LINEAR ALGEBRA C-3

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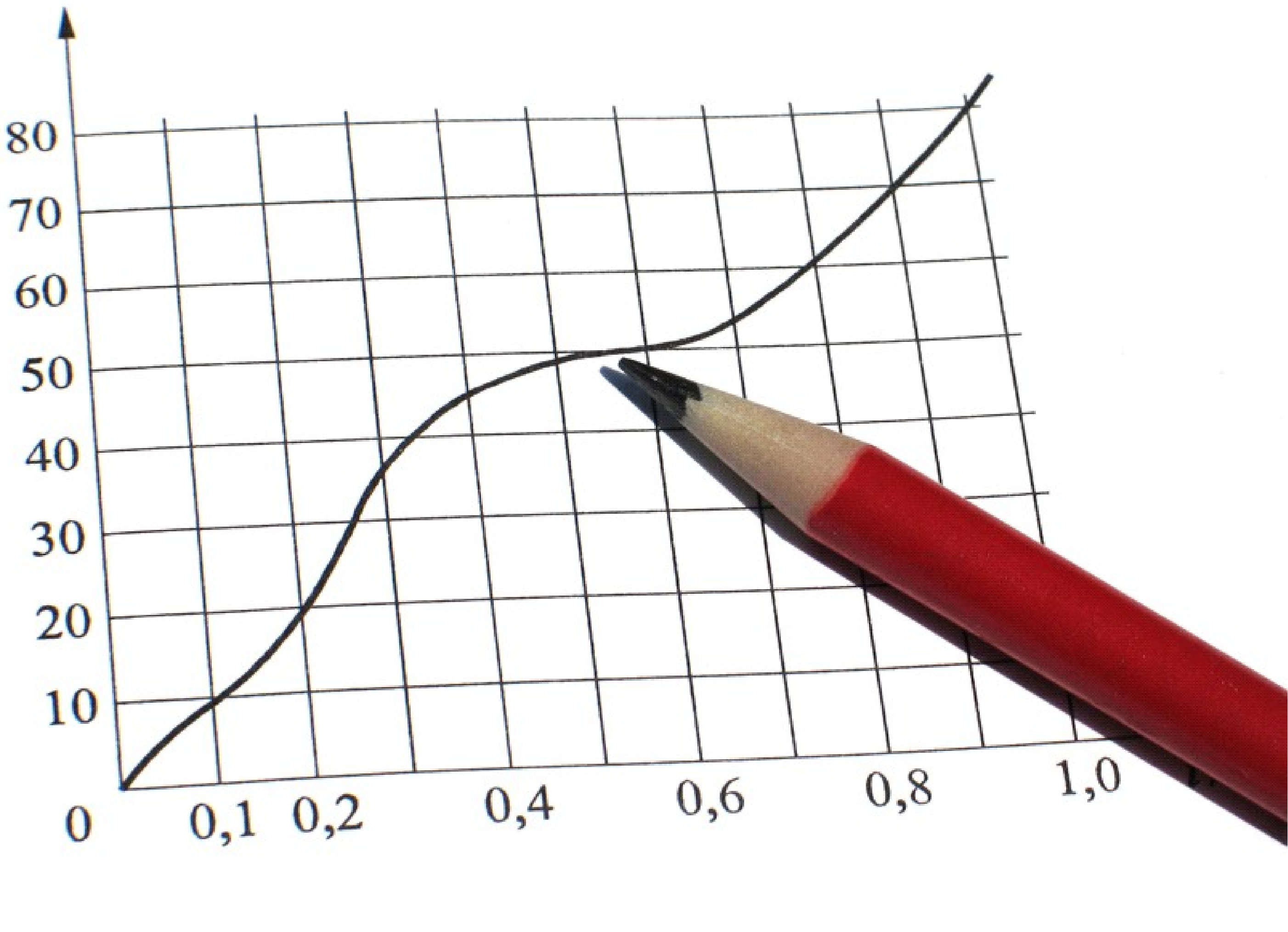

Leif Mejlbro

Linear Algebra Examples c-3

The Eigenvalue Problem and Euclidean Vector Space

Linear Algebra Examples c-3 – The Eigenvalue Problem and Euclideam Vector Space

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Introduction

Here we collect all tables of contents of all the books on mathematics I have written so far for the publisher. In the rst list the topics are grouped according to their headlines, so the reader quickly can get an idea of where to search for a given topic.In order not to make the titles too long I have in the numbering added

a for a compendium

b for practical solution procedures (standard methods etc.)

c for examples.

The ideal situation would of course be that all major topics were supplied with all three forms of books, but this would be too much for a single man to write within a limited time.

After the rst short review follows a more detailed review of the contents of each book. Only Linear Algebra has been supplied with a short index. The plan in the future is also to make indices of every other book as well, possibly supplied by an index of all books. This cannot be done for obvious reasons during the rst couple of years, because this work is very big, indeed.

It is my hope that the present list can help the reader to navigate through this rather big collection of books.

Finally, since this list from time to time will be updated, one should always check when this introduction has been signed. If a mathematical topic is not on this list, it still could be published, so the reader should also check for possible new books, which have not been included in this list yet.

Unfortunately errors cannot be avoided in a rst edition of a work of this type. However, the author has tried to put them on a minimum, hoping that the reader will meet with sympathy the errors which do occur in the text.

Leif Mejlbro
5th October 2008

1 The eigenvalue problem

Example 1.1 *Find the eigenvalues and the corresponding eigenvectors of the following matrix*

$$\begin{pmatrix} 1 & -1 & -1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}.$$

The equation of eigenvalues is

$$\begin{aligned} 0 &= \begin{vmatrix} 1-\lambda & -1 & -1 \\ 1 & -1-\lambda & 0 \\ 1 & 0 & -1-\lambda \end{vmatrix} \underset{S_2}{=} \\ & \begin{vmatrix} 1 & 0 \\ 1 & -(1+\lambda) \end{vmatrix} - (1+\lambda) \begin{vmatrix} 1-\lambda & -1 \\ 1 & -(1+\lambda) \end{vmatrix} \\ &= -(1+\lambda)\{1+\lambda^2-1+1\} = -(\lambda+1)\{\lambda^2+1\}. \end{aligned}$$

The complex eigenvalues are $\lambda = -1$, i, $-i$.

If $\lambda = -1$, then we get the matrix of coefficients

$$\mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix} 2 & -1 & -1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

An element of the kernel is $(0,1,-1)$, so $(0,1,-1)$ is an eigenvector corresponding to $\lambda = -1$.

When $\lambda = i$, we get the following matrix of coefficients

$$\begin{aligned} \mathbf{A} - \lambda\mathbf{I} &= \begin{pmatrix} 1-i & -1 & -1 \\ 1 & -1-i & 0 \\ 1 & 0 & -1-i \end{pmatrix} \begin{matrix} \sim \\ R_1 := R_1 - (1-i)R_3 \\ R_2 := R_2 - R_3 \end{matrix} \\ & \begin{pmatrix} 0 & -1 & 1 \\ 0 & -1-i & 1+i \\ 1 & 0 & -1-i \end{pmatrix} \underset{R_2 := R_2 - (1+i)R_1}{\sim} \\ & \begin{pmatrix} 0 & -1 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1-i \end{pmatrix}. \end{aligned}$$

An element $\mathbf{x}$ of the kernel satisfies $x_1 = (1+i)x_3$ and $x_2 = x_3$, hence an eigenvector corresponding to $\lambda = i$ is $(1+i, 1, 1)$.

Since the matrix is real, an eigenvector corresponding to $\lambda = -i$ is found by complex conjugation, i.e. the eigenvector is $(1-i, 1, 1)$, corresponding to $\lambda = -i$.

The proof of the latter claim is easy. In fact, if we conjugate

$$(\mathbf{A} - i\mathbf{I}) \begin{pmatrix} 1+i \\ 1 \\ 1 \end{pmatrix} = \mathbf{0},$$

we obtain

$$(\mathbf{A}+i\mathbf{I})\begin{pmatrix}1-i\\1\\1\end{pmatrix}=\mathbf{0},$$

and the claim follows.

Example 1.2 *Find the eigenvalue and the corresponding eigenvectors of the following matrix*

$$\begin{pmatrix}2-i & 0 & i\\ 0 & 1-i & 0\\ i & 0 & 2-i\end{pmatrix}.$$

The equation of the eigenvalues is

$$\begin{aligned}0 &= \begin{vmatrix}2-i-\lambda & 0 & i\\ 0 & 1-i-\lambda & 0\\ i & 0 & 2-i-\lambda\end{vmatrix}\\ &= (1-i-\lambda)\begin{vmatrix}2-i-\lambda & i\\ i & 2-i-\lambda\end{vmatrix}\\ &= (1-i-\lambda)\{(2-i-\lambda)^2-i^2\}\\ &= (1-i-\lambda)(2-\lambda)(2-2i-\lambda),\end{aligned}$$

hence the three eigenvalues are $\lambda_1=2$, $\lambda_2=1-i$ and $\lambda_3=2-2i$.

Since the matrix is complex, the conjugation argument of Example 1.1 cannot be applied.

For $\lambda_1 = 2$, we get the matrix of coefficients

$$\mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix} -i & 0 & i \\ 0 & -1-i & 0 \\ i & 0 & -i \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

corresponding to e.g. the eigenvector $(1,0,1)$.

If $\lambda_2 = 1-i$, we get the matrix of coefficients

$$\begin{pmatrix} 1 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 1 \end{pmatrix},$$

corresponding to e.g. the eigenvector $(0,1,0)$.

If $\lambda_3 = 2-2i$, we get the matrix of coefficients

$$\begin{pmatrix} i & 0 & i \\ 0 & -1+i & 0 \\ i & 0 & i \end{pmatrix},$$

corresponding to e.g. the eigenvector $(1,0,-1)$.

Example 1.3 *Find the eigenvalues and the corresponding eigenvectors of the following matrix*

$$\begin{pmatrix} 5 & 6 & -10 & 7 \\ -5 & -4 & 9 & -6 \\ -3 & -2 & 6 & -4 \\ -3 & -3 & 7 & -5 \end{pmatrix}.$$

If one shall compute an $(n \times n)$ determinant, where $n \geq 4$, and one does not have MAPLE or any similar programme at hand, one should follow the following strategy: One should only perform the simplest row or column operations, such that one obtains at least some zeros in the determinant. Then expand after a row or a column which contains as many zeros as possible. The new subdeterminants of order $(n-1) \times (n-1)$ are then treated separately. This procedure is recommended in order to minimize the errors and maximize the simplicity of the determinant.

The equation of the eigenvalues is

$$
\begin{aligned}
0 &= |\mathbf{A}-\lambda\mathbf{I}| = \begin{vmatrix} 5-\lambda & 6 & -10 & 7 \\ -5 & -4-\lambda & 9 & -6 \\ -3 & -2 & 6-\lambda & -4 \\ -3 & -3 & 7 & -5-\lambda \end{vmatrix} \quad \begin{matrix} R_1 := R_1 + R_2 \\ R_3 := R_3 - R_4 \end{matrix} \\
&= \begin{vmatrix} -\lambda & 2-\lambda & -1 & 1 \\ -5 & -4-\lambda & 9 & -6 \\ 0 & 1 & -\lambda-1 & 1+\lambda \\ -3 & -3 & 7 & -5-\lambda \end{vmatrix} \\
&\underset{R_3}{=} -\begin{vmatrix} -\lambda & -1 & 1 \\ -5 & 9 & -6 \\ -3 & 7 & -5-\lambda \end{vmatrix} - (\lambda+1)\begin{vmatrix} -\lambda & 2-\lambda & 1 \\ -5 & -4-\lambda & -6 \\ -3 & -3 & -5-\lambda \end{vmatrix} \\
&\qquad -(\lambda+1)\begin{vmatrix} -\lambda & 2-\lambda & -1 \\ -5 & -4-\lambda & 9 \\ -3 & -3 & 7 \end{vmatrix}.
\end{aligned}
$$

CALCULATIONS:

$$
\begin{aligned}
\begin{vmatrix} -\lambda & -1 & 1 \\ -5 & 9 & -6 \\ -3 & 7 & -5-\lambda \end{vmatrix} &= 9\lambda(\lambda+5) - 18 - 35 + 27 - 42\lambda + 5(\lambda+5) \\
&= 9\lambda^2 + 45\lambda - 26 - 42\lambda + 5\lambda + 25 \\
&= 9\lambda^2 + 8\lambda - 1 = (\lambda+1)(9\lambda-1),
\end{aligned}
$$

$$
\begin{aligned}
\begin{vmatrix} -\lambda & 2-\lambda & 1 \\ -5 & -(\lambda+4) & -6 \\ -3 & -3 & -(\lambda+5) \end{vmatrix} &= -\lambda(\lambda+4)(\lambda+5) - 18(\lambda-2) + 15 - 3(\lambda+4) \\
&\qquad +18\lambda + 5(\lambda+5)(\lambda-2) \\
&= -\lambda(\lambda^2+9\lambda+20) - 18\lambda + 36 + 15 - 3\lambda - 12 \\
&\qquad +18\lambda + 5(\lambda^2+3\lambda-10) \\
&= -\lambda^3 - 9\lambda^2 - 20\lambda - 3\lambda + 39 + 5\lambda^2 + 15\lambda - 50 \\
&= -\lambda^3 - 4\lambda^2 - 8\lambda - 11,
\end{aligned}
$$

$$
\begin{aligned}
\begin{vmatrix} -\lambda & 2-\lambda & -1 \\ -5 & -(\lambda+4) & 9 \\ -3 & -3 & 7 \end{vmatrix} &= 7\lambda(\lambda+4) + 27(\lambda-2) - 15 + 3(\lambda+4) - 27\lambda - 35(\lambda-2) \\
&= 7\lambda^2 + 28\lambda + 27\lambda - 54 - 15 + 3\lambda + 12 - 27 - 35\lambda + 70 \\
&= 7\lambda^2 - 4\lambda + 13.
\end{aligned}
$$

Hence by insertion

$$
\begin{aligned}
0 &= |\mathbf{A}-\lambda\mathbf{I}| \\
&= -(\lambda+1)\{9\lambda - 1 - \lambda^3 - 4\lambda^2 - 8\lambda - 11 + 7\lambda^2 - 4\lambda + 13\} \\
&= -(\lambda+1)\{-\lambda^3 + 3\lambda^2 - 3\lambda + 1\} \\
&= (\lambda+1)\{\lambda^3 - 3\lambda^2 + 3\lambda - 1\} = (\lambda+1)(\lambda-1)^3.
\end{aligned}
$$

Remark 1.1 In my original draft I used some very sophisticated row and column operations which made the calculations much shorter. Unfortunately, this method was not very instructive, so I chose instead to use this longer, though also more standardized method. $\Diamond$

The eigenvalues are $\lambda = 1$ of the algebraic multiplicity 3 and $\lambda = -1$ of the algebraic multiplicity 1.

Whenever the algebraic multiplicity as here by $\lambda = 1$ is bigger than 1, one should always be very careful with the calculations, because the geometric multiplicity is not necessarily equal to the algebraic multiplicity. If $\lambda = 1$ we reduce the matrix of coefficients in the following way

$$\mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix} 4 & 6 & -10 & 7 \\ -5 & -5 & 9 & -6 \\ -3 & -2 & 5 & -4 \\ -3 & -3 & 7 & -6 \end{pmatrix} \begin{array}{l} \sim \\ R_1 := R_1 + R_2 \\ R_3 := R_3 - R_4 \\ R_4 := 3R_2 - 5R_4 \end{array}$$

$$\begin{pmatrix} -1 & 1 & -1 & 1 \\ -5 & -5 & 9 & -6 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & -8 & 12 \end{pmatrix} \begin{array}{l} \sim \\ R_1 := -R_1 \\ R_2 := R_2 - 5R_1 \end{array}$$

$$\begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & -10 & 14 & -11 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & -8 & 12 \end{pmatrix} = \begin{array}{l} \sim \\ R_1 := R_1 + R_3 \\ R_2 := R_3 \\ R_3 := R_2 + 10R_3 \end{array}$$

$$\begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & -6 & 9 \\ 0 & 0 & -8 & 12 \end{pmatrix} \begin{array}{l} \sim \\ R_3 := -R_3/3 \\ R_4 := R_4/4 - R_3/3 \end{array}$$

$$\begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 2 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{array}{l} \sim \\ R_1 := R_1 + R_3/2 \\ R_2 := R_2 + R_3 \end{array}$$

$$\begin{pmatrix} 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 2 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The rank is 3, thus the kernel (= the eigenspace corresponding to $\lambda = 1$) is of dimension $4 - 3 = 1$. The geometric multiplicity is $1 < 3 =$ the algebraic multiplicity .
The conditions of obtaining a zero vector are

$$x_1 = \frac{1}{2}\,x_4, \qquad x_2 = x_4, \qquad x_3 = \frac{3}{2}\,x_4,$$

hence we see by choosing $x_4 = 2$ that an eigenvector is $(1, 2, 3, 2)$ and that all eigenvectors corresponding to $\lambda = 1$ is a scalar multiple of this vector.

If $\lambda = -1$, the matrix of coefficients is reduced to

$$\mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix} 6 & 6 & -10 & 7 \\ -5 & -3 & 9 & -6 \\ -3 & -2 & 7 & -4 \\ -3 & -3 & 7 & -4 \end{pmatrix} \sim \begin{pmatrix} 6 & 6 & -10 & 7 \\ -5 & -3 & 9 & -6 \\ 0 & 1 & 0 & 0 \\ -3 & -3 & 7 & -4 \end{pmatrix}$$
$$\sim \begin{pmatrix} 6 & 0 & -10 & 7 \\ -5 & 0 & 9 & -6 \\ 0 & 1 & 0 & 0 \\ -3 & 0 & 7 & -4 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 1 \\ -5 & 0 & 9 & -6 \\ 0 & 1 & 0 & 0 \\ -3 & 0 & 7 & -4 \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 0 & 4 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & \frac{3}{4} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{4} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

of rank 3. An element the kernel (= the eigenvector of $\lambda = -1$) fulfils

$$x_1 = -\frac{3}{4}\,x_4, \qquad x_2 = 0, \qquad x_3 = \frac{1}{4}\,x_4.$$

Choosing $x_4 = 4$, we get the eigenvector $(-3, 0, 1, 4)$.

Example 1.4 *Find the eigenvalue and the corresponding eigenvectors of the following matrix*

$$\begin{pmatrix} -1 & -1 & -6 & 3 \\ 1 & -2 & -3 & 0 \\ -1 & 1 & 0 & 1 \\ -1 & -1 & -5 & 3 \end{pmatrix}.$$

The equation of the eigenvalues is

$$\begin{aligned} 0 \quad &= \quad |\mathbf{A}-\lambda\mathbf{I}| = \begin{vmatrix} -1-\lambda & -1 & -6 & 3 \\ 1 & -2-\lambda & -3 & 0 \\ -1 & 1 & -\lambda & 1 \\ -1 & -1 & -5 & 3-\lambda \end{vmatrix} = \begin{vmatrix} \lambda+1 & 1 & 6 & -3 \\ -1 & \lambda+2 & 3 & 0 \\ 1 & -1 & \lambda & -1 \\ 1 & 1 & 5 & \lambda-3 \end{vmatrix} \\ &\underset{+R_2}{=} \begin{vmatrix} \lambda & \lambda+3 & 9 & -3 \\ -1 & \lambda+2 & 3 & 0 \\ 0 & \lambda+1 & \lambda+3 & -1 \\ 0 & \lambda+3 & 8 & \lambda-3 \end{vmatrix} \\ &= \quad \lambda \begin{vmatrix} \lambda+3 & 3 & 0 \\ \lambda+1 & \lambda+3 & -1 \\ \lambda+3 & 8 & \lambda-3 \end{vmatrix} + \begin{vmatrix} \lambda+3 & 9 & -3 \\ \lambda+1 & \lambda+3 & -1 \\ \lambda+3 & 8 & \lambda-3 \end{vmatrix}. \end{aligned}$$

CALCULATIONS:

$$\begin{aligned} &\begin{vmatrix} \lambda+2 & 3 & 0 \\ \lambda+1 & \lambda+3 & -1 \\ \lambda+3 & 8 & \lambda-3 \end{vmatrix} = \begin{vmatrix} \lambda+2 & 3 & 0 \\ -1 & \lambda & -1 \\ 1 & 5 & \lambda-3 \end{vmatrix} = \begin{vmatrix} \lambda+2 & 3 & 0 \\ -1 & \lambda & -1 \\ 0 & \lambda+5 & \\ \lambda-4 & & \end{vmatrix} \\ &= (\lambda+2)\begin{vmatrix} \lambda & -1 \\ \lambda+5 & \lambda-4 \end{vmatrix} + \begin{vmatrix} 3 & 0 \\ \lambda+5 & \lambda-4 \end{vmatrix} \\ &= (\lambda+2)(\lambda^2-4\lambda+\lambda+5)+3(\lambda-4) \\ &= (\lambda+2)(\lambda^2-3\lambda+5)+3\lambda-12 \\ &= \lambda^3-3\lambda^2+5\lambda+2\lambda^2-6\lambda+10+3\lambda-12 \\ &= \lambda^3-\lambda^2+2\lambda-2 = (\lambda-1)(\lambda^2+2), \end{aligned}$$

and

$$\begin{aligned} &\begin{vmatrix} \lambda+3 & 9 & -3 \\ \lambda+1 & \lambda+3 & -1 \\ \lambda+3 & 8 & \lambda-3 \end{vmatrix} = \begin{vmatrix} \lambda & 9 & -3 \\ \lambda & \lambda+3 & -1 \\ 2\lambda & 8 & \lambda-3 \end{vmatrix} = \lambda \begin{vmatrix} 1 & 9 & -3 \\ 1 & \lambda+3 & -1 \\ 2 & 8 & \lambda-3 \end{vmatrix} \\ &= \lambda \begin{vmatrix} 1 & 0 & -3 \\ 0 & \lambda-6 & 2 \\ 0 & -10 & \lambda+3 \end{vmatrix} = \lambda \begin{vmatrix} \lambda-6 & 2 \\ -10 & \lambda+3 \end{vmatrix} \\ &= \lambda\{(\lambda-6)(\lambda+3)+20\} = \lambda\{\lambda^2-3\lambda-18+20\} \\ &= \lambda\{\lambda^2-3\lambda+2\} = \lambda(\lambda-1)(\lambda-2). \end{aligned}$$

We get by insertion

$$\begin{aligned}0 &= |\mathbf{A}-\lambda\mathbf{I}| \\ &= \lambda(\lambda-1)(\lambda^2)+\lambda(\lambda-1)(\lambda-2) \\ &= \lambda(\lambda-1)(\lambda^2+\lambda)=\lambda^2(\lambda-1)(\lambda+1).\end{aligned}$$

The eigenvalues are $\lambda = 0$ (algebraic multiplicity 2) and $\lambda = \pm 1$ (each of algebraic multiplicity 1).

