f_2}{\delta x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\delta f_m}{\delta x_1} & \frac{\delta f_m}{\delta x_2} & \cdots & \frac{\delta f_m}{\delta x_n} \end{bmatrix}$$

where $J(x) = \frac{\delta f(x)}{\delta x}$ is termed the Jacobian matrix of f with respect to x.

61. The second derivative of scalar valued function $f(x) = f(x_1, x_2, \cdots, x_n)$ is

$$\frac{\delta^2 f}{\delta x^2} = H(x) = \begin{bmatrix} \frac{\delta^2 f}{\delta x_1^2} & \frac{\delta^2 f}{\delta x_1 \delta x_2} & \cdots & \frac{\delta^2 f}{\delta x_1 \delta x_n} \\ \frac{\delta^2 f}{\delta x_2 \delta x_1} & \frac{\delta^2 f}{\delta x_2^2} & \cdots & \frac{\delta^2 f}{\delta x_2 \delta x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\delta^2 f}{\delta x_n \delta x_1} & \frac{\delta^2 f}{\delta x_n \delta x_2} & \cdots & \frac{\delta^2 f}{\delta x n^2} \end{bmatrix}$$

where $H(x)$ is called the Hessian matrix.

68. $\dfrac{\delta}{\delta x}[\mathrm{tr}(Ax)] = A^T$

69. $\dfrac{\delta}{\delta x}[\mathrm{tr}(x^T Q x)] = (Q + Q^T)x$

70. For $f = f(x)$ and $g = g(x)$

$$\frac{\delta}{\delta x}(f^T g) = \frac{\delta f}{\delta x} g + \frac{\delta g}{\delta x} f$$

71. $\dfrac{\delta}{\delta t}(x^T(t)Qx(t)) = x^T(Q^T + Q)\dot{x}$

72. $\dfrac{\delta}{\delta t}[\mathrm{tr}(A(t)C)] = \mathrm{tr}\left(\dfrac{\delta A}{\delta t} C\right);$ C is a constant.

73. $\dfrac{\delta}{\delta t}[\mathrm{tr}(A^{-1}(t)C)] = -\left[\mathrm{tr}\left(A^{-1}\dfrac{\delta A}{\delta t}A^{-1}C\right)\right];$ C is a constant.

74. $\dfrac{\delta}{\delta A}[\mathrm{tr}(AB)]$

$$= \frac{\delta}{\delta A}\mathrm{tr}(A^T B^T)$$

$$= \frac{\delta}{\delta A}\mathrm{tr}(B^T A^T)$$

$$= \frac{\delta}{\delta A}\mathrm{tr}(BA) = B^T$$

75. $\dfrac{d(x^T)}{dx} = 1;$ x is an $n \times 1$ vector.

76. $\dfrac{d(x^T b)}{dx} = b;$ b is an $n \times 1$ vector.

77. $\dfrac{d}{dx}(x^T x) = 2x$

78. $\dfrac{d(x^T Ax)}{dx} = 2Ax;$ A is an $n \times n$ symmetric matrix.

79. If A is an $n \times n$ matrix, the characteristics equation will be

$$|\lambda I - A| = \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n = 0$$

The Cayley-Hamilton theorem states that

$$A^n + a_1 A^{n-1} + \cdots + a_{n-1}A + a_n I = 0$$

A.5 MATLAB

Functions

color	Color control models	**matfun**	Matrix functions
datafun	Data analysis	**ops**	Operators
plotxy	2-D graphics	**plotxyz**	3-D graphics
elemat	Matrix manipulation	**polyfun**	Polynomial function
funfun	Function functions	**sparfun**	Sparse matrix functions
general	General commands	**graphics**	Graphics functions
lang	Language constructs	**specfun**	Specialized math functions
specmat	Special matrices	**strfun**	Character string functions

General Purpose Command

path	Control MATLAB's path	**type**	Display M-file contents
which	Locate functions and files	**clear**	Clear memory
save	Save workspace variables	**length**	Length of vector
size	Size of matrix	**who**	List variables
load	Retrieve variable	**cd**	Change working directory
dir	Directory listing	**quit**	Terminate MATLAB

Language Constructs

function	Add new functions	**break**	Terminate execution loop
for	Repeat statements	**else**	used with if
if	Conditional execution	**elseif**	used with if
return	Return to invoking function	**end**	Terminate loops
while	Repeat conditionally	**menu**	Generate menu
input	Prompt for user input	**pause**	Wait for user response

Matrix and Math Functions

eye	Identity matrix	**ones**	Ones matrix
rand	Random number	**zeros**	Zeros matrix
abs	Absolute value	**acos**	Inverse cosine
exp	Exponential	**angle**	Phase angle
log	Natural logarithm	**sin**	Sine
cos	Cosine	**conj**	Complex conjugate
sqrt	Square root	**rem**	Remainder after division
det	Determinant of matrix	**rank**	L1 rows or cols
norm	Matrix or vector norm	**inv**	Matrix inverse
rref	Row-reduced form	**orth**	Orthogonalization
cross	Vector cross product	**dot**	Vector dot product

Graphic Functions

plot	Linear plot	**bar**	Bar graph
hist	Histogram plot	**title**	Graph title

Index

Control Engineering

Theory and Practice

M.N. Bandyopadhyay

This book offers a comprehensive introduction to the subject of control engineering. Both continuous- and discrete-time control systems are treated, although the emphasis is on continuous-time systems. A chapter each is devoted to in-depth analysis of non-linear control systems, control system components, and optimal control theory. The book also introduces students to the modern concepts of neural fuzzy and adaptive learning systems.

KEY FEATURES INCLUDE:

- Detailed coverage of state space techniques for continuous- and discrete-time control systems.
- Lucid explanations of controllability and observability concepts and tests.
- Simple treatment of compensation design techniques.
- Extensive treatment of optimal system design.
- Presentation of several practical applications of control engineering.
- Many solved examples in each chapter to reinforce understanding of the topics covered.
- Several typical solved problems of greater analytical nature and numerous objective type questions and exercises in concluding chapters.

Well-written and easy to follow, the book is addressed to the needs of undergraduate engineering students, for a first-course in control theory, of electrical, electronics, mechanical, industrial, aerospace, and chemical disciplines.

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