If $\lambda = 0$, the matrix of coefficients is reduced to

$$\begin{aligned}\mathbf{A}-\lambda\mathbf{I} &= \begin{pmatrix} -1 & -1 & -6 & 3 \\ 1 & -2 & -3 & 0 \\ -1 & 1 & 0 & 1 \\ -1 & -1 & -5 & 3 \end{pmatrix} \begin{matrix} \sim \\ +R_2 \end{matrix} \begin{pmatrix} 0 & -3 & -9 & 3 \\ 1 & -2 & -3 & 0 \\ 0 & -1 & -3 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \\ &\sim \begin{pmatrix} 1 & -2 & -3 & 0 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &\sim \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.\end{aligned}$$

The rank is 3, hence the kernel (= the eigenspace corresponding to $\lambda = 0$) has dimension $4 - 3 = 1 < 2 =$ the algebraic multiplicity. We may choose the eigenvector $(2, 1, 0, 1)$.

If $\lambda = 1$, then the matrix of coefficients is reduces to

$$\begin{aligned}\mathbf{A}-\lambda\mathbf{I} &= \begin{pmatrix} -2 & -1 & -6 & 3 \\ 1 & -3 & -3 & 0 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & -5 & 2 \end{pmatrix} \begin{matrix} \sim \\ R_1 := R_1 + 2R_2 \\ R_3 := R_3 + R_2 \\ R_4 := R_4 + R_2 \end{matrix} \begin{pmatrix} 0 & -7 & -12 & 3 \\ 1 & -3 & -3 & 0 \\ 0 & -2 & -4 & 1 \\ 0 & -4 & -8 & 2 \end{pmatrix} \\ &\sim \begin{pmatrix} 1 & -3 & -3 & 0 \\ 0 & 2 & 4 & -1 \\ 0 & 7 & 12 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -3 & -3 & 0 \\ 0 & 2 & 4 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &\sim \begin{pmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.\end{aligned}$$

If $\mathbf{x}$ lies in the kernel, then

$$x_1 = 3x_3, \qquad x_2 = 0, \qquad x_4 = 4x_3,$$

and an eigenvector is e.g. $(3, 0, 1, 4)$.

If $\lambda = -1$, the matrix of coefficients is reduced to

$$\mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix} 0 & -1 & -6 & 3 \\ 1 & -1 & -3 & 0 \\ -1 & 1 & 1 & 1 \\ -1 & -1 & -5 & 4 \end{pmatrix} \begin{matrix} \sim \\ R_3 := R_2 + R_3 \\ R_4 := R_2 + R_4 \end{matrix} \begin{pmatrix} 0 & -1 & -6 & 3 \\ 0 & -1 & -3 & 0 \\ 0 & 0 & -2 & -1 \\ 0 & -2 & -8 & 4 \end{pmatrix}$$

$$\begin{matrix} \sim \\ R_1 := R_2 - R_1 \\ R_2 := -R_1 \\ R_4 := R_4 - 2R_1 \end{matrix} \sim \begin{pmatrix} 1 & 0 & 3 & -3 \\ 0 & 1 & 6 & -3 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 4 & -2 \end{pmatrix} \begin{matrix} \sim \\ R_1 := R_1 + 3R_3 \\ R_2 := R_2 + 3R_3 \\ R_4 := R_4 + 2R_3 \end{matrix} \begin{pmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The rank is 3, thus the kernel = the eigenspace of $\lambda = -1$ has dimension $4 - 3 = 1$. A vector in the kernel must fulfil

$$x_1 = 3x_3, \qquad x_2 = 0, \qquad x_4 = 2x_3,$$

hence an eigenvector is $(3, 0, 1, 2)$.

Summing up we get:

Eigenvalue 0 with the eigenvector $(2,1,0,1)$.

Eigenvalue 1 with the eigenvector $(3,0,1,4)$.

Eigenvalue -1 with the eigenvector $(3,0,1,2)$.

The space has 4 dimensions, however, modulo a scalar factor there are only three eigenvectors. The eigenvalue 0 has the algebraic multiplicity 2 and the geometric multiplicity 1.

Example 1.5 *A linear map $f : \mathbb{R}^4 \to \mathbb{R}^4$ is in the usual basis given by the matrix*

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 4 & -2 \\ 2 & -1 & 0 & 1 \\ 2 & -1 & -1 & 2 \end{pmatrix}.$$

Find the eigenvalues and the eigenvectors of f.

The equation of the eigenvalues is

$$\begin{aligned}
0 &= |\mathbf{A}-\lambda\mathbf{I}| = \begin{vmatrix} 1-\lambda & 0 & 2 & -1 \\ 0 & 1-\lambda & 4 & -2 \\ 2 & -1 & -\lambda & 1 \\ 2 & -1 & -1 & 2-\lambda \end{vmatrix} \\
&= \begin{vmatrix} 1-\lambda & 0 & 2 & -1 \\ 0 & 1-\lambda & 4 & -2 \\ 0 & 0 & 1-\lambda & \lambda-1 \\ 2 & -1 & -1 & 2-\lambda \end{vmatrix} \\
&= (1-\lambda)\begin{vmatrix} 1-\lambda & 4 & -2 \\ 0 & 1-\lambda & \lambda-1 \\ -1 & -1 & 2-\lambda \end{vmatrix} - 2\begin{vmatrix} 0 & 2 & -1 \\ 1-\lambda & 4 & -2 \\ 0 & 1-\lambda & \lambda-1 \end{vmatrix} \\
&= (1-\lambda)^2\begin{vmatrix} 1-\lambda & 4 & -2 \\ 0 & 1 & -1 \\ -1 & -1 & 2-\lambda \end{vmatrix} + 2(1-\lambda)^2\begin{vmatrix} 2 & -1 \\ 1 & -1 \end{vmatrix} \\
&= (1-\lambda)^2\begin{vmatrix} 1-\lambda & 4 & 2 \\ 0 & 1 & 0 \\ -1 & -1 & 1-\lambda \end{vmatrix} - 2(1-\lambda)^2 \\
&= (1-\lambda)^2\begin{vmatrix} 1-\lambda & 2 \\ -1 & 1-\lambda \end{vmatrix} - 2(1-\lambda)^2 \\
&= (1-\lambda)^2\{(1-\lambda)^2+2-2\} = (\lambda-1)^4.
\end{aligned}$$

It follows that $\lambda = 1$ is the only eigenvalue and its algebraic multiplicity is 4.

Then we reduce the matrix of coefficients,

$$\mathbf{A}-\lambda\mathbf{I}=\begin{pmatrix} 0 & 0 & 2 & -1 \\ 0 & 0 & 4 & -2 \\ 2 & -1 & -1 & 1 \\ 2 & -1 & -1 & 1 \end{pmatrix} \begin{array}{l} R_1 := R_3 \\ R_2 := R_1 \\ R_3 := R_2 - 2R_1 \\ R_4 := R_3 - R_4 \end{array}$$

$$\sim \begin{pmatrix} 2 & -1 & -1 & 1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{array}{c} \sim \\ R_1 := R_1 + R_2 \end{array} \begin{pmatrix} 2 & -1 & 1 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The rank is 2, hence the kernel = the eigenspace has dimension 2. A vector in the kernel satisfies with $x_2 = 2s$ and $x_3 = 2t$ as the chosen parameters,

$$2x_1 - x_2 + x_3 = 2x_1 - 2s + 2t = 0 \quad \text{and} \quad 2x_3 - x_4 = 4t - x_4 = 0,$$

hence

$$x_1 = s - t, \quad x_2 = 2s, \quad x_3 = 2t, \quad x_4 = 4t,$$

and whence.

$$\mathbf{x} = (s-t, 2s, 2t, 4t) = s(1,2,0,0) + t(-1,0,2,4).$$

Two linearly independent eigenvectors which span the eigenspace corresponding to the eigenvalue $\lambda = 1$, are e.g.

$$(1,2,0,0) \quad \text{and} \quad (-1,0,2,4).$$

It follows again that the algebraic multiplicity is bigger that the geometric multiplicity .

Example 1.6 *A linear map $f : \mathbb{R}^3 \to \mathbb{R}^3$ maps*

$$(1,2,1) \to (1,2,1); \quad (2,1,0) \to (-4,-2,0); \quad (1,1,1) \to (0,0,0).$$

Find the determinant of reduction, the trace of the matrix and the determinant of the matrix.
*(*HINT*: It is not necessary explicitly to find the matrix of the map).*

It follows from the given that
$(1,2,1)$ is an eigenvector corresponding to the eigenvalue 1,
$(2,1,0)$ if an eigenvector corresponding to the eigenvalue -2,
$(1,1,1)$ is an eigenvector corresponding to the eigenvalue 0.
The dimension of $\mathbb{R}^3$ is 3, and there are 3 different eigenvalues, hence they must all have multiplicity 1. This means that the determinant of reduction = the characteristic polynomial is

$$(1-\lambda)(-2-\lambda)(0-\lambda) = -\lambda(\lambda-1)(\lambda+2) = -(\lambda^3+\lambda^2-2\lambda) = -\lambda^3-\lambda^2+2\lambda.$$

The trace of the matrix is the sum of the eigenvalues

$$\operatorname{tr}\mathbf{A} = 1 - 2 + 0 = -1.$$

The determinant of the matrix is obtained by putting $\lambda = 0$ into the characteristic polynomial, thus the value is 0.

Example 1.7 *A linear map* $f : \mathbb{R}^3 \to \mathbb{R}^3$ *is given by the matrix equation*

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 & 4 & 0 \\ 0 & 4 & 5 \\ 4 & 3 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Prove that $\lambda = 9$ *is an eigenvalue of* f*, and find the corresponding eigenvectors.*

The easiest way is to prove that $\mathbf{A} - 9\mathbf{I}$ has rank < 3. Then by reduction,

$$\begin{aligned} \mathbf{A} - 9\mathbf{I} &= \begin{pmatrix} -8 & 4 & 0 \\ 0 & -5 & 5 \\ 4 & 3 & -5 \end{pmatrix} \begin{matrix} \sim \\ R_2 := -R_2/5 \\ R_3 := R_2 + R_3 \end{matrix} \begin{pmatrix} -8 & 4 & 0 \\ 0 & 1 & -1 \\ 4 & -2 & 0 \end{pmatrix} \\ &\begin{matrix} \sim \\ R_1 := R_3/2 \\ R_3 := R_1 + 2R_3 \end{matrix} \begin{pmatrix} 2 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

The rank is $2 < 3$, thus $\lambda = 9$ is an eigenvalue with the eigenspace of dimension 1. An eigenvector $\mathbf{x}$ satisfies

$$2x_1 = x_2 \quad \text{and} \quad x_2 = x_3,$$

thus $(1, 2, 2)$ is an eigenvector corresponding to $\lambda = 9$.

Example 1.8 *Assume that two* $(n \times n)$ *matrices* $\mathbf{A}$ *and* $\mathbf{B}$ *have* n *linearly independent vectors as common eigenvectors. Prove that*

$$\mathbf{AB} = \mathbf{BA}.$$

We get

$$\mathbf{V}^{-1}\mathbf{AV} = \mathbf{\Lambda}_1 \quad \text{and} \quad \mathbf{V}^{-1}\mathbf{BV} = \mathbf{\Lambda}_2,$$

where the columns of $\mathbf{V}$ are the common eigenvectors for $\mathbf{A}$ and $\mathbf{B}$, and $\mathbf{\Lambda}_1$ and $\mathbf{\Lambda}_2$ are diagonal matrices, thus

$$\mathbf{\Lambda}_1\mathbf{\Lambda}_2 = \mathbf{\Lambda}_2\mathbf{\Lambda}_1.$$

Then

$$\begin{aligned} \mathbf{\Lambda}_1\mathbf{\Lambda}_2 &= \left(\mathbf{V}^{-1}\mathbf{AV}\right)\left(\mathbf{V}^{-1}\mathbf{BV}\right) = \mathbf{V}^{-1}(\mathbf{AB})\mathbf{V} \\ &= \mathbf{\Lambda}_2\mathbf{\Lambda}_1 = \left(\mathbf{V}^{-1}\mathbf{BV}\right)\left(\mathbf{V}^{-1}\mathbf{AV}\right) = \mathbf{V}^{-1}(\mathbf{BA})\mathbf{V}, \end{aligned}$$

hence $\mathbf{AB} = \mathbf{BA}$.

Example 1.9 *Let f denote the linear map of $\mathbb{R}^3$ into $\mathbb{R}^3$, which in the usual basis of $\mathbb{R}^3$ has the matrix*

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & -2 \\ 0 & 1 & 0 \\ 0 & 1 & 3 \end{pmatrix}.$$

Find the eigenvalues and the corresponding eigenvectors of f.
Find also the vectors $\mathbf{x} \in \mathbb{R}^3$, for which $f(\mathbf{x}) = \mathbf{x}$.

The eigenvalues are given by

$$\begin{aligned} 0 &= \det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 1-\lambda & -1 & -2 \\ 0 & 1-\lambda & 0 \\ 0 & 1 & 3-\lambda \end{vmatrix} \\ &= (1-\lambda)\begin{vmatrix} 1-\lambda & 0 \\ 1 & 3-\lambda \end{vmatrix} = -(\lambda-1)^2(\lambda-3), \end{aligned}$$

hence the eigenvalues are $\lambda = 1$ (algebraic multiplicity 2) and $\lambda = 3$.

If $\lambda = 1$, then the matrix of coefficients is reduced to

$$\mathbf{A} - \mathbf{I} = \begin{pmatrix} 0 & -1 & -2 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

of rank 1, and the kernel has the dimension 2. It follows immediately that the kernel = the eigenspace of $\lambda = 1$ is spanned by $(1, 0, 0)$ and $(0, 2, -1)$, which therefore are two linearly independent eigenvectors corresponding to $\lambda = 1$.

If $\lambda = 3$, then the matrix of coefficients is reduced to

$$\mathbf{A} - 3\mathbf{I} = \begin{pmatrix} -2 & -1 & -2 \\ 0 & -2 & 0 \\ 0 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} -2 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The rank is 2, so the kernel = the eigenspace of $\lambda = 3$ has dimension $3 - 2 = 1$. An eigenvector is e.g. $(1, 0, -1)$.

All solutions of $f(\mathbf{x}) = \mathbf{x}$ form the eigenspace corresponding to $\lambda = 1$. This eigenspace was generated by the vectors $(1, 0, 0)$ and $(0, 2, -1)$, hence the set of solutions is

$$\{s(1, 0, 0) + t(0, 2, -1) \mid s, t \in \mathbb{R}\} = \{(s, 2t, -t) \mid s, t \in \mathbb{R}\}.$$

Example 1.10 *A linear map $f : \mathbb{R}^3 \to \mathbb{R}^3$ is given by the matrix equation*

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Find the eigenvalues and the corresponding eigenvectors of the map f.

Since we have a lower triangular matrix, the eigenvalues are the diagonal elements, i.e. $\lambda = 1$ is the only eigenvalue, and it has the algebraic multiplicity 3.

Since

$$\mathbf{A} - \mathbf{I} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$

has the rank 1, the eigenspace has dimension $3 - 1 = 2$, and it is e.g. spanned by the eigenvectors $(1, 0, 0)$ and $(0, 0, 1)$.

Example 1.11 *Find the eigenvalues and all eigenvectors of the matrix*

$$\mathbf{A} = \begin{pmatrix} -2 & 0 & 3 \\ 0 & 4 & 0 \\ -6 & 0 & 7 \end{pmatrix}.$$

Does there exist a proper column $\mathbf{v} \in \mathbb{R}^{3\times 1}$*, such that* $\mathbf{A}\mathbf{v} = 2\mathbf{v}$*?*

We infer the eigenvalues from the equation

$$\begin{aligned} 0 &= \det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -2-\lambda & 0 & 3 \\ 0 & 4-\lambda & 0 \\ -6 & 0 & 7-\lambda \end{vmatrix} = -(\lambda-4)\begin{vmatrix} \lambda+2 & -3 \\ 6 & \lambda-7 \end{vmatrix} \\ &= -(\lambda-4)\{\lambda^2 - 5\lambda - 14 + 18\} = -(\lambda-4)\{\lambda^2 - 5\lambda + 4\} \\ &= -(\lambda-4)(\lambda-1)(\lambda-4) = -(\lambda-1)(\lambda-4)^2. \end{aligned}$$

The eigenvalues are $\lambda = 1$, and $\lambda = 4$ (of algebraic multiplicity 2).

If $\lambda = 1$, then the matrix of coefficients is reduced to

$$\mathbf{A} - \mathbf{I} = \begin{pmatrix} -3 & 0 & 3 \\ 0 & 3 & 0 \\ -6 & 0 & 6 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The rank is 2, so the eigenspace has dimension $3 - 2 = 1$, generated by the eigenvector $(1, 0, -1)$.

If $\lambda = 4$, then the matrix of coefficients is reduced to

$$\mathbf{A} - 4\mathbf{I} = \begin{pmatrix} -6 & 0 & 3 \\ 0 & 0 & 0 \\ -6 & 0 & 3 \end{pmatrix} \sim \begin{pmatrix} 2 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The rank is 1, so the dimension of the eigenspace is $3 - 1 = 2$. Two linearly independent eigenvectors are

$$(1, 0, 2) \quad \text{and} \quad (0, 1, 0).$$

The answer is "no". Because if it was true, then $\lambda = 2$ would be an eigenvalue, which $\lambda = 2$ is not.

Example 1.12 *Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map, which in the usual basis of $\mathbb{R}^3$ is determined by the matrix*

$$\mathbf{A} = \begin{pmatrix} 3 & 4 & 4 \\ 6 & 6 & 6 \\ -6 & -7 & -7 \end{pmatrix}.$$

1. *Prove that the vectors $\mathbf{v}_1 = (1, 0, -1)$, $\mathbf{v}_2 = (0, 1, -1)$ and $\mathbf{v}_3 = (1, 2, -2)$ are eigenvectors of f, and find the corresponding eigenvalues.*
2. *Prove that $f(\mathbb{R}^3) = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_3\}$.*

1. We get by insertion,

$$\mathbf{A}\mathbf{v}_1 = \begin{pmatrix} 3 & 4 & 4 \\ 6 & 6 & 6 \\ -6 & -7 & -7 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = -\mathbf{v}_1, \quad \text{eigenvalue } -1;$$

$$\mathbf{A}\mathbf{v}_2 = \begin{pmatrix} 3 & 4 & 4 \\ 6 & 6 & 6 \\ -6 & -7 & -7 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0\,\mathbf{v}_2, \quad \text{eigenvalue } 0;$$

$$\mathbf{A}\mathbf{v}_3 = \begin{pmatrix} 3 & 4 & 4 \\ 6 & 6 & 6 \\ -6 & -7 & -7 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ -6 \end{pmatrix} = 3\,\mathbf{v}_3, \quad \text{eigenvalue } 3.$$

2. Since $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ belong to different eigenvalues, they are linearly independent, so we conclude that $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ form a basis of $\mathbb{R}^3$. Then we conclude from

$$f(\alpha\mathbf{v}_1 + \beta\mathbf{v}_2 + \gamma\mathbf{v}_3) = -\alpha\mathbf{v}_1 + 3\gamma\mathbf{v}_3,$$

that $f(\mathbb{R}^3) = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_3\}$ of dimension 2.

Example 1.13 *Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map, which in the usual basis of $\mathbb{R}^3$ has the matrix*

$$\mathbf{F} = \begin{pmatrix} 2 & 0 & -3 \\ 0 & 5 & 0 \\ 4 & 0 & 9 \end{pmatrix}.$$

1. *Find all eigenvalues and all the corresponding eigenvectors for f.*
2. *Check if there exists a basis for $\mathbb{R}^3$, such that the matrix of f with respect to this basis is a diagonal matrix.*

1. We find the eigenvalues from the equation

$$\begin{aligned} 0 &= \det(\mathbf{F} - \lambda\mathbf{I}) = \begin{vmatrix} 2-\lambda & 0 & -3 \\ 0 & 5-\lambda & 0 \\ 4 & 0 & 9-\lambda \end{vmatrix} = -(\lambda-5)\begin{vmatrix} \lambda-2 & 3 \\ -4 & \lambda-9 \end{vmatrix} \\ &= -(\lambda-5)\{\lambda^2 - 11\lambda + a8 + 12\} = -(\lambda-5)(\lambda^2 - 11\lambda + 30) \\ &= -(\lambda-5)^2(\lambda-6). \end{aligned}$$

The eigenvalues are $\lambda = 5$ (algebraic multiplicity 2) and $\lambda = 6$.

If $\lambda = 5$, then the matrix of coefficients is reduced to

$$\mathbf{A} - 5\mathbf{I} = \begin{pmatrix} -3 & 0 & -3 \\ 0 & 0 & 0 \\ 4 & 0 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The rank is 1, hence the kernel = the the eigenspace is of dimension 2. Two linearly independent eigenvectors are e.g. $(1, 0, -1)$ and $(0, 1, 0)$.

If $\lambda = 6$, then the matrix of coefficients is reduced to

$$\mathbf{A} - 6\mathbf{I} = \begin{pmatrix} -4 & 0 & -3 \\ 0 & -1 & 0 \\ 4 & 0 & 3 \end{pmatrix} \sim \begin{pmatrix} 4 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It follows that e.g. $(3, 0, -4)$ is a corresponding eigenvector.

2. Since we have $3(= \dim \mathbb{R}^3)$ linearly independent eigenvectors, these must form a basis for $\mathbb{R}^3$, and the matrix with respect to $(1, 0, -1)$, $(0, 1, 0)$, $(3, 0, -4)$ is the diagonal matrix

$$\begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 6 \end{pmatrix}.$$

Example 1.14 *Given the matrices*

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 2 & -3 \\ 1 & -1 & a \end{pmatrix} \qquad \textit{where } a \in \mathbb{R}, \ \textit{and } \mathbf{v} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$$

1. *Prove that 3 is an eigenvalue of* $\mathbf{A}$ *for every* a*, and find for every* a *its geometric multiplicity.*
2. *Find* a*, such that* $\mathbf{v}$ *is an eigenvector of* $\mathbf{A}$*, and prove that* $\mathbf{A}$ *can be diagonalized for such an* a*.*

1. Let $\lambda = 3$. We reduce

$$\mathbf{A} - \lambda\mathbf{I} = \mathbf{A} - 3\mathbf{I} = \begin{pmatrix} -1 & 1 & 3 \\ 1 & -1 & -3 \\ 1 & -1 & a-3 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & -3 \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix}.$$

If $a \neq 0$, then the rank is $2 < 3$, thus $\lambda = 3$ is an eigenvalue of geometric multiplicity $3 - 2 = 1$. An eigenvector is e.g. $(1, 1, 0)$.

If $a = 0$, then the rank is $1 < 3$, hence $\lambda = 3$ is an eigenvalue of the geometric multiplicity $3 - 1 = 2$ Two linearly independent eigenvectors are e.g. $(3, 0, 1)$ and $(0, 3, -1)$.

2. We compute

$$\mathbf{A}\mathbf{v} = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 2 & -3 \\ 1 & -1 & a \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ a-2 \end{pmatrix} = -2\,\mathbf{v} = \begin{pmatrix} 2 \\ -2 \\ -2 \end{pmatrix}$$

for $a = 0$, in which case we have the three linearly independent eigenvectors:

$(3, 0, 1)$ and $(0, 3, -1)$ for $\lambda = 3$,

$(-1, 1, 1)$ for $\lambda = -2$.

Then $\mathbf{A}$ can be diagonalized for $a = 0$.

Example 1.15 *Prove that the matrices*

$$\mathbf{A} = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 2 & -1 \\ -3 & -2 & 3 \end{pmatrix}$$

have the same characteristic polynomial, and yet they are not similar.

We compute

$$\begin{aligned}
\det(\mathbf{A} - \lambda\mathbf{I}) &= \begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = \begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 0 & \lambda-1 & 1-\lambda \end{vmatrix} \\
&= (1-\lambda)\begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 0 & -1 & 1 \end{vmatrix} = (1-\lambda)\begin{vmatrix} 2-\lambda & 3 & 1 \\ 1 & 4-\lambda & 1 \\ 0 & 0 & 1 \end{vmatrix} \\
&= (1-\lambda)\begin{vmatrix} 2-\lambda & 3 \\ 1 & 4-\lambda \end{vmatrix} = (1-\lambda)\{(\lambda-2)(\lambda-4)-3\} \\
&= (1-\lambda)\{\lambda^2-6\lambda+5\} = -(\lambda-1)(\lambda-1)(\lambda-5) \\
&= -(\lambda-1)^2(\lambda-5),
\end{aligned}$$

and

$$\begin{aligned}
\det(\mathbf{B} - \lambda\mathbf{I}) &= \begin{vmatrix} 2-\lambda & 1 & -1 \\ 0 & 2-\lambda & -1 \\ -3 & -2 & 3-\lambda \end{vmatrix} = \begin{vmatrix} 2-\lambda & 0 & -1 \\ 0 & 1-\lambda & -1 \\ -3 & 1-\lambda & 3-\lambda \end{vmatrix} \\
&= (1-\lambda)\begin{vmatrix} 2-\lambda & 0 & -1 \\ 0 & 1 & -1 \\ -3 & 1 & 3-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 2-\lambda & 0 & -1 \\ 0 & 1 & 0 \\ -3 & 1 & 4-\lambda \end{vmatrix} \\
&= (1-\lambda)\begin{vmatrix} 2-\lambda & -1 \\ -3 & 4-\lambda \end{vmatrix} = (1-\lambda)\{(\lambda-2)(\lambda-4)-3\} \\
&= -(\lambda-1)^2(\lambda-5),
\end{aligned}$$

hence $\mathbf{A}$ and $\mathbf{B}$ have the same characteristic polynomial.

If they are not similar, then $\lambda = 1$, which has the algebraic multiplicity 2, must have different geometric multiplicity for $\mathbf{A}$ and $\mathbf{B}$.

We reduce for $\lambda = 1$,

$$\mathbf{A} - \mathbf{I} = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$\mathbf{B} - \mathbf{I} = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ -3 & -2 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since $\mathbf{A} - \mathbf{I}$ has rank 1, and $\mathbf{B} - \mathbf{I}$ has rank 2, the matrices $\mathbf{A}$ and $\mathbf{B}$ cannot be similar.

Example 1.16 *Check if any of the following matrices are similar:*

$$\mathbf{A}=\begin{pmatrix}3&1&-3\\-4&-2&6\\-1&-1&5\end{pmatrix},\quad \mathbf{B}=\begin{pmatrix}-1&0&3\\2&2&-5\\-1&0&2\end{pmatrix},\quad \mathbf{C}=\begin{pmatrix}1&1&0\\0&1&1\\1&0&1\end{pmatrix},$$

$$\mathbf{D}=\begin{pmatrix}2&0&0\\1&1&1\\0&-1&3\end{pmatrix}.$$

A necessary (though not sufficient) condition of similarity is that the characteristic polynomials are identical. We therefore compute

$$\begin{aligned}\det(\mathbf{A}-\lambda\mathbf{I}) &= \begin{vmatrix}3-\lambda&1&-3\\-4&-2-\lambda&6\\-1&-1&5-\lambda\end{vmatrix} \underset{S_2:=S_2-S_1}{=} \begin{vmatrix}3-\lambda&\lambda-2&-3\\-4&2-\lambda&6\\-1&0&5-\lambda\end{vmatrix}\\
&= (2-\lambda)\begin{vmatrix}3-\lambda&-1&-3\\-4&1&6\\-1&0&5-\lambda\end{vmatrix} \underset{R_1:=R_1+R_2}{=}\\
&= -(\lambda-2)\begin{vmatrix}-(\lambda+1)&0&3\\-4&1&6\\-1&0&-(\lambda-5)\end{vmatrix} = -(\lambda-2)\begin{vmatrix}\lambda+1&-3\\1&\lambda-5\end{vmatrix}\\
&= -(\lambda-2)\left(\lambda^2-4\lambda+2\right),\end{aligned}$$

$$\begin{aligned}\det(\mathbf{B}-\lambda\mathbf{I}) &= \begin{vmatrix}-1-\lambda&0&3\\2&2-\lambda&-5\\-1&0&2-\lambda\end{vmatrix} = (2-\lambda)\begin{vmatrix}-(\lambda+1)&3\\-1&-(\lambda-2)\end{vmatrix}\\
&= -(\lambda-2)\begin{vmatrix}\lambda&-3\\1&\lambda-2\end{vmatrix} = -(\lambda-2)\left(\lambda^2-\lambda+1\right),\end{aligned}$$

$$\begin{aligned}\det(\mathbf{C}-\lambda\mathbf{I}) &= \begin{vmatrix}1-\lambda&1&0\\0&1-\lambda&1\\1&0&1-\lambda\end{vmatrix} \underset{R_2:=R_1+R_2+R_3}{=}\\
&\quad \begin{vmatrix}1-\lambda&1&0\\2-\lambda&2-\lambda&2-\lambda\\1&0&1-\lambda\end{vmatrix}\\
&= (2-\lambda)\begin{vmatrix}1-\lambda&1&0\\1&1&1\\1&0&1-\lambda\end{vmatrix} = -(\lambda-2)\begin{vmatrix}1-\lambda&1&0\\\lambda&0&1\\1&0&1-\lambda\end{vmatrix}\\
&= (\lambda-2)\begin{vmatrix}\lambda&1\\1&1-\lambda\end{vmatrix} = -(\lambda-2)\begin{vmatrix}\lambda&-1\\1&\lambda-1\end{vmatrix}\\
&= -(\lambda-2)\left(\lambda^2-\lambda+1\right),\end{aligned}$$

$$\begin{aligned}
\det(\mathbf{D}-\lambda\mathbf{I}) &= \begin{vmatrix} 2-\lambda & 0 & 0 \\ 1 & 1-\lambda & 1 \\ 0 & -1 & 3-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 1-\lambda & 1 \\ -1 & 3-\lambda \end{vmatrix} \\
&= -(\lambda-2)\begin{vmatrix} \lambda-1 & -1 \\ 1 & \lambda-3 \end{vmatrix} = -(\lambda-2)\left(\lambda^2-4\lambda+4\right) \\
&= -(\lambda-2)^3.
\end{aligned}$$

Summing up we have

$$\begin{aligned}
\det(\mathbf{A}-\lambda\mathbf{I}) &= -(\lambda-2)\left(\lambda^2-4\lambda+2\right), \\
\det(\mathbf{B}-\lambda\mathbf{I}) &= -(\lambda-2)\left(\lambda^2-\lambda+1\right), \\
\det(\mathbf{C}-\lambda\mathbf{I}) &= -(\lambda-2)\left(\lambda^2-\lambda+1\right), \\
\det(\mathbf{D}-\lambda\mathbf{I}) &= -(\lambda-2)^3.
\end{aligned}$$

The only possibility of similarity is between $\mathbf{B}$ and C, because they have the same characteristic polynomial. This has the complex eigenvalues

$$2, \qquad \frac{1}{2}+\frac{i}{2}\sqrt{3}, \qquad \frac{1}{2}-\frac{i}{2}\sqrt{3},$$

which each has the algebraic, hence also the geometric, multiplicity 1, both $\mathbf{B}$ and $\mathbf{C}$ can be diagonalized in the complex domain with the same diagonal matrix Δ, thus $\mathbf{B}$ and $\mathbf{C}$ must be similar.

Example 1.17 *Given the matrix*

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ b & c & 2 & 0 \\ b & c & 2 & 0 \\ \alpha & \beta & \gamma & 2 \end{pmatrix}.$$

Find a necessary and sufficient condition for that **A** *can be diagonalized.*

Clearly, the eigenvalues $\lambda = 1$ and $\lambda = 2$ have both the algebraic multiplicity 2.

If $\lambda = 1$, then

$$\mathbf{A} - \mathbf{I} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ c & c & 1 & 0 \\ \alpha & \beta & \gamma & 2 \end{pmatrix}.$$

The latter two rows are clearly linearly independent (they are both $\neq \mathbf{0}$ and the latter two coordinates can only be brought to disappear by trivial linear combinations) and the can not produce the second row, unless $a = 0$, thus the rank can only be 2, if $a = 0$.

Thus, a necessary condition is that $a = 0$. We therefore put $a = 0$ and then find for $\lambda = 2$ that

$$\mathbf{A} - 2\mathbf{I} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ b & c & 0 & 0 \\ \alpha & \beta & \gamma & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma & 0 \end{pmatrix}.$$

The rank is 2, if and only if $\gamma = 0$.

Hence, a necessary condition for that A can be diagonalized is that both $a = 0$ and $\gamma = 0$. Conversely, if both $a = 0$ and $\gamma = 0$, then the same argument as above shows that for the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ b & c & 2 & 0 \\ \alpha & \beta & 0 & 2 \end{pmatrix} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \begin{matrix} b & c \\ \alpha & \beta \end{matrix} & 2\mathbf{I} \end{pmatrix}$$

both eigenvalues, $\lambda = 1$ and $\lambda = 2$ have the geometric multiplicity 2, and the matrix can be diagonalized.

Hence, the necessary and sufficient condition for that **A** can be diagonalized is that $a = 0$ and $\gamma = 0$.

Example 1.18 *Given the matrix*

$$\mathbf{A} = \begin{pmatrix} 3 & -7 & 9 \\ 2 & -4 & 2 \\ -1 & 2 & -5 \end{pmatrix}.$$

1. *Prove that $\lambda = -2$ is a triple root of the characteristic polynomial.*
2. *Find all eigenvectors of $\mathbf{A}$, and explain why $\mathbf{A}$ cannot be diagonalized.*
3. *Given a linear map $f : \mathbb{R}^3 \to \mathbb{R}^3$ with respect to the usual basis by the matrix equation*

   $$\mathbf{v} = \mathbf{A}\mathbf{x}.$$

   We consider in $\mathbb{R}^3$ the subspace U, which is spanned by the vectors $\mathbf{u}_1 = (2, 0, -1)$ and $\mathbf{u}_2 = (0, 4, 3)$.
   Prove that $f(U) = U$.

1. The characteristic polynomial is

$$\begin{aligned}
\det((\mathbf{A} - \lambda\mathbf{I}) &= \begin{vmatrix} 3-\lambda & -7 & 9 \\ 2 & -4-\lambda & 2 \\ -1 & 2 & -5-\lambda \end{vmatrix} = \begin{vmatrix} 3-\lambda & -(\lambda+4) & \lambda+6 \\ 2 & -(\lambda+2) & 0 \\ -1 & 1 & -(\lambda+4) \end{vmatrix} \\
&= \begin{vmatrix} -(\lambda+4) & \lambda+6 \\ 1 & -(\lambda+4) \end{vmatrix} - (\lambda+2) \begin{vmatrix} 3-\lambda & \lambda+6 \\ -1 & -(\lambda+4) \end{vmatrix} \\
&= -2\left\{(\lambda+4)^2 - (\lambda+6)\right\} - (\lambda+2)\left\{(\lambda-3)(\lambda+4) + \lambda + 6\right\} \\
&= -2\left\{\lambda^2 + 7\lambda + 10\right\} - (\lambda+2)\left\{\lambda^2 + 2\lambda - 6\right\} \\
&= -2(\lambda+2)(\lambda+5) - (\lambda+2)\left\{\lambda^2 + 2\lambda - 6\right\} \\
&= -(\lambda+2)\left\{2\lambda + 10 + \lambda^2 + 2\lambda - 6\right\} \\
&= -(\lambda+2)\left\{\lambda^2 + 4\lambda + 4\right\} = -(\lambda+2)(\lambda+2)^2 \\
&= -(\lambda+2)^3.
\end{aligned}$$

   It follows that $\lambda = -2$ is a triple root of the characteristic polynomial, hence $\lambda = -2$ is the only eigenvalue (of algebraic multiplicity 3).

2. We get by reduction,

$$\begin{aligned}
\mathbf{A} + 2\mathbf{I} &= \begin{pmatrix} 5 & -7 & 9 \\ 2 & -2 & 2 \\ -1 & 2 & -3 \end{pmatrix} \begin{matrix} \sim \\ R_1 := -R_3 \\ R_2 := R_2 + 2R_3 \\ R_3 := R_1 + 5R_3 \end{matrix} \\
&\sim \begin{pmatrix} 1 & -2 & 3 \\ 0 & 2 & -4 \\ 0 & 3 & -6 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix},
\end{aligned}$$

   thus the rank is 2. This proves that $\lambda = -2$ has the geometric multiplicity $3 - 2 = 1 < 3 =$ the algebraic multiplicity. It follows that $\mathbf{A}$ cannot be diagonalized.

3. We compute

$$f(\mathbf{u}_1) = \begin{pmatrix} 3 & -7 & 9 \\ 2 & -4 & 2 \\ -1 & 2 & -5 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \\ 3 \end{pmatrix}$$

and

$$f(\mathbf{u}_2) = \begin{pmatrix} 3 & -7 & 9 \\ 2 & -4 & 2 \\ -1 & 2 & -5 \end{pmatrix} \begin{pmatrix} 0 \\ 4 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ -10 \\ -7 \end{pmatrix}.$$

We infer from

$$f(\mathbf{u}_1) = \begin{pmatrix} -3 \\ 2 \\ 3 \end{pmatrix} = -\frac{3}{2}\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 \\ 4 \\ 3 \end{pmatrix} = -\frac{3}{2}\mathbf{u}_1 + \frac{1}{2}\mathbf{u}_2$$

and

$$f(\mathbf{u}_2) = \begin{pmatrix} -1 \\ -10 \\ -7 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} - \frac{5}{2}\begin{pmatrix} 0 \\ 4 \\ 3 \end{pmatrix} = -\frac{1}{2}\mathbf{u}_1 - \frac{5}{2}\mathbf{u}_2,$$

that $f(\mathbf{u}_1), f(\mathbf{u}_2) \in U$, so $f(U) \subseteq U$.

Since

$$\begin{vmatrix} -\frac{3}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{5}{2} \end{vmatrix} = \frac{1}{4}\begin{vmatrix} -3 & 1 \\ -1 & -5 \end{vmatrix} = \frac{1}{4}(15+1) = 4 \neq 0,$$

it follows that $f(\mathbf{u}_1)$ and $f(\mathbf{u}_2)$ are linearly independent, so they span U. Finally, it follows by the linearity that $f(U) = U$.

Example 1.19 *Let a linear map $f : \mathbb{R}^3 \to \mathbb{R}^3$ be given by its matrix*

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & -3 & 3 \\ -1 & -1 & 0 \end{pmatrix}$$

with respect to the usual basis of $\mathbb{R}^3$.

1. *Find the eigenvalues and the corresponding eigenvectors of f. Explain why $\mathbf{A}$ is not similar to a diagonal matrix.*
2. *Let $\mathbf{b}_1 = (-1, 1, 1)$. Find all vectors $\mathbf{b}_2 \in \mathbb{R}^3$, which satisfy the equation*

   $$f(\mathbf{b}_2) = \mathbf{b}_1 - \mathbf{b}_2.$$

3. *Prove that there exists a basis $(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$ of $\mathbb{R}^3$, such that f in this basis has the matrix*

   $$\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix},$$

   and find such a basis.
   *(*HINT*: Use as the first two basis vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ from 2.)*

1. We compute the characteristic polynomial,

$$\begin{aligned} \det(\mathbf{A} - \lambda\mathbf{I}) &= \begin{vmatrix} -\lambda & 1 & -1 \\ -1 & -(\lambda+3) & 3 \\ -1 & -1 & -\lambda \end{vmatrix} \\ &= -\lambda^2(\lambda+3) - 3 - 1 + (\lambda+3) - \lambda - 3\lambda \\ &= (\lambda+3)(1-\lambda^2) - 4\lambda - 4 \\ &= -(\lambda+1)\{(\lambda-1)(\lambda+3)+4\} \\ &= -(\lambda+1)\{\lambda^2+2+1\} = -(\lambda+1)^2, \end{aligned}$$

hence $\lambda = -1$ is the only eigenvalue (of algebraic multiplicity 3).

We infer from the reduction

$$\mathbf{A} + \mathbf{I} = \begin{pmatrix} 1 & 1 & -1 \\ -1 & -2 & 3 \\ -1 & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

that the eigenspace has the dimension 1, thus $\lambda = -1$ has the geometric multiplicity $1 < 3 =$ the algebraic multiplicity. This shows that $\mathbf{A}$ is not similar to a diagonal matrix.

An eigenvector corresponding to $\lambda = -1$ must satisfy

$$x_1 + x_3 = 0 \quad \text{and} \quad -x_2 + 2x_3 = 0,$$

e.g. $(-1, 2, 1) = \mathbf{b}_1$.

2. The equation $f(\mathbf{b}_2) = \mathbf{b}_1 - \mathbf{b}_2$ is of course equivalent to the equation

$$(\mathbf{A} + \mathbf{I})\mathbf{b}_2 = \mathbf{b}_1,$$

the corresponding homogeneous equation of which has the solutions $k \cdot \mathbf{b}_1$.

We are only missing a particular solution, so we reduce,

$$\begin{aligned}(\mathbf{A} + \mathbf{I} \mid \mathbf{b}_1) &= \left(\begin{array}{rrr|r} 1 & 1 & -1 & -1 \\ -1 & -2 & 3 & 2 \\ -1 & -1 & 1 & 1 \end{array}\right) \\ &\sim \left(\begin{array}{rrr|r} 1 & 1 & -1 & -1 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{rrr|r} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right).\end{aligned}$$

One particular solution is (of course) $\mathbf{b}_2 = (0, -1, 0)$, thus the complete solution is

$$\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} + k \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \qquad k \in \mathbb{R}.$$

3. Choosing $\mathbf{b}_1 = (-1, 2, 1)$ and $\mathbf{b}_2 = (0, -1, 0)$, we get

$$f(\mathbf{b}_1) = -\mathbf{b}_1 \quad \text{and} \quad f(\mathbf{b}_2) = \mathbf{b}_1 - \mathbf{b}_2,$$

which is taking care of the first two columns of the matrix.

Finally, we shall choose $\mathbf{b}_3$, such that

$$f(\mathbf{b}_3) = \mathbf{b}_2 - \mathbf{b}_3,$$

i.e. $(\mathbf{A} + \mathbf{I})\mathbf{b}_3 = \mathbf{b}_2$. We reduce

$$\begin{aligned}(\mathbf{A} + \mathbf{I} \mid \mathbf{b}_2) &= \left(\begin{array}{rrr|r} 1 & 1 & -1 & 0 \\ -1 & -2 & 3 & -1 \\ -1 & -1 & 1 & 0 \end{array}\right) \\ &= \left(\begin{array}{rrr|r} 1 & 1 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{rrr|r} 1 & 0 & 1 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right).\end{aligned}$$

Hence

$$x_1 + x_3 = -1 \quad \text{and} \quad x_2 - 2x_3 = 1.$$

If we choose $x_2 = 1$, then $x_3 = 0$ and $x_1 = -1$, thus

$$\mathbf{b}_3 = (-1, 1, 0).$$

CHECK:

$$(\mathbf{A}+\mathbf{I})\mathbf{b}_3 = \left(\begin{array}{rrr|r} 1 & 1 & -1 & -1 \\ -1 & -2 & 3 & 1 \\ -1 & -1 & 1 & 0 \end{array}\right) = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} = \mathbf{b}_2,$$

and $f(\mathbf{b}_3) = \mathbf{b}_2 - \mathbf{b}_3$ as required. $\Diamond$

We may choose the basis

$$\mathbf{b}_1 = (-1, 2, 1), \quad \mathbf{b}_2 = (0, -1, 0), \quad \mathbf{b}_3 = (-1, 1, 0).$$

Example 1.20 *Let f denote the linear map of $\mathbb{R}^3$ into $\mathbb{R}^3$, which in the usual basis of $\mathbb{R}^3$ is given by the matrix*

$$\mathbf{A} = \begin{pmatrix} 5 & 1 & -1 \\ -4 & 1 & 2 \\ 4 & 0 & -1 \end{pmatrix}.$$

1. *Find all eigenvalues and the corresponding eigenvectors of f.*
2. *Does there exist another basis of $\mathbb{R}^3$, such that the matrix of f with respect to this new basis is a diagonal matrix?*
3. *Let $\mathbf{v}_1 = (1, -1, 1)$ and $\mathbf{v}_2 = (1, -2, 2)$. Find a vector $\mathbf{v}_3$, such that*

   $$f(\mathbf{v}_3) = \mathbf{v}_2 + \mathbf{v}_3,$$

   and prove that $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ form a basis of $\mathbb{R}^3$.
4. *Find the matrix of f with respect to the basis $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$.*

1. We first compute

$$\begin{aligned}
\det(\mathbf{A}-\lambda\mathbf{I}) &= \begin{vmatrix} 5-\lambda & 1 & -1 \\ -4 & 1-\lambda & 2 \\ 4 & 0 & -1-\lambda \end{vmatrix} \\
&= 4\begin{vmatrix} 1 & -1 \\ 1-\lambda & 2 \end{vmatrix} - (\lambda+1)\begin{vmatrix} 5-\lambda & 1 \\ -4 & 1-\lambda \end{vmatrix} \\
&= 4\{2+1-\lambda\} - (\lambda+1)\{(\lambda-1)(\lambda-5)+4\} \\
&= -4(\lambda-3) - (\lambda+1)\{\lambda^2-6\lambda+9\} \\
&= -4(\lambda-3) - (\lambda+1)(\lambda-3)^2 \\
&= -(\lambda-3)\{(\lambda+1)(\lambda-3)+4\} = -(\lambda-3)\{\lambda^2-2\lambda+1\} \\
&= -(\lambda-1)^2(\lambda-3).
\end{aligned}$$

We infer that the eigenvalues are $\lambda = 1$ (of multiplicity 2) and $\lambda = 3$.

If $\lambda = 1$, then by reduction

$$\begin{aligned}\mathbf{A} - \lambda\mathbf{I} &= \mathbf{A} - \mathbf{I} = \begin{pmatrix} 4 & 1 & -1 \\ -4 & 0 & 2 \\ 4 & 0 & -2 \end{pmatrix} \sim \begin{pmatrix} 4 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \\ &\sim \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.\end{aligned}$$

The rank is 2, hence the eigenspace has the dimension $3-2 = 1$. An eigenvector is $(1, -2, 2) = \mathbf{v}_2$.

If $\lambda = 3$, then by reduction

$$\begin{aligned}\mathbf{A} - \lambda\mathbf{I} &= \mathbf{A} - 3\mathbf{I} = \begin{pmatrix} 2 & 1 & -1 \\ -4 & -2 & 2 \\ 4 & 0 & -4 \end{pmatrix} \sim \begin{pmatrix} 2 & 1 & -1 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \\ &\sim \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}.\end{aligned}$$

One eigenvector is e.g. $(1, -1, 1) = \mathbf{v}_1$.

2. Now, $\lambda = 1$ has the algebraic multiplicity 2 and the geometric multiplicity 1. Hence there does not exist another basis, such that the matrix of f is a diagonal matrix.

3. We now write the equation as $(\mathbf{A} - \mathbf{I})\mathbf{v}_3 = \mathbf{v}_2$. Then the corresponding homogeneous equation has the solution $k \cdot \mathbf{v}_2$, so we just have to find a particular solution. We reduce

$$\begin{aligned}(\mathbf{A} - \mathbf{I} \mid \mathbf{v}_2) &= \left(\begin{array}{rrr|r} 4 & 1 & -1 & 1 \\ -4 & 0 & 2 & -2 \\ 4 & 0 & -2 & 2 \end{array}\right) \sim \left(\begin{array}{rrr|r} 4 & 1 & -1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right) \\ &\sim \left(\begin{array}{rrr|r} 4 & 0 & -2 & -2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right).\end{aligned}$$

We see that we can choose $\mathbf{v}_3 = (0, 0, -1)$.

It follows from

$$\begin{aligned}\det\{\mathbf{v}_1 \;\; \mathbf{v}_2 \;\; \mathbf{v}_3\} &= \begin{vmatrix} 1 & 1 & 0 \\ -1 & -2 & 0 \\ 1 & 2 & -1 \end{vmatrix} = - \begin{vmatrix} 1 & 1 \\ -1 & -2 \end{vmatrix} \\ &= \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1 \neq 0,\end{aligned}$$

that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ form a basis of $\mathbb{R}^3$.

4. Then it follows from

$$f(\mathbf{v}_1) = 3\mathbf{v}_1, \quad f(\mathbf{v}_2) = \mathbf{v}_2, \quad f(\mathbf{v}_3) = \mathbf{v}_2 + \mathbf{v}_3,$$

that the matrix of the map with respect to the basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is given by

$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Example 1.21 *Given the matrices*

$$\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \textit{and} \quad \mathbf{A} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a & b & 0 & 0 \\ c & d & 0 & 0 \end{pmatrix},$$

where a, b, c, d are constants.

1. *Prove that if $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{C}^{2\times 1}$ is an eigenvector of $\mathbf{M}$ corresponding to the eigenvalue λ, and if μ satisfies $\mu^2 = \lambda$, then $\begin{pmatrix} x \\ y \\ \mu x \\ \mu y \end{pmatrix} \in \mathbb{C}^{4\times 1}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue μ.*

2. *In particular, let*

$$\mathbf{M} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix},$$

and let $f : \mathbb{C}^4 \to \mathbb{C}^4$ be the linear map, which with respect to the usual basis is given by the matrix $\mathbf{A}$, corresponding to $\mathbf{M}$. Find the eigenvalues and the eigenvectors of f.

3. *Find a basis $(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4)$ of $\mathbb{C}^4$, such that the matrix of f with respect to this basis is a diagonal matrix, and find this diagonal matrix.*

1. When we put $\mathbf{z} = \begin{pmatrix} x \\ y \\ \mu x \\ \mu y \end{pmatrix}$, then

$$\mathbf{A}\mathbf{z} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a & b & 0 & 0 \\ c & d & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ \mu x \\ \mu y \end{pmatrix} = \begin{pmatrix} \mu x \\ \mu y \\ \lambda x \\ \lambda y \end{pmatrix} = \mu \begin{pmatrix} x \\ y \\ \mu x \\ \mu y \end{pmatrix} = \mu \mathbf{z},$$

and the claim is proved.

2. The characteristic polynomial of $\mathbf{M}$ is

$$\det(\mathbf{M}-\lambda\mathbf{I}) = \begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = (\lambda-1)^2 - 2^2 = (\lambda+1)(\lambda-3),$$

thus the eigenvalues are $\lambda = -1$ and $\lambda = 3$.

If $\lambda = -1$, then

$$\mathbf{M}-\lambda\mathbf{I} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix},$$

hence an eigenvector is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

If $\lambda = 3$, then

$$\mathbf{M}-\lambda\mathbf{I} = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix},$$

hence an eigenvector is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

We now apply 1. We get from $i^2 = -1$ that $\mu = \pm i$. Then it follows that $\mu = i$ is an eigenvalue with the eigenvector $\begin{pmatrix} 1 \\ -1 \\ i \\ -i \end{pmatrix}$, and $\mu = -i$ is an eigenvalue with the eigenvector $\begin{pmatrix} 1 \\ -1 \\ -i \\ i \end{pmatrix}$.

Analogously, $\mu = \sqrt{3}$ is an eigenvalue with the eigenvector $\begin{pmatrix} 1 \\ 1 \\ \sqrt{3} \\ \sqrt{3} \end{pmatrix}$, and $\mu = -\sqrt{3}$ is an eigenvalue with the eigenvector $\begin{pmatrix} 1 \\ 1 \\ -\sqrt{3} \\ -\sqrt{3} \end{pmatrix}$.

3. If we choose

$$\begin{array}{ll} \mathbf{b}_1 = (1,-1,i,-i), & f(\mathbf{b}_1) = i\,\mathbf{b}_1, \\ \mathbf{b}_2 = (1,-1,-i,i), & f(\mathbf{b}_2) = -i\,\mathbf{b}_2, \\ \mathbf{b}_3 = (1,1,\sqrt{3},\sqrt{3}), & f(\mathbf{b}_3) = \sqrt{3}\,\mathbf{b}_3, \\ \mathbf{b}_4 = (1,1,-\sqrt{3},-\sqrt{3}, & f(\mathbf{b}_4) = -\sqrt{3}\,\mathbf{b}_4, \end{array}$$

then the matrix of the map with respect to this basis becomes the following diagonal matrix

$$\begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \\ 0 & 0 & 0 & -\sqrt{3} \end{pmatrix}.$$

Example 1.22 *Let (r, s) denote a pair of numbers, where s is bigger than r, and r is one of the numbers 1, 2, 3, and s is one of the numbers 2, 3, 4. Let $\mathbf{A}$ denote the matrix, which we get when we in the matrix*

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

replace the zero of row number r and of column number s by the cypher 1. For which (r, s) can $\mathbf{A}$ be diagonalized?

It $s \in \{2, 3\}$, then it is easily seen that $\lambda = 1$ has algebraic multiplicity 3 and geometric multiplicity 2, hence $\mathbf{A}$ cannot be diagonalized in this case.

If $s = 4$, then we reduce

$$\mathbf{A} - \mathbf{I} = \begin{pmatrix} 0 & 0 & 0 & \star \\ 0 & 0 & 0 & \star \\ 0 & 0 & 0 & \star \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where the $\star$ denotes that we on such a place may have either 0 or 1 and that precisely one of the $\star$ takes on the value 1. Then $\mathbf{A} - \mathbf{I}$ has rank 1, and the geometric multiplicity $= 3 =$ the algebraic multiplicity, i.e. $\mathbf{A}$ can be diagonalized if $s = 4$ and $r \in \{1, 2, 3\}$.

Example 1.23 *Given the matrices*

$$\mathbf{A} = \begin{pmatrix} 7 & -2 & 2 \\ 1 & 4 & 2 \\ -1 & 2 & 4 \end{pmatrix}, \quad \mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \quad \textit{and} \quad \mathbf{v}_2 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}.$$

1. *Prove that $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent eigenvectors of $\mathbf{A}$.*

2. *Find all the eigenvalues and all the corresponding eigenvectors of $\mathbf{A}$.*

3. *Find a regular matrix $\mathbf{V}$ and a diagonal matrix Λ, such that*

$$\mathbf{V}^{-1}\mathbf{A}\mathbf{V} = \mathbf{\Lambda}.$$

1. Clearly, $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent. Then

$$\mathbf{A}\mathbf{v}_1 = \begin{pmatrix} 7 & -2 & 2 \\ 1 & 4 & 2 \\ -1 & 2 & 4 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 12 \\ 6 \\ 0 \end{pmatrix} = 6 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = 6\mathbf{v}_1,$$

and

$$\mathbf{A}\mathbf{v}_2 = \begin{pmatrix} 7 & -2 & 2 \\ 1 & 4 & 2 \\ -1 & 2 & 4 \end{pmatrix} \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -12 \\ 0 \\ 6 \end{pmatrix} = 6 \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} = 6\mathbf{v}_2,$$

hence $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent eigenvectors corresponding to the same eigenvalue $\lambda = 6$.

2. When we compute the characteristic polynomial we can apply that we already know that $(\lambda-6)^2$ is a factor of this polynomial,

$$\begin{aligned}
\det(\mathbf{A}-\lambda\mathbf{I}) &= \begin{vmatrix} 7-\lambda & -2 & 2 \\ 1 & 4-\lambda & 2 \\ -1 & 2 & 4-\lambda \end{vmatrix} = \begin{vmatrix} 6-\lambda & 0 & 6-\lambda \\ 0 & 6-\lambda & 6-\lambda \\ -1 & 2 & 4-\lambda \end{vmatrix} \\
&= (6-\lambda)^2 \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & 2 & 4-\lambda \end{vmatrix} = (\lambda-6)^2 \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 5-\lambda \end{vmatrix} \\
&= (\lambda-6)^2\{5-\lambda-2\} = (\lambda-3)(\lambda-6)^2.
\end{aligned}$$

The eigenvalue $\lambda = 6$ has according to 1) the two linearly independent eigenvectors $\mathbf{v}_1 = (2,1,0)$ and $\mathbf{v}_2 = (-2,0,1)$.

If $\lambda = 3$, we reduce

$$\begin{aligned}
\mathbf{A}-\lambda\mathbf{I} &= \begin{pmatrix} 4 & -2 & 2 \\ 1 & 1 & 2 \\ -1 & 2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 2 \\ 0 & 3 & 3 \\ 0 & 6 & 6 \end{pmatrix} \\
&\sim \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.
\end{aligned}$$

We may choose the eigenvector $(1,1,-1) = \mathbf{v}_3$.

3. Since $\mathbf{\Lambda} = \operatorname{diag}(6,6,3)$, and $\mathbf{V}$, then
$$\mathbf{V} = (\mathbf{v}_1\ \ \mathbf{v}_2\ \ \mathbf{v}_3) = \begin{pmatrix} 2 & -2 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}.$$

Remark 1.2 For the sake of completeness we add the reduction
$$\begin{aligned}(\mathbf{V} \mid \mathbf{I}) &= \left(\begin{array}{rrr|rrr} 2 & -2 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{rrr|rrr} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & -2 & -1 & 1 & -2 & 0 \end{array}\right) \\ &\sim \left(\begin{array}{rrr|rrr} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & -3 & 1 & -2 & 2 \end{array}\right) \sim \left(\begin{array}{rrr|rrr} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -\frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{array}\right) \\ &\sim \left(\begin{array}{rrr|rrr} 1 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{2}{3} \\ 0 & 1 & 0 & -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{array}\right),\end{aligned}$$
thus
$$\mathbf{V}^{-1} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{pmatrix}. \qquad \Diamond$$

Example 1.24 *Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map, which in the usual basis of $\mathbb{R}^3$ has the matrix*
$$\mathbf{A} = \begin{pmatrix} 3 & -1 & 1 \\ 7 & -5 & 1 \\ 6 & -6 & 2 \end{pmatrix}.$$

1. *Prove that $\mathbf{v}_1 = (0,1,1)$ is an eigenvalue of f, and check if 2 is an eigenvalue of f.*
2. *Prove that $\mathbf{v}_1 = (0,1,1)$, $\mathbf{v}_2 = (1,1,0)$ and $\mathbf{e}_3 = (0,0,1)$ form a basis of $\mathbb{R}^3$, and find the matrix of f with respect to the basis $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_2)$.*

1. By insertion,
$$\mathbf{A}\mathbf{v}_1 = \begin{pmatrix} 3 & -1 & 1 \\ 7 & -5 & 1 \\ 6 & -6 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -4 \\ -4 \end{pmatrix} = -4\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = -4\mathbf{v}_1,$$
hence $\mathbf{v}_1$ is an eigenvector corresponding to the eigenvalue $\lambda = -4$.

Then by reduction,
$$\mathbf{A} - 2\mathbf{I} = \begin{pmatrix} 1 & -1 & 1 \\ 7 & -7 & 1 \\ 6 & -6 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & -1 & 0 \end{pmatrix}$$
which is of rank 2. Hence $\lambda = 2$ is an eigenvalue, and a corresponding eigenvector is $(1,1,0) = \mathbf{v}_2$.

2. From

$$\alpha\mathbf{v}_1+\beta\mathbf{v}_2+\gamma\mathbf{e}_3=\alpha\begin{pmatrix}0\\1\\1\end{pmatrix}+\beta\begin{pmatrix}1\\1\\0\end{pmatrix}+\gamma\begin{pmatrix}0\\0\\1\end{pmatrix}=\mathbf{0},$$

follows successively that $\beta=0$, $\alpha=0$ and $\gamma=0$, and $(\mathbf{v}_1,\mathbf{v}_2,\mathbf{e}_3)$ form a basis of $\mathbb{R}^3$.

It follows from

$$f(\mathbf{e}_3)=\begin{pmatrix}3&-1&1\\7&-5&1\\6&-6&2\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}1\\1\\2\end{pmatrix}=\begin{pmatrix}1\\1\\0\end{pmatrix}+2\begin{pmatrix}0\\0\\1\end{pmatrix}=\mathbf{v}_2+2\mathbf{e}_3,$$

that the matrix of the map with respect to $(\mathbf{v}_1,\mathbf{v}_2,\mathbf{e}_3)$ is

$$\begin{pmatrix}-4&0&0\\0&2&1\\0&0&2\end{pmatrix}.$$

Example 1.25 *Given the matrix*

$$\mathbf{A}=\begin{pmatrix}1&0&0\\1&1&1\\1&0&2\end{pmatrix}.$$

1. *Find all eigenvalues and all corresponding eigenvectors of* $\mathbf{A}$.
2. *Find a regular matrix* $\mathbf{V}$ *and a diagonal matrix* $\mathbf{\Lambda}$, *such that*

$$\mathbf{V}^{-1}\mathbf{\Lambda}\mathbf{V}=\mathbf{A}.$$

1. If we expand after the first row, then

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}1-\lambda&0&0\\1&1-\lambda&1\\1&0&2-\lambda\end{vmatrix}=(1-\lambda)^2(2-\lambda),$$

and the eigenvalues are $\lambda=1$ (multiplicity 2) and $\lambda=2$.

If $\lambda=1$, then we get by reduction

$$\mathbf{A}-\lambda\mathbf{I}=\mathbf{A}-\mathbf{I}=\begin{pmatrix}0&0&0\\1&0&1\\1&0&1\end{pmatrix}\sim\begin{pmatrix}1&0&1\\0&0&0\\0&0&0\end{pmatrix}$$

of rank 1, so the eigenspace has dimension $3-1=2$ with the linearly independent eigenvectors $\mathbf{v}_1=(1,0,-1)$ and $\mathbf{v}_2=(0,1,0)$.

If $\lambda=2$, then we get by reduction

$$\mathbf{A}-\lambda\mathbf{I}=\mathbf{A}-2\mathbf{I}=\begin{pmatrix}-1&0&0\\1&-1&1\\1&0&0\end{pmatrix}\sim\begin{pmatrix}1&0&0\\0&1&-1\\0&0&0\end{pmatrix}$$

of rank 2, hence the eigenspace has dimension $3-2=1$, and an eigenvector is $\mathbf{v}_3=(0,1,1)$.

2. It is immediately seen that

$$\mathbf{V}=(\mathbf{v}_1\;\;\mathbf{v}_2\;\;\mathbf{v}_3)=\begin{pmatrix}1&0&0\\0&1&1\\-1&0&1\end{pmatrix}\quad\text{and}\quad\mathbf{\Lambda}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&2\end{pmatrix}.$$

Example 1.26 *The linear map* $f : \mathbb{R}^3 \to \mathbb{R}^3$ *is in the usual basis of* $\mathbb{R}^3$ *given by the matrix*

$$\mathbf{A} = \begin{pmatrix} 16 & -13 & -2 \\ 18 & -15 & -2 \\ -24 & 24 & 4 \end{pmatrix}.$$

1. *Compute* tr $\mathbf{A}$.
2. *Prove that* $(1, 2, -4)$ *is an eigenvector of* f*, and find the corresponding eigenvalue.*
3. *Prove that 3 is an eigenvalue of* f*, and find a proper corresponding eigenvector.*
4. *Explain why* $\mathbf{A}$ *can be diagonalized and find a diagonal matrix, which is similar to* $\mathbf{A}$.

1. The trace is equal to the sum of the diagonal elements,

$$\text{spor } \mathbf{A} = 16 - 15 + 4 = 5.$$

2. We get by putting $\mathbf{v}_1 = (1, 2, -4)$,

$$\begin{aligned} \mathbf{A}\mathbf{v}_1 &= \begin{pmatrix} 16 & -13 & -2 \\ 18 & -15 & -2 \\ -24 & 24 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} = \begin{pmatrix} 16-26+8 \\ 18-30+8 \\ -24+48-16 \end{pmatrix} \\ &= \begin{pmatrix} -2 \\ -4 \\ 8 \end{pmatrix} = -2 \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} = -2\,\mathbf{v}_1, \end{aligned}$$

hence $\mathbf{v}_1$ is an eigenvector corresponding to the eigenvalue -2.

3. We reduce

$$\mathbf{A} - \lambda\mathbf{I} = \mathbf{A} - 3\mathbf{I} = \begin{pmatrix} 13 & -13 & -2 \\ 18 & -18 & -2 \\ -24 & 24 & 1 \end{pmatrix} \sim \begin{pmatrix} -35 & 35 & 0 \\ -30 & 30 & 0 \\ -24 & 24 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

of rank 2, thus the eigenspace has dimension $3 - 2 = 1$, and an eigenvector is e.g. $\mathbf{v}_2 = (1, 1, 0)$.

4. Using that $\lambda_1 + \lambda_2 + \lambda_3 = \text{ tr } \mathbf{A}$, we get

$$\lambda_3 = \text{ tr } \mathbf{A} - \lambda_1 - \lambda_2 = 5 - (-2) - 3 = 4.$$

The three different eigenvalues are all of multiplicity 1, hence $\mathbf{A}$ is similar to a diagonal matrix

$$\Lambda = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

ADDITION. For the sake of completeness we reduce

$$\mathbf{A} - \lambda_3\mathbf{I} = \mathbf{A} - 4\mathbf{I} = \begin{pmatrix} 12 & -13 & -2 \\ 18 & -19 & -2 \\ -24 & 24 & 0 \end{pmatrix} \sim \begin{pmatrix} -12 & 13 & 2 \\ 6 & -6 & 0 \\ 1 & -1 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 2 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

of rank 2, hence the eigenspace has dimension $3-2=1$, and an eigenvector is e.g. $\mathbf{v}_3=(2,2,-1)$, and

$$\mathbf{V}=(\mathbf{v}_1\ \ \mathbf{v}_2\ \ \mathbf{v}_3)=\begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 2 \\ -4 & 0 & -1 \end{pmatrix}.$$

Example 1.27 *Given the matrix*

$$\mathbf{A}=\begin{pmatrix} -2 & -5 & -3 \\ -2a-4 & -5a-2 & -3a-2 \\ 4a+8 & 10a+6 & 6a+5 \end{pmatrix},$$

where a is any real number.

1. *Find the trace of* $\mathbf{A}$.
2. *Prove that 0 is an eigenvalue of* $\mathbf{A}$.
3. *It is given that 1 is an eigenvalue of* $\mathbf{A}$.
   Prove that a is an eigenvalue of $\mathbf{A}$.
4. *Find all $a\in\mathbb{R}$, for which* $\mathbf{A}$ *can be diagonalized.*

1. The trace is

$$\lambda_1+\lambda_2+\lambda_3=\ \text{tr}\ \mathbf{A}=-2-5a-2+6a+5=a+1.$$

2. We reduce

$$\begin{aligned}\mathbf{A} &= \begin{pmatrix} -2 & -5 & -3 \\ -2a-4 & -5a-2 & -3a-2 \\ 4a+8 & 10a+6 & 6a+5 \end{pmatrix} \\ &\sim \begin{pmatrix} -2 & -5 & -3 \\ -2a-4 & -5a-2 & -3a-2 \\ 0 & 2 & 1 \end{pmatrix} \sim \begin{pmatrix} -2 & -5 & -3 \\ -4 & -2 & -2 \\ 0 & 2 & 1 \end{pmatrix} \\ &\sim \begin{pmatrix} -2 & 1 & 0 \\ -4 & 2 & 0 \\ 0 & 2 & 1 \end{pmatrix} \sim \begin{pmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}.\end{aligned}$$

The rank is 2, so the eigenspace has dimension $3-2=1$, and an eigenvector is e.g. $\mathbf{v}_1=(1,2,-4)$, and $\lambda_1=0$.

3. **Remark 1.3** We first check that $\lambda_2=1$ is an eigenvalue. Then by a reduction,

$$\mathbf{A}-\mathbf{I}=\begin{pmatrix} -3 & -5 & -3 \\ -2a-4 & -5a-3 & -3a-2 \\ 4a+8 & 10a+6 & 6a+4 \end{pmatrix} \sim \begin{pmatrix} -3 & -5 & -3 \\ a-4 & -3 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

of rank 2, hence $\lambda_2 = 1$ is an eigenvalue. It follows that

$$\begin{vmatrix} -5 & -3 \\ -3 & -2 \end{vmatrix} = 1 \neq 0,$$

hence the rank is 2 for every choice of a, i.e. the eigenspace has always dimension 1. $\Diamond$

If $\lambda_2 = 1$ is an eigenvalue, then it follows from 1) that

$$\lambda_3 = \text{ tr } \mathbf{A} - \lambda_1 - \lambda_2 = (a+1) - 0 - 1 = a,$$

and the last eigenvalue is $\lambda_3 = a$.

4. If $a = 0$ or $a = 1$, then a is a double root in the characteristic polynomial. According to the remark above, the geometric multiplicity is always 1 for $\lambda = 1$, and since the geometric multiplicity also is 1 for $\lambda = 0$ by the reduction of 1), we conclude that the matrix cannot be diagonalized for $a = 0$ or for $a = 1$.

   On the other hand, if $a \neq 0$ and $a \neq 1$, then the three eigenvalues $\{0, 1, a\}$ are all different and all of multiplicity 1, and we can diagonalize the matrix in this case.

Example 1.28 *Given the real matrices*

$$\mathbf{A} = \begin{pmatrix} 1 & a-1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \textit{and} \quad \mathbf{B} = \begin{pmatrix} 1 & 2 & -2 \\ 0 & \sqrt{a} & 0 \\ 0 & 0 & -\sqrt{a} \end{pmatrix}, \quad \textit{where } a \geq 0.$$

1. *Find for every $a \geq 0$ all eigenvalues of $\mathbf{A}$. Indicate for each of the eigenvalues its algebraic and geometric multiplicity .*
2. *For which $a \geq 0$ is $\mathbf{A}$ similar to a diagonal matrix? Find for each of these a a diagonal matrix, which is similar to $\mathbf{A}$.*
3. *Prove that $\mathbf{A}$ and $\mathbf{B}$ have the same characteristic polynomial.*
4. *Find all $a \geq 0$, for which $\mathbf{A}$ is similar to $\mathbf{B}$.*

1. We first compute the characteristic polynomial,

$$\begin{aligned} \det(\mathbf{A}-\lambda\mathbf{I}) &= \begin{vmatrix} 1-\lambda & a-1 & 0 \\ 1 & -1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 1-\lambda & a-1 \\ 1 & -1-\lambda \end{vmatrix} \\ &= -(\lambda-1)\{\lambda^2-1-a+1\} = -(\lambda-1)\{\lambda^2-a\} \\ &= -(\lambda-1)(\lambda-\sqrt{a})(\lambda+\sqrt{a}). \end{aligned}$$

The eigenvalues are $\{1, \sqrt{a}, -\sqrt{a}\}$.

If $a \notin \{0,1\}$, then both the algebraic and the geometric multiplicity are 1 for each of the eigenvalues. In particular, $\mathbf{A}$ can be diagonalized, if $a \notin \{0,1\}$.

If $a = 0$, then the eigenvalue $\lambda = 0$ has algebraic multiplicity 2.
It follows from the reduction

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

that the rank is 2, hence the geometric multiplicity is 1. Since the algebraic and the geometric multiplicity are not identical, on cannot diagonalize $\mathbf{A}$ for $a = 0$.

If $a = 1$, then $\lambda = 1$ is an eigenvalue of algebraic multiplicity 2. It follows from the reduction

$$\mathbf{A}-\mathbf{I} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

that the rank is 1, hence the geometric multiplicity is 2. Thus we can in this case diagonalize $\mathbf{A}$.

Summing up we get

(a) $a > 0$, $a \neq 1$.
The eigenvalues are $\{1, \sqrt{a}, -\sqrt{a}\}$, are all of algebraic and geometric multiplicity 1.

(b) $a = 0$.
The eigenvalues are $\lambda_1 = 1$ of algebraic and geometric multiplicity 1, and $\lambda_2 = 0$ of algebraic multiplicity 2 and geometric multiplicity 1.

(c) $a = 1$.
The eigenvalues are $\lambda_1 = -1$ of algebraic and geometric multiplicity 1, and $\lambda_2 = 1$ of algebraic and geometric multiplicity 2.

2. Since the algebraic and geometric multiplicities are identical for $a > 0$, and not for $a = 0$, the matrix is similar to a diagonal matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{a} & 0 \\ 0 & 0 & -\sqrt{a} \end{pmatrix},$$

if and only if $a > 0$.

3. Now, $\mathbf{B}$ is an upper triangular matrix, hence the characteristic polynomial of $\mathbf{B}$ is

$$\det(\mathbf{B} - \lambda\mathbf{I}) = -(\lambda - 1)(\lambda - \sqrt{a})(\lambda + \sqrt{a}) = \det(\mathbf{A} - \lambda\mathbf{I}),$$

and the claim is proved.

4. If $a \notin \{0, 1\}$, then $\mathbf{B}$ can be diagonalized, so $\mathbf{B}$ is similar to the matrix of 2), and thus also similar to $\mathbf{A}$ for $a \notin \{0, 1\}$.

If $a = 0$, then

$$\mathbf{B} = \begin{pmatrix} 1 & 2 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

of rank 2. Since $\mathbf{A}$ for $a = 0$ has rank 1, the two matrices are not similar for $a = 0$.

If $a = 1$ and $\lambda = 1$, then

$$\mathbf{B} - \lambda\mathbf{I} = \begin{pmatrix} 0 & 2 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

of rank 2, and since $\mathbf{A} - \lambda\mathbf{I}$ for $a = 1$ and $\lambda = 1$ has rank 1, they are not similar for $a = 1$.

The matrices $\mathbf{A}$ and $\mathbf{B}$ are similar for $a > 0$, $a \neq 1$.

Example 1.29 *Given for every $a \in \mathbb{R}$ the matrix*

$$\mathbf{A} = \begin{pmatrix} 2a+1 & -1-1 \\ 2a+2 & -a-2 \end{pmatrix}.$$

1. *Prove that a is an eigenvalue of $\mathbf{A}$.*
2. *Find all eigenvalues and eigenvectors of $\mathbf{A}$.*
3. *Explain why $\mathbf{A}$ can be diagonalized for every $a \in \mathbb{R}$, and find a diagonal matrix Λ and a regular matrix $\mathbf{V}$, such that $\mathbf{\Lambda} = \mathbf{V}^{-1}\mathbf{AV}$.*
4. *Find first Λ^{14}, and then $\mathbf{A}^{14}$.*

1. We get by a reduction,

$$\mathbf{A} - a\mathbf{I} = \begin{pmatrix} a+1 & -a-1 \\ 2a+2 & -2a-2 \end{pmatrix} \sim \begin{pmatrix} a+1 & -(a+1) \\ 0 & 0 \end{pmatrix}$$

of rank *at most* $1 < 2$, hence a is an eigenvalue of $\mathbf{A}$.

2. Since

$$\lambda_1 + \lambda_2 = a + \lambda_2 = \text{ tr } \mathbf{A} = (2a+1) - a - 2 = a - 1,$$

the second eigenvalue is $\lambda_2 = -1$.

If $a \neq -1$, then we have the two simple eigenvalues $\{a, -1\}$, where an eigenvector corresponding to a is $(1, 1)$.

If $\lambda_2 = -1$, we have the reduction

$$\mathbf{A} - \lambda_2\mathbf{I} = \begin{pmatrix} 2a+2 & -a-1 \\ 2a+2 & -a-1 \end{pmatrix} \sim \begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix},$$

and a corresponding eigenvector is $(1, 2)$.

If $a = -1$, then

$$\mathbf{A} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

is already diagonalized, and we have the two linearly independent eigenvectors $(1, 0)$ and $(0, 1)$.

3. We have in 2) proved for every $a \in \mathbb{R}$ that $\mathbf{A}$ has two linearly independent eigenvectors. Since the dimension is 2, we can always diagonalize $\mathbf{A}$, and $\mathbf{A}$ is similar to

$$\mathbf{\Lambda} = \begin{pmatrix} a & 0 \\ 0 & -1 \end{pmatrix}, \qquad a \in \mathbb{R}.$$

If $a \neq -1$, then

$$\mathbf{V} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \quad \text{med} \quad \mathbf{V}^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}.$$

For $a = -1$ er

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{V}^{-1}.$$

4. Obviously,

$$\mathbf{\Lambda}^{14} = \begin{pmatrix} a^{14} & 0 \\ 0 & (-1)^{14} \end{pmatrix} = \begin{pmatrix} a^{14} & 0 \\ 0 & 1 \end{pmatrix}.$$

If $a = -1$, then $\mathbf{\Lambda}^{14} = \mathbf{A}^{14} = \mathbf{I}$.

If $a \neq -1$, then

$$\begin{aligned} \mathbf{A}^{14} &= \left(\mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}\right)^{-14} = \mathbf{V}\mathbf{\Lambda}^{14}\mathbf{V}^{-1} \\ &= \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} a^{14} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} a^{14} & 1 \\ a^{14} & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2a^{14}-1 & -a^{14}+1 \\ 2a^{14}-2 & -a^{14}+2 \end{pmatrix}. \end{aligned}$$

We note that $\lim_{a \to -1} \mathbf{A}^{14} = \mathbf{I}$.

Example 1.30 *Given the matrix*

$$\mathbf{A} = \begin{pmatrix} 0 & -1 & 0 \\ a & a+1 & a-2 \\ 0 & 0 & 2 \end{pmatrix}, \qquad \text{where } a \in \mathbb{R}.$$

1. *Find for every a all eigenvalues and the corresponding eigenvectors of* $\mathbf{A}$.

2. *Find all a, for which* $\mathbf{A}$ *is similar to a diagonal matrix, and find for each of these a a diagonal matrix which is similar to* $\mathbf{A}$, *and also a regular matrix, which diagonalizes* $\mathbf{A}$ *to the indicated diagonal matrix.*

1. The characteristic polynomial is

$$\begin{aligned} \det(\mathbf{A} - \lambda\mathbf{I}) &= \begin{vmatrix} -\lambda & -1 & 0 \\ 1 & 1+1-\lambda & a-2 \\ 0 & 0 & 2-\lambda \end{vmatrix} = -(\lambda-2) \begin{vmatrix} -\lambda & -1 \\ a & a+1-\lambda \end{vmatrix} \\ &= -(\lambda-2)\{\lambda^2 - (a+1)\lambda + a\} = -(\lambda-1)(\lambda-2)(\lambda-a), \end{aligned}$$

thus the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 2$ and $\lambda_3 = a$.

If $\lambda = 1$, then we get by reduction (no matter the choice of a),

$$\mathbf{A} - \mathbf{I} = \begin{pmatrix} -1 & -1 & 0 \\ a & a & a-2 \\ 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

of rank 2 with the eigenvector $(1, -1, 0)$.

If $\lambda = 2$, then by reduction

$$\mathbf{A} - 2\mathbf{I} = \begin{pmatrix} -2 & -1 & 0 \\ a & a-1 & a-2 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} -2 & -1 & 0 \\ a-2 & a-2 & a-2 \\ 0 & 0 & 0 \end{pmatrix}$$

of *at most* rank 2.

If $a \neq 2$, then the rank is 2, and the eigenspace has dimension $3 - 2 = 1$. An eigenvector is e.g. $(1, -2, 1)$.

If $a = 2$, then

$$\mathbf{A} - 2\mathbf{I} = \begin{pmatrix} -2 & -1 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

of rank 1, thus the eigenspace has dimension 2. Two linearly independent eigenvectors are e.g. $(1, -2, 0)$ and $(0, 0, 1)$.

If $a \notin \{1, 2\}$, then we get by reduction

$$\mathbf{A} - a\mathbf{I} = \begin{pmatrix} -a & -1 & 0 \\ a & 1 & a-2 \\ 0 & 0 & 2-a \end{pmatrix} \sim \begin{pmatrix} a & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

of rank 2, thus the eigenspace has dimension $3 - 2 = 1$. An eigenvector is e.g. $(1, -a, 0)$.

2. It follows from 1) that the algebraic and the geometric multiplicities are equal for $a \neq 1$, while they are differfent for $a = 1$, where $\lambda = 1$ has algebraic multiplicity 2 and geometric multiplicity 1.

   Hence, the matrix $\mathbf{A}$ is similar to

$$\mathbf{\Lambda} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & a \end{pmatrix} \qquad \text{for } a \neq 1.$$

   If $a \notin \{1, 2\}$, then we diagonalize $\mathbf{A}$ by

$$\mathbf{V} = (\mathbf{v}_1 \;\; \mathbf{v}_2 \;\; \mathbf{v}_3) = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -2 & -a \\ 0 & 1 & 0 \end{pmatrix}.$$

   If $a = 2$, we diagonalize $\mathbf{A}$ by

$$\mathbf{V} = (\mathbf{v}_1 \;\; \mathbf{v}_2 \;\; \mathbf{v}_3) = \begin{pmatrix} 1 & 1 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Example 1.31 *Given the matrix*

$$\mathbf{A} = \begin{pmatrix} 17 & -18 \\ 9 & -10 \end{pmatrix}.$$

1. *Find a diagonal matrix* $\mathbf{\Lambda}$ *and a regular matrix* $\mathbf{V}$*, such that*

$$\mathbf{\Lambda} = \mathbf{V}^{-1}\mathbf{A}\mathbf{V}.$$

2. *Find a matrix* $\mathbf{D}$*, such that* $\mathbf{D}^3 = \mathbf{\Lambda}$*, and then find a matrix* $\mathbf{C}$*, such that* $\mathbf{C}^3 = \mathbf{A}$*.*

1. The characteristic polynomial is

$$\begin{aligned} \det(\mathbf{A} - \lambda\mathbf{I}) &= \lambda^2 - \text{ spor } \mathbf{A} \cdot \lambda + \det \mathbf{A} = \lambda^2 - 7\lambda - 8 \\ &= (\lambda - 8)(\lambda + 1). \end{aligned}$$

   The eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 8$.

We infer from

$$\mathbf{A} - \lambda_1 \mathbf{I} = \begin{pmatrix} 18 & -18 \\ 9 & -9 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix},$$

that $\mathbf{v}_1 = (1,1)$ is an eigenvector corresponding to $\lambda_1 = -1$.

Since

$$\mathbf{A} - \lambda_2 \mathbf{I} = \begin{pmatrix} 9 & -18 \\ 9 & -18 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix},$$

then $\mathbf{v}_2 = (2,1)$ is an eigenvector corresponding to $\lambda_2 = 8$.

It follows that

$$\mathbf{\Lambda} = \begin{pmatrix} -1 & 0 \\ 0 & 8 \end{pmatrix} \quad \text{and} \quad \mathbf{V} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \text{ med } \mathbf{V}^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix},$$

where $\mathbf{\Lambda} = \mathbf{V}^{-1}\mathbf{A}\mathbf{V}$.

2. We can obviously choose

$$\mathbf{D} = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}.$$

If we put

$$\begin{aligned} \mathbf{C} &= \mathbf{V}\mathbf{D}\mathbf{V}^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 2 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -4 & 3 \\ -6 & 5 \end{pmatrix}, \end{aligned}$$

then

$$\mathbf{C}^3 = \left(\mathbf{V}\mathbf{D}\mathbf{V}^{-1}\right)^3 = \mathbf{V}\mathbf{D}^3\mathbf{V}^{-1} = \mathbf{A}.$$

Example 1.32 *Given the matrices*

$$\mathbf{A} = \begin{pmatrix} 0 & -1 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad \textit{and} \quad \mathbf{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Prove that $\mathbf{A}$ *and* $\mathbf{D}$ *are similar.*

We note that $\mathbf{D}$ is a diagonal matrix with the characteristic polynomial

$$-(\lambda - 1)(\lambda - 2)^2.$$

The characteristic polynomial of $\mathbf{A}$ is

$$\begin{aligned}\det(\mathbf{A}-\lambda\mathbf{I}) &= \begin{vmatrix} -\lambda & -1 & 0 \\ 2 & 3-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} = \begin{vmatrix} -\lambda & -1 & 0 \\ 2-\lambda & 2-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} \\ &= (\lambda-2)^2 \begin{vmatrix} -\lambda & -1 \\ 1 & 1 \end{vmatrix} = -(\lambda-1)(\lambda-2)^2,\end{aligned}$$

thus the same characteristic polynomial as for $\mathbf{D}$.

If $\lambda = 2$, then

$$\mathbf{A}-2\mathbf{I} = \begin{pmatrix} -2 & -1 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

of rank 1, hence the geometric multiplicity is $3-1=2=$ the algebraic multiplicity.

If $\lambda = 1$, then the geometric and the algebraic multiplicities are trivially equal. Consequently, $\mathbf{A}$ is similar to a diagonal matrix, i.e. to $\mathbf{D}$.

Example 1.33 *Let* $f : \mathbb{R}^3 \to \mathbb{R}^3$ *be the linear map, which in the usual basis of* $\mathbb{R}^3$ *has the matrix*

$$\mathbf{A} = \begin{pmatrix} 0 & 2 & -1 \\ -4 & 5 & -1 \\ -4 & 3 & 1 \end{pmatrix}.$$

It is further given that the vectors

$$\mathbf{b}_1 = (1,2,2), \quad \mathbf{b}_2 = (0,1,1), \quad \mathbf{b}_3 = (0,1,2)$$

form a basis of $\mathbb{R}^3$

1. *Prove that the matrix*

$$\mathbf{B} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

is the matrix of f *with respect to the basis* $(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$ *i* $\mathbb{R}^3$.

2. *Find the eigenvalues of* f.

1. We get by insertion

$$\begin{aligned}
\mathbf{A}\mathbf{b}_1 &= \begin{pmatrix} 0 & 2 & -1 \\ -4 & 5 & -1 \\ -4 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 0+4-2 \\ -4+10-2 \\ -4+6+2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 4 \end{pmatrix} = 2\,\mathbf{b}_1, \\
\mathbf{A}\mathbf{b}_2 &= \begin{pmatrix} 0 & 2 & -1 \\ -4 & 5 & -1 \\ -4 & 3 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \mathbf{b}_1 + \mathbf{b}_2, \\
\mathbf{A}\mathbf{b}_3 &= \begin{pmatrix} 0 & 2 & -1 \\ -4 & 5 & -1 \\ -4 & 3 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \mathbf{b}_2 + 2\mathbf{b}_3,
\end{aligned}$$

thus the matrix is

$$\mathbf{B} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

2. We have trivially, $\det(\mathbf{B} - \lambda\mathbf{I}) = -(\lambda - 2)^3$, hence $\lambda = 2$ is the only eigenvalue of algebraic multiplicity 3. It follows from the structure of $\mathbf{B}$ that the geometric multiplicity is only 1, and that $\mathbf{b}_1$ is an eigenvector.

2 Systems of differential equations

Example 2.1 *1. Solve the system of differential equations*

$$\begin{aligned}\frac{dx_1}{dt}(t) &= x_1(t)+3x_2(t),\\ \frac{dx_2}{dt}(t) &= 4x_1(t)+5x_2(t).\end{aligned}$$

2. Find the solution, which satisfies the initial conditions

$$x_1(0)=1, \qquad x_2(0)=10.$$

1. The corresponding matrix is

$$\mathbf{A}=\begin{pmatrix}1 & 3\\ 4 & 5\end{pmatrix},$$

of the characteristic polynomial

$$\begin{aligned}\det(\mathbf{A}-\lambda\mathbf{I}) &= \begin{vmatrix}1-\lambda & 3\\ 4 & 5-\lambda\end{vmatrix} = \lambda^2-6\lambda+5-12\\ &= \lambda^2-6\lambda-7=(\lambda+1)(\lambda-7).\end{aligned}$$

The eigenvalues are $\lambda=-1$ with the eigenvector $(3,-2)$, and $\lambda=7$ with the eigenvector $(1,2)$.

The complete solution is

$$\begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix}=\begin{pmatrix}3e^{-t} & e^{7t}\\ -e^{-t} & 2e^{7t}\end{pmatrix}\begin{pmatrix}c_1\\ c_2\end{pmatrix},$$

where c_1 and c_2 are arbitrary constants.

2. We get for $t=0$ that

$$\begin{pmatrix}x_1(0)\\ x_2(0)\end{pmatrix}=\begin{pmatrix}3 & 1\\ -2 & 2\end{pmatrix}\begin{pmatrix}c_1\\ c_2\end{pmatrix}=\begin{pmatrix}1\\ 10\end{pmatrix},$$

where

$$\begin{aligned}\left(\begin{array}{rr|r}3 & 1 & 1\\ -2 & 2 & 10\end{array}\right) &\sim \left(\begin{array}{rr|r}1 & 3 & 11\\ -2 & 2 & 10\end{array}\right)\sim\left(\begin{array}{rr|r}1 & 3 & 11\\ 0 & 8 & 32\end{array}\right)\\ &\sim \left(\begin{array}{rr|r}1 & 3 & 11\\ 0 & 1 & 4\end{array}\right)\sim\left(\begin{array}{rr|r}1 & 0 & -1\\ 0 & 1 & 4\end{array}\right),\end{aligned}$$

hence $c_1=-1$ and $c_2=4$, corresponding to the solution

$$\begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix}=\begin{pmatrix}3e^{-t} & e^{7t}\\ -2e^{-t} & 2e^{7t}\end{pmatrix}\begin{pmatrix}-1\\ 4\end{pmatrix}=\begin{pmatrix}-3e^{-t}+4e^{7t}\\ 2e^{-t}+8e^{7t}\end{pmatrix}.$$

Example 2.2 1. *Solve the systems of differential equations*

$$\begin{aligned}
\frac{dx_1}{dt}(t) &= 4x_1(t) &&+ x_3(t),\\
\frac{dx_2}{dt}(t) &= -2x_1(t) + x_2(t) &&\\
\frac{dx_3}{dt}(t) &= -2x_1(t) &&+ x_3(t).
\end{aligned}$$

2. *Find the solution, which satisfies the initial conditions*

$$x_1(0) = -1, \qquad x_2(0) = 1, \qquad x_3(0) = 1.$$

1. The corresponding matrix

$$\mathbf{A} = \begin{pmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

has the characteristic polynomial

$$\begin{aligned}
\det(\mathbf{A} - \lambda\mathbf{I}) &= \begin{vmatrix} 4-\lambda & 0 & 1 \\ -2 & 1-\lambda & 0 \\ -2 & 0 & 1-\lambda \end{vmatrix} = -(\lambda-1)\begin{vmatrix} 4-\lambda & 1 \\ -2 & 1-\lambda \end{vmatrix}\\
&= -(\lambda-1)\{(\lambda-1)(\lambda-4)+2\} = -(\lambda-1)\{\lambda^2-5\lambda+6\}\\
&= -(\lambda-1)(\lambda-2)(\lambda-3).
\end{aligned}$$

The simple eigenvalues are $\lambda = 1,\, 2,\, 3$.
If $\lambda = 1$, then $\mathbf{v}_1 = (0,1,0)$ is trivially an eigenvector.
If $\lambda = 2$, then

$$\mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix} 2 & 0 & 1 \\ -2 & -1 & 0 \\ -2 & 0 & -1 \end{pmatrix} \sim \begin{pmatrix} 2 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

hence $\mathbf{v}_2 = (1,-2,-2)$ is an eigenvector.
If $\lambda = 3$, the

$$\mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix} 1 & 0 & 1 \\ -2 & -2 & 0 \\ -2 & 0 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

hence $\mathbf{v}_3 = (1,-1,1)$ is an eigenvector.

The complete solution is

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = \begin{pmatrix} 0 & e^{2t} & e^{3t} \\ e^t & -2e^{2t} & -e^{3t} \\ 0 & -2e^{2t} & -e^{3t} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix},$$

where c_1, c_2, c_3 are arbitrary constants.

2. If we put $t=0$, then

$$\begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & -2 & -1 \\ 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

It follows from the reductions

$$\left(\begin{array}{rrr|r} 0 & 1 & 1 & -1 \\ 1 & -2 & -1 & 1 \\ 0 & -2 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{rrr|r} 0 & 1 & 1 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & 2 & -1 & 0 \end{array}\right) \sim \left(\begin{array}{rrr|r} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -\frac{1}{3} \\ 0 & 0 & 1 & -\frac{2}{3} \end{array}\right)$$

that $c_1 = 1$, $c_2 = -\dfrac{1}{3}$ and $c_3 = -\dfrac{2}{3}$, hence we get the solution

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = \begin{pmatrix} 0 \\ e^t \\ 0 \end{pmatrix} - \frac{1}{3} \begin{pmatrix} e^{2t} \\ -2e^{2t} \\ -2e^{-2t} \end{pmatrix} - \frac{1}{2} \begin{pmatrix} e^{3t} \\ -e^{3t} \\ -e^{3t} \end{pmatrix}.$$

Example 2.3 *Solve the system of differential equations*

$$\begin{aligned}\frac{dx_1}{dt} &= 4x_1(t)+2x_2(t)+2x_3(t),\\ \frac{dx_2}{dt} &= 2x_1(t)+4x_2(t)+2x_3(t),\\ \frac{dx_3}{dt} &= 2x_1(t)+2x_2(t)+4x_3(t).\end{aligned}$$

The corresponding matrix is

$$\mathbf{A}=\begin{pmatrix}4&2&2\\2&4&2\\2&2&4\end{pmatrix}.$$

Remark 2.1 If we add the three equations, then we immediately get

$$\frac{d}{dt}\{x_1(x)+x_2(t)-x_3(t)\}=8\{x_1(t)+x_2(t)+x_3(t)\},$$

and analogously by a subtraction,

$$\frac{d}{dt}\{x_1(t)-x_2(t)\}=2x_1(t)-2x_2(t)=2\{x_1(t)-x_2(t)\},$$

and

$$\frac{d}{dt}\{x_2(t)-x_3(t)\}=2x_2(t)-2x_3(t)=2\{x_2(t)-x_3(t)\}.$$

These equations are immediately solved, thus we could by this nonstandard observation save us a lot of trouble. $\Diamond$

The characteristic polynomial is

$$\begin{aligned}\det(\mathbf{A}-\lambda\mathbf{I}) &= \begin{vmatrix}4-\lambda&2&2\\2&4-\lambda&2\\2&2&4-\lambda\end{vmatrix}=\begin{vmatrix}2-\lambda&2-\lambda&0\\0&2-\lambda&\lambda-2\\2&2&4-\lambda\end{vmatrix}\\ &= (\lambda-2)^2\begin{vmatrix}-1&1&0\\0&-1&1\\2&2&4-\lambda\end{vmatrix}=(\lambda-2)^2\begin{vmatrix}-1&1&0\\0&-1&1\\0&4&4-\lambda\end{vmatrix}\\ &= -(\lambda-2)^2\begin{vmatrix}-1&1\\4&4-\lambda\end{vmatrix}=-(\lambda-2)^2\{\lambda-4-4\}\\ &= -(\lambda-2)^2(\lambda-8).\end{aligned}$$

The eigenvalues are $\lambda=8$ and $\lambda=2$ (of multiplicity 2), which we also found in the remark above.

If $\lambda=8$, then

$$\begin{aligned}\mathbf{A}-8\mathbf{I} &= \begin{pmatrix}-4&2&2\\2&-4&2\\2&2&-4\end{pmatrix}\sim\begin{pmatrix}-2&1&1\\1&-2&1\\1&1&-2\end{pmatrix}\\ &\sim\begin{pmatrix}0&3&-3\\0&-3&3\\1&1&-2\end{pmatrix}\sim\begin{pmatrix}1&1&-2\\0&1&-1\\0&0&0\end{pmatrix},\end{aligned}$$

and we conclude that an eigenvector may be chosen as e.g. $(1,1,1)$.

If $\lambda = 2$, then

$$\mathbf{A} - 2\mathbf{I} = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

of rank 1. Two linearly independent vectors are e.g. $(1,-1,0)$ and $(1,0,-1)$.

The complete solution is

$$\begin{aligned} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} &= \begin{pmatrix} e^{8t} & e^{2t} & e^{2t} \\ e^{8t} & -e^{2t} & 0 \\ e^{8t} & 0 & -e^{2t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \\ &= c_1 e^{8t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + c_3 e^{2t} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \end{aligned}$$

where c_1, c_2, c_3 are arbitrary constants.

Example 2.4 *1. Find all* complex *solutions of the system of differential equations*

$$\begin{aligned} \frac{dx_1}{dt}(t) &= 3x_1(t) + x_2(t) + 3x_3(t), \\ \frac{dx_2}{dt}(t) &= 3x_1(t) + 3x_2(t) + x_3(t), \\ \frac{dx_3}{dt}(t) &= -x_1(t) + 2x_3(t). \end{aligned}$$

2. Find all real solutions of the system of differential equations.

1. The corresponding matrix

$$\mathbf{A} = \begin{pmatrix} 3 & 1 & 3 \\ 3 & 3 & 1 \\ -1 & 0 & 2 \end{pmatrix}$$

has the characteristic polynomial

$$\begin{aligned} \det(\mathbf{A} - \lambda\mathbf{I}) &= \begin{vmatrix} 3-\lambda & 1 & 3 \\ 3 & 3-\lambda & 1 \\ -1 & 0 & 2-\lambda \end{vmatrix} \\ &= -\begin{vmatrix} 1 & 3 \\ 3-\lambda & 1 \end{vmatrix} - (\lambda-2) \begin{vmatrix} 3-\lambda & 1 \\ 3 & 3-\lambda \end{vmatrix} \\ &= -1 + 3(3-\lambda) - (\lambda-2)\{(\lambda-3)^2 - 3\} \\ &= -(\lambda-2)(\lambda-3)^2 + 3(3-\lambda+\lambda-2) - 1 \\ &= -(\lambda-2)(\lambda-3)^2 + 2. \end{aligned}$$

It follows by insertion of $\lambda = 4$ that $\det(\mathbf{A} - 4\mathbf{I}) = 0$, thus $\lambda = 4$ is an eigenvalue. Hence, the factor $(\lambda - 4)$ can be isolated. Then

$$\begin{aligned}
-\det(\mathbf{A} - \lambda\mathbf{I}) &= (\lambda-2)(\lambda-3)^2 - 2 \\
&= (\lambda-4)(\lambda-3)^2 + 2(\lambda-3)^2 - 2 \\
&= (\lambda-4)(\lambda-3)^2 + 2(\lambda-4)(\lambda-2) \\
&= (\lambda-4)\{(\lambda-3)^2 + 2(\lambda-2)\} \\
&= (\lambda-4)\{\lambda^2 - 6\lambda + 9 + 2\lambda - 4\} = (\lambda-4)\{\lambda^2 - 4\lambda + 5\} \\
&= (\lambda-4)(\lambda-2-i)(\lambda-2+i).
\end{aligned}$$

The three complex roots are

$$\lambda_1 = 4, \qquad \lambda_2 = 2+i, \qquad \lambda_3 = 2-i.$$

If $\lambda_1 = 4$, then

$$\mathbf{A} - \lambda_1\mathbf{I} = \begin{pmatrix} -1 & 1 & 3 \\ 3 & -1 & 1 \\ -1 & 0 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 5 \\ 0 & -1 & -5 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{pmatrix},$$

and an eigenvector is $(2, 5, -1)$.

If $\lambda_2 = 2+i$, then

$$\begin{aligned}
\mathbf{A} - \lambda_2\mathbf{I} &= \begin{pmatrix} 1-i & 1 & 3 \\ 3 & 1-i & 1 \\ -1 & 0 & -i \end{pmatrix} \sim \begin{pmatrix} 2 & 1+i & 3+3i \\ 3 & 1-i & 1 \\ -1 & 0 & -i \end{pmatrix} \\
&\sim \begin{pmatrix} 1 & 0 & i \\ 0 & 1+i & 3+i \\ 0 & 1-i & 1-3i \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & i \\ 0 & 2 & 4-2i \\ 0 & 2 & 4-2i \end{pmatrix} \\
&\sim \begin{pmatrix} 1 & 0 & i \\ 0 & 1 & 2-i \\ 0 & 0 & 0 \end{pmatrix}.
\end{aligned}$$

A complex eigenvector is e.g. $\mathbf{v}_2 = (-i, -2+i, 1)$.

Now, $\mathbf{A}$ is real, hence by a complex conjugation,

$$\mathbf{A}\overline{\mathbf{v}_2} = \overline{\mathbf{A}\mathbf{v}_2} = \overline{\lambda_2\mathbf{v}_2} = \overline{\lambda_2}\,\overline{\mathbf{v}_2},$$

proving that $\lambda_3 = \overline{\lambda_2} = 2 - i$ is an eigenvalue and that

$$\mathbf{v}_3 = \overline{\mathbf{v}_2} = (i, -2-i, 1)$$

is a corresponding eigenvector.

All complex solutions are then

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = \begin{pmatrix} 2e^{4t} & -i\,e^{(2+i)t} & i\,e^{(2-i)t} \\ 5e^{4y} & (-2+i)e^{(2+i)t} & (-2-i)e^{(2-i)t} \\ -e^{4t} & e^{(2+i)t} & e^{(2-i)t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}.$$

2. Since

$$
\begin{aligned}
\begin{pmatrix} -i\,e^{(2+i)t} \\ (-2+i)e^{(2+i)t} \\ e^{(2+i)t} \end{pmatrix}
&= e^{2t}\begin{pmatrix} -i\{\cos t + i\,\sin t\} \\ (-2+i)\{\cos t + i\,\sin t\} \\ \cos t + i\,\sin t \end{pmatrix} \\
&= e^{2t}\begin{pmatrix} \sin t - i\,\cos t \\ -2\cos t - \sin t + i\{\cos t - 2\sin t\} \\ \cos t + i\,\sin t \end{pmatrix} \\
&= \begin{pmatrix} e^{2t}\sin t \\ -(2\cos t + \sin t)e^{2t} \\ e^{2t}\cos t \end{pmatrix} + i \begin{pmatrix} -e^{2t}\cos t \\ e^{2t}\{\cos t - 2\sin t\} \\ e^{2t}\sin t \end{pmatrix},
\end{aligned}
$$

all the real solutions are

$$
\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = \begin{pmatrix} 2e^{4t} & -e^{2t}\sin t & -e^{2t} \\ 5e^{4t} & -e^{2t}(2\cos t + \sin t) & e^{2t}(\cos t - 2\sin t) \\ -e^{4t} & e^{2t}\cos t & e^{2t}\sin t \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix},
$$

where c_1, c_2, c_3 are arbitrary constants.

3 Euclidean vector space

Example 3.1 *Given in space the usual rectangular coordinate system. Find all $a \in \mathbb{R}$ for which the vector of coordinates $(a, 2a, 3a^2)$ is perpendicular to the vector of the coordinates $(1, -1, a)$.*

The condition is that the inner product of the two vectors is 0, so we compute

$$\begin{aligned} 0 &= (a, 2a, 3a^2) \cdot (1, -1, a) = a\{1 - 2 + 3a^2\} \\ &= a(3a^2 - 1) = 3a\left(a - \frac{1}{\sqrt{3}}\right)\left(a + \frac{1}{\sqrt{3}}\right). \end{aligned}$$

The possible values are $a = 0$ [where $(a, 2a, 3a^2) = (0, 0, 0)$, and by convention we say that $\mathbf{0}$ is perpendicular to any other vector] and

$$a = \pm\frac{1}{\sqrt{3}}.$$

Example 3.2 *In an usual rectangular coordinate system in the space are given the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ of the coordinates $(1, 3, -1)$, $(0, 2, 4)$ and $(2, 1, -1)$, resp.. Find $k \in \mathbb{R}$, such that the vector $\vec{a} + k\vec{b}$ is perpendicular to the vector $\vec{c}$.*

The condition is

$$\begin{aligned} 0 &= (\vec{a} + k\vec{b}) \cdot \vec{c} = \vec{a} \cdot \vec{c} + k\vec{b} \cdot \vec{c} \\ &= (1, 3, -1) \cdot (2, 1, -1) + k(0, 2, 4) \cdot (2, 1, -1) \\ &= 2 + 3 + 1 + k\{0 + 2 - 4\} = 6 - 2k, \end{aligned}$$

thus $k = 3$, corresponding to

$$\vec{a} + k\vec{b} = (1, 3, -1) + 3(0, 2, 4) = (1, 9, 11).$$

Example 3.3 *Let the vectors $\vec{a}$ and $\vec{b}$ be given by their coordinates $(3, 1, 2)$ and $(4, -8, 1)$ resp. with respect to the basis vectors in an usual rectangular coordinate system of positive orientation.*
The vector $\vec{a}$ is split into a contribution along the straight line given by the vector $\vec{b}$ and a contribution in a plane which is perpendicular to this vector.
Find the coordinates of the components of the vector.

First notice that

$$|\vec{b}| = \sqrt{4^2 + (-8)^2 + 1^2} = \sqrt{16 + 64 + 1} = \sqrt{81} = 9,$$

thus

$$\vec{e} = \frac{1}{9}\,\vec{b}$$

is a unit vector in the direction of $\vec{b}$. Then the projection of $\vec{a}$ onto the line given by $\vec{b}$ is determined by

$$\begin{aligned}
(\vec{a}\cdot\vec{e})\vec{e} &= \frac{1}{81}\,(\vec{a}\cdot\vec{b})\vec{b} = \frac{1}{81}\{(3,1,2)\cdot(4,-8,1)\}(4,-8,1)\\
&= \frac{1}{81}\{12-8+2\}(4,-8,1) = \frac{2}{27}\,(4,-8,1)\\
&= \left(\frac{8}{27},-\frac{16}{27},\frac{2}{27}\right),
\end{aligned}$$

and a perpendicular vector is of course

$$\vec{a}-(\vec{a}\cdot\vec{e})\vec{e} = (3,1,2)-\left(\frac{8}{27},-\frac{16}{27},\frac{2}{27}\right) = \left(\frac{73}{27},\frac{43}{27},\frac{52}{27}\right).$$

Example 3.4 *Given in the space a coordinate system $(O;\vec{a}_1,\vec{a}_2,\vec{a}_3)$, where $|\vec{a}_1|=1$, $|\vec{a}_2|=2$, $|\vec{a}_3|=3$ and $\angle(\vec{a}_2,\vec{a}_3)=\angle(\vec{a}_3,\vec{a}_1)=\angle(\vec{a}_1,\vec{a}_2)=60°$. Given in this coordinate system the vectors $\vec{a}$ and $\vec{b}$ with the coordinates (a_1,a_2,a_3) and (b_1,b_2,b_3), respectively.*
Find the scalar product $\vec{a}\cdot\vec{b}$ of the two vectors.

First notice that if $i\neq j$, then

$$\vec{a}_i\cdot\vec{a}_j = |\vec{a}_i|\cdot|\vec{a}_j|\cos(\angle(\vec{a}_i,\vec{a}_j)) = |\vec{a}_i|\cdot|\vec{a}_j|\cos\frac{\pi}{3} = \frac{1}{2}|\vec{a}_i|\cdot|\vec{a}_j|.$$

Hence,

$$\begin{aligned}
\vec{a}\cdot\vec{b} &= \{a_1\vec{a}_1+a_2\vec{a}_2+a_3\vec{a}_3\}\cdot\{b_1\vec{a}_1+b_2\vec{a}_2+b_3\vec{a}_3\}\\
&= a_1b_1|\vec{a}_1|^2+a_1b_2\vec{a}_1\cdot a_2+a_1b_3\vec{a}_1\cdot\vec{a}_3\\
&\quad +a_2b_1\vec{a}_2\cdot\vec{a}_1+a_2b_2|\vec{a}_2|^2+a_2b_3\vec{a}_2\cdot\vec{a}_3\\
&\quad +a_3b_1\vec{a}_3\cdot a_1+a_3b_2\vec{a}_3\cdot\vec{a}_2+a_3b_3|\vec{a}_3|^2\\
&= (a_1\ \ a_2\ \ a_3)\begin{pmatrix}1 & 1 & \frac{3}{2}\\ 1 & 4 & 3\\ \frac{3}{2} & 3 & 9\end{pmatrix}\begin{pmatrix}b_1\\ b_2\\ b_3\end{pmatrix}.
\end{aligned}$$

Then notice that the element at place number (i,j) in the matric is $\vec{a}_i\cdot\vec{a}_j$, thus is general

$$\vec{a}\cdot\vec{b} = (a_1\ \ a_12\ \ a_3)\begin{pmatrix}\vec{a}_1\cdot\vec{a}_1 & \vec{a}_1\cdot\vec{a}_2 & \vec{a}_1\cdot\vec{a}_3\\ \vec{a}_2\cdot\vec{a}_1 & \vec{a}_2\cdot\vec{a}_2 & \vec{a}_2\cdot\vec{a}_3\\ \vec{a}_3\vec{a}_1 & \vec{a}_3\cdot\vec{a}_2 & \vec{a}_3\cdot\vec{a}_3\end{pmatrix}.$$

We note that this matrix is always symmetric, because the inner product is symmetric.

Example 3.5 *Consider in the vector space $\mathbb{P}_2(\mathbb{R})$ of all real polynomials of at most degree 2 with the scalar product*

$$\langle P, Q\rangle = \int_{-1}^{1} P(x)Q(x)\,dx, \qquad \textit{where } P(x),\ Q(x) \in \mathbb{P}_2(\mathbb{R}).$$

Find the angles between the vectors below in $\mathbb{P}_2(\mathbb{R})$:

$$P_1(x) = 1, \qquad P_2(x) = x, \qquad P_3(x) = 1 - x.$$

It follows from

$$\begin{aligned}
&\|P_1\|^2 = \textstyle\int_{-1}^{1} 1^2\,dx = 2, && \|P_1\| = \sqrt{2},\\
&\|P_2\|^2 = \textstyle\int_{-1}^{1} x^2\,dx = \left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{2}{3}, && \|P_2\| = \sqrt{\frac{2}{3}},\\
&\|P_3\|^2 = \textstyle\int_{-1}^{1}(1-x)^2\,dx = \int_0^2 u^2\,du = \left[\frac{u^3}{3}\right]_0^2 = \frac{8}{3}, && \|P_3\| = 2\sqrt{\frac{2}{3}},
\end{aligned}$$

and

$$\begin{aligned}
\langle P_1, P_2\rangle &= \int_{-1}^{1} 1\cdot dx = \left[\frac{x^2}{2}\right]_{-1}^{1} = 0 = \|P_1\|\cdot\|P_2\|\cos(\angle(P_1,P_2)),\\
\langle P_1, P_3\rangle &= \langle 1, 1-x\rangle = \|P_1\|^2 - \langle P_1, P_1 2\rangle = 2 = \|P_1\|\cdot\|P_3\|\cos(\angle(P_1,P_3)),\\
\langle P_2, P_3\rangle &= \int_{-1}^{1} x\cdot(1-x)\,dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^{1} = -\frac{2}{3} = \|P_2\|\cdot\|P_3\|\cos(\angle(P_2,P_3)),
\end{aligned}$$

that

$$\begin{aligned}
&\cos(\angle(P_1,P_2)) = 0, && \text{dvs. } \angle(P_1,P_2) = \frac{\pi}{2},\\
&\cos(\angle(P_1,P_3)) = \frac{2}{\sqrt{2}\cdot 2\sqrt{\frac{2}{3}}} = \frac{\sqrt{3}}{2}, && \text{dvs. } \angle(P_1,P_3) = \frac{\pi}{6},\\
&\cos(\angle(P_2,P_3)) = -\frac{2}{3}\cdot\frac{1}{\sqrt{\frac{2}{3}}\cdot 2\sqrt{\frac{2}{3}}} = -\frac{2}{3}\cdot\frac{1}{2\cdot\frac{2}{3}} = -\tfrac{1}{2}, && \text{dvs. } \angle(P_2,P_3) = \frac{2\pi}{3}.
\end{aligned}$$

Example 3.6 *Let $C^0(I)$ be the vector space of all continuous and real functions defined on $I = [0;2\pi]$.*

1. *Prove that $\langle f,g\rangle = \int_0^{2\pi} f(t)g(t)\,dt$ is a scalar product, for f, $g \in C^0(I)$.*
2. *Write down explicitly the Cauchy-Schwarz inequality in this case*
3. *Let U be the subspace of $C^0(I)$, consisting of all linear combinations*
$$\sum_{k=0}^{n}(a_k\cos kt + b_k\sin kt)$$
for fixed n. Prove that the functions
$$\frac{1}{\sqrt{2\pi}},\ \frac{1}{\sqrt{\pi}}\cos t,\ \frac{1}{\sqrt{\pi}}\sin t,\ \dots,\ \frac{1}{\sqrt{\pi}}\cos nt,\ \frac{1}{\sqrt{\pi}}\sin nt$$
form an orthonormal basis of U.
4. *Find the length*
$$\left\|\sum_{k=0}^{n}(a_k\cos kt + b_k\sin kt)\right\|.$$

1. Obviously,
$$\langle f,g\rangle = \int_0^{2\pi} f(t)g(t)\,dt$$
is bilinear, and $\langle f,g\rangle = \langle g,f\rangle$. Finally,
$$\langle f,f\rangle = \int_0^{2\pi} f(t)^2\,dt \ge 0,$$
and if $f(t_0)\neq 0$ for some $t_0\in[0,2\pi]$, then if follows from the continuity that $f(t)\neq 0$ in a neighbourhood of t_0, hence $f(t)^2>0$ in the same neighbourhood, i.e. $\langle f,f\rangle > 0$.
Therefore, if $\langle f,f\rangle = 0$, then $f(t)\equiv 0$.
Thus we have proved that $\langle f,g\rangle$ is a scalar product.

2. The Cauchy-Schwarz inequality

$$|\langle f,g\rangle| \le \|f\|\cdot\|g\|,$$

is here written

$$\left|\int_0^{2\pi} f(t)g(t)\,dt\right| \le \sqrt{\int_0^{2\pi} f(t)^2\,dt}\cdot\sqrt{\int_0^{2\pi} g(t)^2\,dt}.$$

3. Then compute

$$\left\langle \frac{1}{\sqrt{2\pi}},\frac{1}{\sqrt{2\pi}}\right\rangle = \frac{1}{(\sqrt{2\pi})^2}\int_0^{2\pi} 1^2\,dt = \frac{2\pi}{2\pi} = 1,$$

and

$$\begin{aligned}\left\langle \frac{1}{\sqrt{\pi}}\sin kt,\frac{1}{\sqrt{\pi}}\sin kt\right\rangle &= \frac{1}{\pi}\int_0^{2\pi}\sin^2 kt\,dt \\ = \ \frac{1}{\pi}\int_0^{2\pi}\frac{1}{2}\left\{\sin^2 kt+\cos^2 kt\right\}dt \\ &= \frac{1}{\pi}\int_0^{2\pi}\frac{1}{2}\,dt = 1 = \left\langle \frac{1}{\sqrt{\pi}}\cos kt,\frac{1}{\sqrt{\pi}}\cos kt\right\rangle, \quad k=1,\dots,n,\end{aligned}$$

and the vectors are all unit vectors.

It follows from the trigonometric formulæ

$$\sin(A+B) = \sin A\cdot\cos B + \cos A\cdot\sin B,$$

$$\sin(A-B) = \sin A\cdot\cos B - \cos A\cdot\sin B,$$

by addition and division by 2 that

$$\sin A\cdot\cos B = \frac{1}{2}\{\sin(A+b)+\sin(A-B)\}.$$

If we put $A=kt$ and $B=mt$, $k\ne m$, then

$$\begin{aligned}\left\langle \frac{1}{\sqrt{\pi}}\sin kt,\frac{1}{\sqrt{\pi}}\cos mt\right\rangle &= \frac{1}{\pi}\int_0^{2\pi}\sin kt\cdot\cos mt\,dt \\ &= \frac{1}{2\pi}\int_0^{2\pi}\{\sin(k+m)t+\sin(k-m)t\}dt \\ &= \frac{1}{2\pi}\left[\frac{-1}{k+m}\cos(k+m)t-\frac{1}{k-m}\cos(k-m)t\right]_0^{2\pi} = 0,\end{aligned}$$

and

$$\begin{aligned}\left\langle \frac{1}{\sqrt{\pi}}\sin kt,\frac{1}{\sqrt{\pi}}\cos kt\right\rangle &= \frac{1}{\pi}\int_0^{2\pi}\sin kt\cdot\cos kt\,dt \\ &= \frac{1}{2\pi}\int_0^{2\pi}\sin 2kt\,dt = \frac{1}{4k\pi}[-\cos 2kt]_0^{2\pi} = 0,\end{aligned}$$

and we have orthogonality, if we take the scalar product between a sinus-vector and a cosinus-vector.

Analogously, we get from the trigonometric formulæ

$$\cos(A-B) = \cos A \cdot \cos B + \sin A \cdot \sin B,$$

$$\cos(A+B) = \cos A \cdot \cos B - \sin A \cdot \sin B,$$

that

$$\cos A \cdot \cos B = \frac{1}{2}\{\cos(A-B) + \cos(A+B)\},$$

$$\sin A \cdot \sin B = \frac{1}{2}\{\cos(A-B) - \cos(A+B)\}.$$

Choose $A = kt$ and $B = mt$, where $k \neq m$. Then

$$\begin{aligned}\left\langle \frac{1}{\sqrt{\pi}} \cos kt, \frac{1}{\sqrt{\pi}} \cos mt \right\rangle &= \frac{1}{\pi} \int_0^{2\pi} \cos kt \cdot \cos mt \, dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} \{\cos(k-m)t + \cos(k+m)t\} \, dt \\ &= \frac{1}{2\pi} \left[\frac{1}{k-m} \sin(k-m)t + \frac{1}{k+m} \sin(k+m)t \right]_0^{2\pi} = 0,\end{aligned}$$

and

$$\begin{aligned}\left\langle \frac{1}{\sqrt{\pi}} \sin kt, \frac{1}{\sqrt{\pi}} \sin mt \right\rangle &= \frac{1}{\pi} \int_0^{2\pi} \sin kt \cdot \sin mt \, dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} \{\cos(k-m)t - \cos(k+m)t\} \, dt \\ &= \frac{1}{2\pi} \left[\frac{1}{k-m} \sin(k-m)t - \frac{1}{k+m} \sin(k+m)t \right]_0^{2\pi} = 0,\end{aligned}$$

and we have proved that

$$\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos t, \frac{1}{\sqrt{\pi}} \sin t, \dots, \frac{1}{\sqrt{\pi}} \cos nt, \frac{1}{\sqrt{\pi}} \sin nt$$

form an orthonormal basis of U.

4. The length is

$$\begin{aligned}&\left\| \sum_{k=0}^{n} (a_k \cos kt + b_k \sin kt) \right\| \\ &\quad = \left\| \left(a_0 \sqrt{2\pi}\right) \frac{1}{\sqrt{2\pi}} + \sum_{k=1}^{n} \sqrt{\pi} \left\{ a_k \cdot \frac{\cos kt}{\sqrt{\pi}} + b_k \cdot \frac{\sin kt}{\sqrt{\pi}} \right\} \right\| \\ &\quad = \sqrt{\pi} \cdot \sqrt{2a_0^2 + \sum_{k=1}^{n} \{a_k^2 + b_k^2\}}.\end{aligned}$$

Example 3.7 *Let $\varphi_1, \varphi_2, \dots, \varphi_n$ be an orthonormal basis of a subspace U of an Euclidean vector space V, and let $a_1, a_2, \dots, a_n$ be scalars.*

1. *Choosing $f \in V$ and $\varphi = \sum_{k=1}^n a_k\varphi_k \in U$, prove* Parseval's equation

$$\|f-\varphi\|^2 = \|f\|^2 + \sum_{k=1}^n (a_k-c_k)^2 - \sum_{k=1}^n c_k^2,$$

where c_k denotes the scalar product of f and φ_k.

(The numbers c_k are called the Fourier coefficients *of f).*

2. *Prove that $\inf_{\varphi\in U}\|f-\varphi\|$ happens if $a_1 = c_1, \dots, a_n = c_n$.*

(The method of the least squares).

3. *Prove* Bessel's inequality:

$$\sum_{k=1}^n c_k^2 \le \|f\|^2.$$

1. We get by a computation

$$\begin{aligned}
\|f-\varphi\|^2 &= \langle f-\varphi, f-\varphi\rangle = \|f\|^2 - 2\langle f,\varphi\rangle + \|\varphi\|^2 \\
&= \|f\|^2 - 2\left\langle f, \sum_{k=1}^n a_k\varphi_k\right\rangle + \sum_{k=1}^n a_k^2 \\
&= \|f\|^2 + \sum_{k=1}^n c_k^2 - \sum_{k=1}^n 2a_kc_k + \sum_{k=1}^n a_k^2 - \sum_{k=1}^n c_l^2 \\
&= \|f\|^2 + \sum_{k=1}^n (a_k-c_k)^2 - \sum_{k=1}^n c_k^2.
\end{aligned}$$

2. It follows from the equation of 1) that $\varphi\in U$ only occurs in the $\sum_{k=1}^n (a_k-c_k)\ge 0$. This term is smallest, when $a_k = c_k$, $k = 1, \dots, n$, hence

$$0 \le \inf_{\varphi\in U}\|f-\varphi\|^2 = \|f\|^2 - \sum_{k=1}^n c_k^2.$$

3. Bessel's inequality follows from a rearrangement of the inequality of 2).

Example 3.8 *Given the matrix*

$$\mathbf{A} = \begin{pmatrix} 5 & 1 & 1 \\ 1 & 5 & -1 \\ 1 & -1 & 5 \end{pmatrix}.$$

1. *Find the eigenvalues and corresponding eigenvectors of* $\mathbf{A}$.
2. *A bilinear function* $g : \mathbb{R}^3 \times \mathbb{R}^2 \to \mathbb{R}$ *is in the usual basis of* $\mathbb{R}^3$ *given by the equation*

$$g(\mathbf{x},\mathbf{y}) = (x_1 \;\; x_2 \;\; x_3)\mathbf{A} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}.$$

Prove that g *is a scalar product, and prove that the vectors* $(1,0,0)$ *and* $(1,-4,-1)$ *are orthogonal with respect to this scalar product.*

3. *Find a basis of* $\mathbb{R}^3$*, which is orthogonal with respect to this scalar product.*

1. The characteristic polynomial is

$$\begin{aligned}
\det(\mathbf{A}-\lambda\mathbf{I}) &= \begin{vmatrix} 5-\lambda & 1 & 1 \\ 1 & 5-\lambda & -1 \\ 1 & -1 & 5-\lambda \end{vmatrix} \\
&= (5-\lambda)^3 - 1 - 1 - (5-\lambda) - (5-\lambda) - (5-\lambda) \\
&= (5-\lambda)^3 - 3(5-\lambda) - 2 \\
&= -\{(\lambda-5)^3 - 3(\lambda-5) + 2\} \\
&= -\{(\lambda-5)^3 - (\lambda-5)^2 + (\lambda-5)^2 - (\lambda-5) - 2(\lambda-5) + 2\} \\
&= -\{(\lambda-5) - 1\}\{(\lambda-5)^2 + (\lambda-5) - 2\} \\
&= -(\lambda-6)\{(\lambda-5)+2\}\{(\lambda-5)-1\} \\
&= -(\lambda-6)^2(\lambda-3).
\end{aligned}$$

The eigenvalues are $\lambda = 3 > 0$ and $\lambda = 6 > 0$ (of multiplicity 2).

If $\lambda = 3$, then we reduce

$$\begin{aligned}
\mathbf{A}-\lambda\mathbf{I} &= \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{pmatrix} \\
&\sim \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix},
\end{aligned}$$

and an eigenvector is eg. $(1,-1,-1)$.

If $\lambda = 6$, we reduce

$$\mathbf{A}-\lambda\mathbf{I} = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Two linearly independent eigenvectors are e.g.

$$(1,1,0) \qquad \text{and} \qquad (1,0,1).$$

2. Clearly, g is bilinear , and it follows from $\mathbf{A}^T = \mathbf{A}$ that

$$g(\mathbf{y},\mathbf{x}) = \mathbf{y}^T\mathbf{A}\mathbf{x} = \left(\mathbf{x}^T\mathbf{A}^T\mathbf{y}\right)^T = \mathbf{x}^T\mathbf{A}\mathbf{y} = g(\mathbf{x},\mathbf{y}),$$

thus g is symmetric .

Finally, $\mathbf{A}$ is similar to

$$\mathbf{\Lambda} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix},$$

which has positive elements in the diagonal, so

$$g(\mathbf{x},\mathbf{x}) \geq 0.$$

If $g(\mathbf{x},\mathbf{x}) = 0$, then $\mathbf{x} = \mathbf{0}$.

Thus, we have proved that $g(\mathbf{x},\mathbf{y})$ is a scalar product.

It follows from

$$(1\ \ 0\ \ 0)\begin{pmatrix} 5 & 1 & 1 \\ 1 & 5 & -1 \\ 1 & -1 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ -4 \\ -1 \end{pmatrix} = 5-4-1=0,$$

that the vectors $(1,0,0)$ and $(1,-4,-1)$ are orthogonal with respect to this scalar product.

3. Here we have several possibilities, because we may start from both 1) and 2), and then of course also approach the solution quite differently.

From 1) we get the eigenvectors

$(1,-1,1) \qquad \text{for } \lambda = 3,$

$(1,1,0) \quad \text{and} \quad (1,0,1) \qquad \text{for } \lambda = 6.$

Eigenvectors corresponding to different eigenvalues are orthogonal, which can also be seen from

$$(1\ \ -1\ \ 1)\begin{pmatrix} 5 & 1 & 1 \\ 1 & 5 & -1 \\ 1 & -1 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 6(1,-1,1)\cdot(1,1,0)=0,$$

$$(1\ \ -1\ \ 1)\begin{pmatrix} 5 & 1 & 1 \\ 1 & 5 & -1 \\ 1 & -1 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 6(1,-1,1)\cdot(1,0,1)=0,$$

where $\cdot$ denotes the usual inner product.

Furthermore,

$$(1\ \ 1\ \ 0)\begin{pmatrix} 5 & 1 & 1 \\ 1 & 5 & -1 \\ 1 & -1 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 6(1,1,0)\cdot(1,0,1)=6.$$

The squares of the lengths are

$$(1\ \ -1\ \ 1)\mathbf{A}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 3\cdot 3 = 9,$$

and

$$(1\ \ 1\ \ 0)\mathbf{A}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 6\cdot 2 = 12 = (1\ \ 0\ \ 1)\mathbf{A}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},$$

hence

$\dfrac{1}{3}(1,-1,1)$ is normed,

$\dfrac{1}{2\sqrt{3}}(1,1,0)$ normed,

and

$$(1,0,1)-\frac{1}{2\sqrt{3}}(1,1,0)\mathbf{A}\begin{pmatrix}1\\1\\0\end{pmatrix}\cdot\frac{1}{2\sqrt{3}}(1,1,0)$$
$$=(1,0,1)-\frac{1}{12}\cdot 6\cdot(1,1,0)=(1,0,1)-\frac{1}{2}(1,1,0)=\frac{1}{2}(1,-1,2)$$

is orthogonal onto both of them. This vector is an eigenvector corresponding to the eigenvalue 6, hence the square of the length of $(1,-1,2)$ is given by

$$(1,-1,2)\mathbf{A}\begin{pmatrix}1\\-1\\2\end{pmatrix}=6\cdot\{1+1+4\}=36,$$

and an orthonormal basis is

$$\frac{1}{3}(1,-1,1),\qquad \frac{1}{2\sqrt{3}}(1,1,0),\qquad \frac{1}{6}(1,-1,2).$$

If we instead apply 2), then we know that $(1,0,0)$ and $(1,-4,-1)$ are orthogonal. The squares of the lengths are

$$(1,0,0)\mathbf{A}\begin{pmatrix}1\\0\\0\end{pmatrix}=5,$$

and

$$(1,-4,-1)\begin{pmatrix}5&1&1\\1&5&-1\\1&-1&5\end{pmatrix}\begin{pmatrix}1\\-4\\-1\end{pmatrix}=(0,-18,0)\cdot(1,-4,-1)=72,$$

hence the normed basis vectors are

$$\frac{1}{\sqrt{5}}(1,0,0)\qquad\text{and}\qquad\frac{1}{6\sqrt{2}}(1,-4,-1).$$

Any vector, which is perpendicular to these two vectors with respect to n$\mathbf{A}$, must in the usual Euclidean space be orthogonal to

$$(5,1,1)\qquad\text{and}\qquad(0,-18,0),$$

thus $x_2=0$ and $x_3=-5x_1$.

The square of the length of $(1,0,-5)$ is

$$(1,0,5)\begin{pmatrix}5&1&1\\1&5&-1\\1&-1&5\end{pmatrix}\begin{pmatrix}1\\0\\-5\end{pmatrix}=(0,6,-24)\cdot(1,0,-5)=120,$$

so a unit vector is

$$\frac{1}{2\sqrt{30}}(1,0,-5).$$

We obtain by this construction the orthonormal basen

$$\frac{1}{\sqrt{5}}(1,0,0), \quad \frac{1}{6\sqrt{2}}(1,-4,-1), \quad \frac{1}{2\sqrt{30}}(1,0,-5).$$

Example 3.9 *A real function* $g : \mathbb{R}^{3\times 3} \times \mathbb{R}^{3\times 3} \to \mathbb{R}$ *is given by*

$$g(\mathbf{A},\mathbf{B}) = \ \mathrm{trace}(\mathbf{A}^T\mathbf{B}).$$

Prove that g *is a scalar product in* $\mathbb{R}^{3\times 3}$.

Clearly, g is bilinear. Furthermore, it follows from

$$g(\mathbf{B},\mathbf{A}) = \ \mathrm{trace}(\mathbf{B}^T\mathbf{A}) = \ \mathrm{trace}((\mathbf{B}^T\mathbf{A})^T) = \ \mathrm{trace}(\mathbf{A}^T\mathbf{B}) = g(\mathbf{A},\mathbf{B}),$$

that g is symmetric .

Finally,

$$\mathbf{e}_i^T \mathbf{A}^T \mathbf{A} \mathbf{e}_i = (\mathbf{A}\mathbf{e}_i)^T (\mathbf{A}\mathbf{e}_i) = c_{ii} \geq 0$$

is the i-th diagonal element of $\mathbf{A}^T\mathbf{A}$, so

$$g(\mathbf{A},\mathbf{A}) = \text{ trace}(\mathbf{A}^T\mathbf{A}) = \sum_{i=1}^{3} c_i i \geq 0.$$

If $\text{trace}(\mathbf{A}^T\mathbf{A}) = 0$, then every $c_{ii} = 0$, hence $\mathbf{A}\mathbf{e}_i = \mathbf{0}$, $i = 1, 2, 3$. Since $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is a basis of $\mathbb{R}^3$, we have $\mathbf{A}\mathbf{v} = \mathbf{0}$ for ever $\mathbf{v} \in \mathbb{R}^3$, i.e. $\mathbf{A} = \mathbf{0}$.

Thus we have proved that g is a scalar product.

Example 3.10 *Consider the bilinear function $g : \mathbb{P}_2(\mathbb{R}) \times \mathbb{P}_2(\mathbb{R}) \to \mathbb{R}$, given by*

$$g(P,Q) = P(0)Q(0) + \int_0^1 P'(x)Q'(x)\,dx.$$

1. *Find the matrix description of g with respect to the basis of monomials $(1, x, x^2)$.*
2. *Prove that g defines a scalar product in $\mathbb{P}_2(\mathbb{R})$.*

1. Compute successively

$$g(1,1) = 1, \qquad g(1,x) = 0, \qquad g(1,x^2) = 0,$$
$$g(x,x) = \int_0^1 1\,dx = 1, \qquad g(x,x^2) = \int_0^1 1 \cdot 2x\,dx = 1,$$
$$g(x^2,x^2) = \int_0^1 2x \cdot 2x\,dx = \int_0^1 4x^2\,dx = \frac{4}{3}.$$

The corresponding matrix is

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & \frac{4}{3} \end{pmatrix}.$$

2. Since $\mathbf{A}$ is symmetric, we shall only prove that all eigenvalues are positive. The characteristic polynomial is

$$\begin{aligned} -(\lambda-1)\begin{vmatrix} 1-\lambda & 1 \\ 1 & \frac{4}{3}-\lambda \end{vmatrix} &= -(\lambda-1)\left\{(\lambda-1)\left(\lambda-\frac{4}{3}\right)-1\right\} \\ &= -(\lambda-1)\left\{\lambda^2-\frac{7}{3}\lambda+\frac{4}{3}-1\right\} \\ &= (\lambda-1)\left\{\lambda^2-\frac{7}{3}\lambda+\frac{1}{3}\right\}. \end{aligned}$$

We infer that $\lambda = 1$ and $\lambda_2 + \lambda_3 = \dfrac{7}{3} > 0$ and $\lambda_2 \cdot \lambda_3 = \dfrac{1}{3} > 0$, thus all three eigenvalues are positive, and g defines a scalar product in $\mathbb{P}_2(\mathbb{R})$.

Remark 3.1 This is a proof in the spirit of Linear Algebra. A proof in Calculus would be given in the following way:

Clearly, g is bilinear and symmetric, and

$$g(P,P) = P(0)^2 + \int_0^1 P'(x)^2\,dx \geq 0.$$

Now, if $g(P,P) = 0$, then

$$P(0) = 0 \qquad \text{and} \qquad \int_0^1 P'(x)^2\,dx = 0.$$

It follows from the latter condition that $P'(x) = 0$, hence

$$P(x) = P(0) + \int_0^x P'(t)\,dt = 0 + 0 = 0,$$

and we have proved that g defines a scalar product in $\mathbb{P}_2(\mathbb{R})$ (and even in $C^0([0,1])$). $\Diamond$

Example 3.11 *Let the function $g : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be given by*

$$g(\mathbf{x},\mathbf{y}) = x_1y_1 + 4x_2y_2 + x_1y_2 + x_2y_1,$$

where $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$.
Prove that the function g defines a scalar product in $\mathbb{R}^2$.
In the following we equip $\mathbb{R}^2$ with this scalar product.
Find the lengths of the vectors $\mathbf{v}$ and $\mathbf{w}$ given ny

$$\mathbf{v} = (2,1) \qquad \textit{and} \qquad \mathbf{w} = (-1,1).$$

Find the angle between the vectors $\mathbf{v}$ and $\mathbf{w}$.

We write in matrix form

$$g(\mathbf{x},\mathbf{y}) = (x_1 \;\; x_2)\begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \mathbf{x}^T\mathbf{A}\mathbf{y}.$$

The characteristic polynomial of $\mathbf{A}$,

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 1-\lambda & 1 \\ 1 & 4-\lambda \end{vmatrix} = (\lambda-1)(\lambda-4) - 1 = \lambda^2 - 5\lambda + 3$$

has the positive roots λ_1, λ_2, because $\lambda_1 + \lambda_2 = 5$ and $\lambda_1 \cdot \lambda_2 = 3$, (in fact,

$$\lambda = \frac{5}{2} \pm \frac{1}{2}\sqrt{13} > 0, \Big)$$

so the eigenvalues are positive, and thus $g(\mathbf{x},\mathbf{y})$ is a scalar product.

Then

$$\|v\|^2 = (2 \;\; 1)\begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = (3 \;\; 6)\begin{pmatrix} 2 \\ 1 \end{pmatrix} = 6 + 6 = 12,$$

$$\|w\|^2 = (-1 \;\; 1)\begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = (0 \;\; 3)\begin{pmatrix} -1 \\ 1 \end{pmatrix} = 3,$$

so the lengths become

$$\|v\| = 2\sqrt{3} \qquad \text{and} \qquad \|w\| = \sqrt{3}.$$

Finally,

$$\begin{aligned} g(\mathbf{v},\mathbf{w}) &= \|v\| \cdot \|w\| \cdot \cos(\angle(\mathbf{v},\mathbf{w})) = 6\cos(\angle(\mathbf{v},\mathbf{w})) \\ &= (2 \;\; 1)\begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = (3 \;\; 6)\begin{pmatrix} -1 \\ 1 \end{pmatrix} = 3, \end{aligned}$$

thus

$$\cos(\angle(\mathbf{v},\mathbf{w})) = \frac{1}{2}, \qquad \text{dvs. } \angle(\mathbf{v},\mathbf{w}) = \frac{\pi}{3}.$$

Example 3.12 *A mapping* $g : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ *is given by*

$$g(\mathbf{x}, \mathbf{y}) = 4x_1y_1 - x_1y_2 - x_2y_1 + 2x_2y_2,$$

where $\mathbf{x} = (x_1, x_2)$ *and* $\mathbf{y} = (y_1, y_2)$.
Check, if g *is a scalar product of* $\mathbb{R}^2$.

The corresponding symmetric matrix is

$$\mathbf{A} = \begin{pmatrix} 4 & -1 \\ -1 & 2 \end{pmatrix}$$

with its characteristic polynomial

$$\begin{aligned} \det(\mathbf{A} - \lambda\mathbf{I}) &= \begin{vmatrix} 4-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = (\lambda-4)(\lambda-2) - 1 \\ &= \lambda^2 - 6\lambda + 7 = (\lambda-3)^2 - 2 \end{aligned}$$

and the two positive eigenvalues $3 \pm \sqrt{2}$. Hence we infer that g determines a scalar product in $\mathbb{R}^2$.

Example 3.13 *A bilinear function* $g : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ *is given by*

$$g(\mathbf{x}, \mathbf{y}) = (x_1 \;\; x_2 \;\; x_3) \begin{pmatrix} 4 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 7 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix},$$

where $\mathbf{x} = (x_1, x_2, x_3)$ *and* $\mathbf{y} = (y_1, y_2, y_3)$.

1. *Prove that the function* g *defines a scalar product in* $\mathbb{R}^3$.
   In the following we equip $\mathbb{R}^3$ *with this scalar product.*
2. *Find the lengths of* $\mathbf{v} = (1, -1, 1)$ *and* $\mathbf{w} = (-1, 2, 1)$, *and find* $\cos(\angle(\mathbf{v}, \mathbf{w}))$.
3. *Find a proper vector* $\mathbf{u}$, *which is orthogonal to both* $\mathbf{v}$ *and* $\mathbf{w}$.

1. The matrix $\mathbf{A}$ is symmetric and has the characteristic polynomial

$$\begin{aligned} \det(\mathbf{A} - \lambda\mathbf{I}) &= \begin{vmatrix} 4-\lambda & -1 & 0 \\ -1 & 3-\lambda & 0 \\ 0 & 0 & 7-\lambda \end{vmatrix} = -(\lambda-7)\{(\lambda-4)(\lambda-3) - 1\} \\ &= -(\lambda-7)\{\lambda^2 - 7\lambda + 11\} \\ &= -(\lambda-7)\left\{\left(\lambda - \frac{7}{2}\right)^2 + 11 - \frac{49}{4}\right\} \\ &= -(\lambda-7)\left\{\left(\lambda - \frac{7}{2}\right)^2 - \frac{5}{2}\right\}. \end{aligned}$$

All three roots, $\lambda = 7$ and $\lambda = \dfrac{7}{2} \pm \dfrac{\sqrt{5}}{2}$, are positive, so $g(\mathbf{x}, \mathbf{y})$ defines a scalar product in $\mathbb{R}^3$.

2. We compute

$$\|\mathbf{v}\|^2 = (1,-1,1)\begin{pmatrix} 4 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 7 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = (5,-4,7)\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 16$$

and

$$\|\mathbf{w}\|^2 = (-1,2,1)\begin{pmatrix} 4 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 7 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = (-6,7,7)\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = 27,$$

hence

$$\|\mathbf{v}\| = 4 \qquad \text{and} \qquad \|\mathbf{w}\| = 3\sqrt{3}.$$

Furthermore,

$$\begin{aligned} g(\mathbf{v},\mathbf{w}) &= \|\mathbf{v}\| \cdot \|\mathbf{w}\| \cdot \cos(\angle(\mathbf{v},\mathbf{w})) = 12\sqrt{3}\cos(\angle(\mathbf{v},\mathbf{w})) \\ &= (1,-1,1)\begin{pmatrix} 4 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 7 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = (5,-4,7)\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = -6, \end{aligned}$$

hence

$$\cos(\angle(\mathbf{v},\mathbf{w})) = \frac{-6}{12\sqrt{3}} = -\frac{\sqrt{3}}{6},$$

and whence

$$\angle(\mathbf{v},\mathbf{w} = \arccos\left(-\frac{\sqrt{3}}{6}\right).$$

3. A proper vector, which is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$, must in the usual coordinates fulfil

$$(5,-4,7)\cdot\mathbf{x} = 5x_1 - 4x_2 + 7x_3 = 0,$$

$$(-6,7,7)\cdot\mathbf{x} = -6x_1 + 7x_2 + 7x_3 = 0.$$

We reduce

$$\begin{pmatrix} 5 & -4 & 7 \\ -6 & 7 & 7 \end{pmatrix} \sim \begin{pmatrix} 5 & -4 & 7 \\ 11 & -11 & 0 \end{pmatrix} \sim \begin{pmatrix} 5 & -4 & 7 \\ 1 & -1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 7 \\ 1 & -1 & 0 \end{pmatrix}.$$

Hence a possible vector is $(7,7,-1)$, and we get them all by taking scalar multiples of this one.

Example 3.14 *Let* $\mathbf{x} = (x_1, x_2)$ *and* $\mathbf{y} = (y_1, y_2)$ *be any vectors in* $\mathbb{R}^2$.

1. *Prove that*

$$g(\mathbf{x}, \mathbf{y}) = (x_1 \;\; x_2) \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

defines a scalar product g *in* $\mathbb{R}^2$.

2. *Prove that the vectors*

$$\mathbf{a}_1 = (4,7) \qquad \textit{and} \qquad \mathbf{a}_2 = (-6,2)$$

are orthogonal with respect to this scalar product g.

3. *Find an orthonormal basis of* $\mathbb{R}^2$ *with respect to the scalar product* g.

1. The matrix $\mathbf{A}$ is symmetric and its characteristic polynomial is

$$\begin{aligned}\det(\mathbf{A} - \lambda \mathbf{I}) &= (2-\lambda)(1-\lambda) - 1 = \lambda^2 - 3\lambda + 1 \\ &= \left(\lambda - \frac{3}{2}\right)^2 + 1 - \frac{9}{4} = \left(\lambda - \frac{3}{2}\right)^2 - \frac{5}{4}.\end{aligned}$$

The eigenvalues are $\lambda = \frac{3}{2} \pm \frac{\sqrt{5}}{2} > 0$, from which follows that they are positive, thus g is a scalar product .

2. It follows by insertion that

$$g(\mathbf{a}_1, \mathbf{a}_2) = (4,7) \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} -6 \\ 2 \end{pmatrix} = (1,3) \begin{pmatrix} -6 \\ 2 \end{pmatrix} = 0,$$

hence $(4,7)$ and $(-6,2)$ are orthogonal with respect to g.

3. It follows from

$$g(\mathbf{a}_1, \mathbf{a}_1) = (4,7) \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 7 \end{pmatrix} = (1,3) \begin{pmatrix} 4 \\ 7 \end{pmatrix} = 25$$

and

$$g(\mathbf{a}_2, \mathbf{a}_2) = (-6,2) \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} -6 \\ 2 \end{pmatrix} = (-14,8) \begin{pmatrix} -6 \\ 2 \end{pmatrix} = 100,$$

that

$$\|\mathbf{a}_1\| = \sqrt{g(\mathbf{a}_1, \mathbf{a}_1)} = 5 \quad \text{and} \quad \|\mathbf{a}_2\| = \sqrt{g(\mathbf{a}_2, \mathbf{a}_2)} = 10.$$

An orthonormal basis is e.g.

$$\frac{1}{\|\mathbf{a}_1\|}\mathbf{a}_1 = \left(\frac{4}{5}, \frac{7}{5}\right) \quad \text{and} \quad \frac{1}{\|\mathbf{a}_2\|}\mathbf{a}_2 = \left(-\frac{6}{10}, \frac{2}{10}\right) = \left(-\frac{3}{5}, \frac{1}{5}\right).$$

Example 3.15 *Let* $\mathbf{x} = (x_1, x_2)$ *and* $\mathbf{y} = (y_1, y_2)$ *be given vectors from* $\mathbb{R}^2$. *Let a scalar product in* $\mathbb{R}^2$ *be given by*

$$g(\mathbf{x}, \mathbf{y}) = (x_1 \;\; x_2) \begin{pmatrix} 4 & 5 \\ 5 & 9 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

1. *Find the length of* $\mathbf{e}_1 = (1, 0)$ *with respect to the scalar product* g.
2. *Find every vector in* $\mathbb{R}^2$, *which is orthogonal to* $\mathbf{e}_1$ *with respect to the scalar product* g.
3. *Find an orthonormal basis for* $\mathbb{R}^2$ *with respect to the scalar product* g.

Since $\mathbf{A}$ is symmetric with the characteristic polynomial

$$\begin{aligned} \det(\mathbf{A} - \lambda\mathbf{I}) &= (\lambda - 4)(\lambda - 9) - 25 = \lambda^2 - 13\lambda + 11 \\ &= \left(\lambda - \frac{13}{2}\right)^2 + 11 - \frac{169}{4} = \left(\lambda - \frac{13}{2}\right)^2 - \frac{79}{2}, \end{aligned}$$

which clearly has two real roots λ_1 and λ_2, where $\lambda_1 + \lambda_2 = 13$ and $\lambda_1 \cdot \lambda_2 = 11$, thus $\lambda_1 > 0$ and $\lambda_2 > 0$, so g is a scalar product.

1. It follows by insertion that

$$g(\mathbf{e}_1, \mathbf{e}_1) = 4, \qquad \text{så } \|\mathbf{e}_1\| = \sqrt{g(\mathbf{e}_1, \mathbf{e}_2)} = 2.$$

2. It follows from

$$g(\mathbf{e}_1,\mathbf{x}) = (1,0)\begin{pmatrix} 4 & 5 \\ 5 & 9 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 4x_1 + 5x_2 = 0$$

for $\mathbf{x} = s(5,-4)$, $s \in \mathbb{R}$, that we have found every vector, which is orthogonal to $(1,0)$ with respect to g.

3. It follows from

$$g((5,-4),(5,-4)) = (5,-4)\begin{pmatrix} 4 & 5 \\ 4 & 9 \end{pmatrix}\begin{pmatrix} 5 \\ -4 \end{pmatrix} = (0,-11)\begin{pmatrix} 5 \\ -4 \end{pmatrix} = 44,$$

that

$$\|(5,-4)\| = \sqrt{g((5,-4),(5,-4))} = 2\sqrt{11}.$$

Then an orthonormal basis for $\mathbb{R}^2$ with respect to g is

$$\frac{1}{2}(1,0) \qquad \text{and} \qquad \frac{1}{2\sqrt{11}}(5,-4).$$

Example 3.16 *Given in $\mathbb{R}^2$ for every real a a bilinear function g_a by*

$$g_a(\mathbf{x},\mathbf{y}) = 10x_1y_1 + a^2x_1y_2 + 4x_2y_1 - 2ax_2y_2,$$

where $\mathbf{x} = (x_1,x_2)$ and $\mathbf{y} = (y_1,y_2)$.

1. *Find the matrix equation for g_a with respect to the usual basis in $\mathbb{R}^2$.*
2. *Find the a, for which g_a isn symmetric.*
3. *Prove that g_a is a scalar product in $\mathbb{R}^2$, if and only if $a = -2$.*
4. *Find the lenths of and the angle between the vectors $\mathbf{u} = (4,1)$ and $\mathbf{v} = (4,-3)$, when $\mathbb{R}^2$ is given the scalar product g_{-2}.*

1. It follows immediately that

$$g_a(\mathbf{x},\mathbf{y}) = (x_1 \;\; x_2)\begin{pmatrix} 10 & a^2 \\ 4 & -2a \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

2. The symmetry requires that $a^2 = 4$, so g_a is symmetric for $a = \pm 2$.
3. If $a = 2$ then we get the characteristic polynomial

$$\begin{aligned}\det(\mathbf{A} - \lambda\mathbf{I}) &= (10-\lambda)(-4-\lambda) - 16 = \lambda^2 - 6\lambda - 56 \\ &= (\lambda-3)^2 - 65.\end{aligned}$$

Since $\sqrt{65} > 3$, it follows that $\mathbf{A}$ a negative eigenvalue, hence g_2 is not a scalar product.

Alternatively, the product of the roots -56 is negative, so there is one positive and one negative root. $\Diamond$

If $a = -2$, then we have the characteristic polynomial

$$\begin{aligned}\det(\mathbf{A} - \lambda\mathbf{I}) &= (10-\lambda)(4-\lambda) - 16 = \lambda^2 - 14\lambda + 24 \\ &= (\lambda - 2)(\lambda - 12).\end{aligned}$$

The eigenvalues 2 and 12 are both positive, hence g_{-2} is a scalar product.

4. It follows by a simple computation that

$$\|\mathbf{u}\|^2 = g_{-2}(\mathbf{u},\mathbf{u}) = (4,1)\begin{pmatrix} 10 & 4 \\ 4 & 4 \end{pmatrix}\begin{pmatrix} 4 \\ 1 \end{pmatrix} = (44,20)\begin{pmatrix} 4 \\ 1 \end{pmatrix} = 196 = 14^2,$$

and

$$\|\mathbf{v}\|^2 = g_{-2}(\mathbf{v},\mathbf{v}) = (4,-3)\begin{pmatrix} 10 & 4 \\ 4 & 4 \end{pmatrix}\begin{pmatrix} 4 \\ -3 \end{pmatrix} = (28,4)\begin{pmatrix} 4 \\ -3 \end{pmatrix} = 100 = 10^2,$$

hence $\|\mathbf{u}\| = 14$ and $\|\mathbf{v}\| = 10$.

Finally,

$$\begin{aligned} g_{-2}(\mathbf{u},\mathbf{v}) &= (4,1)\begin{pmatrix} 10 & 4 \\ 4 & 4 \end{pmatrix}\begin{pmatrix} 4 \\ -3 \end{pmatrix} = (4,1)\begin{pmatrix} 28 \\ 4 \end{pmatrix} = 116 \\ &= \|\mathbf{u}\|\cdot\|\mathbf{v}\|\cos(\angle(\mathbf{u},\mathbf{v})) = 14\cdot 10\cdot\cos(\angle(\mathbf{u},\mathbf{v})), \end{aligned}$$

hence

$$\cos(\angle(\mathbf{u},\mathbf{v})) = \frac{116}{140} = \frac{29}{35},$$

and whence

$$\angle(\mathbf{u},\mathbf{v}) = \arccos\left(\frac{29}{35}\right).$$

Example 3.17 *Let $g : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ be the bilinear function, which is given by*

$$g(\mathbf{x},\mathbf{y}) = \mathbf{x}^T\mathbf{G}\mathbf{y},$$

where

$$\mathbf{G} = \begin{pmatrix} 1 & 0 & a \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix},$$

and where $\mathbf{x}$ and $\mathbf{y}$ also denote the coordinate matrices of $\mathbf{x}$ and $\mathbf{y}$ with respect to the usual basis of $\mathbb{R}^3$, and where a is a real constant.

1. *Prove that g is only a scalar product in the vector space $\mathbb{R}^3$ for $a = 1$.*

   We choose in the following $a = 1$.

2. *Prove that $\mathbf{u} = (4, 1, -4)$ and $\mathbf{v}_c = (c, 2, 1)$ are orthogonal with respect to the scalar product g, no matter the choice of c.*

3. *Find with respect to the scalar product g an orthogonal basis of $\mathbb{R}^3$ consisting of the vectors $\mathbf{u}$, $\mathbf{v}_0$ and $\mathbf{v}_c$ for a convenient choice of c.*

1. Since $\mathbf{G}$ is only symmetric for $a = 1$, this is the only possibility. Then we get the characteristic polynomial:

$$\begin{aligned} \det(\mathbf{G}_1 - \lambda\mathbf{I}) &= \begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix} = -(\lambda-2)\begin{vmatrix} 1-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} \\ &= -(\lambda-2)\{\lambda^2 - 3\lambda + 1\} = -(\lambda-2)\left\{\left(\lambda - \frac{3}{2}\right)^2 - \frac{5}{4}\right\}. \end{aligned}$$

Since $\sqrt{5} < 3$, all eigenvalues are positive, thus g is a scalar product for $a = 1$.

2. We get by a computation,

$$g(\mathbf{u},\mathbf{v}_c)=(4,1,-4)\begin{pmatrix}1&0&1\\0&2&0\\1&0&2\end{pmatrix}\begin{pmatrix}c\\2\\1\end{pmatrix}=(0,2,-4)\begin{pmatrix}c\\2\\1\end{pmatrix}=4-4=0,$$

   and the claim is proved.

3. Since $g(\mathbf{u},\mathbf{v}_0)=0=g(\mathbf{u},\mathbf{v}_c)$, we shall only find c, such that

$$0=g(\mathbf{v}_0,\mathbf{v}_c)=(0,2,1)\begin{pmatrix}1&0&1\\0&2&0\\1&0&2\end{pmatrix}\begin{pmatrix}c\\2\\1\end{pmatrix}=(1,4,2)\begin{pmatrix}c\\2\\a\end{pmatrix}=c+10.$$

   This equation is fulfilled, if and only if $c=-10$, hence an orthogonal basis is

$$\mathbf{u}=(4,1,-4),\quad \mathbf{v}_0=(0,2,1),\quad \mathbf{v}_{-10}=(-10,2,1).$$

Example 3.18 *Let the vector space $\mathbb{P}_2(\mathbb{R})$ be given the scalar product*

$$g(P(x),Q(x))=\int_{-1}^{1}P(x)Q(x)\,dx.$$

Given two orthogonal polynomials $P_1(x)=x$ and $P_2(x)=x^2$. Let

$$U=\ \operatorname{span}\{P_1(x),P_2(x)\}.$$

1. Find an orthonormal basis of U.

2. Find the coordinates of $P(x)=x-x^2$ with respect to the orthonormal basis of U.

Remark 3.2 It follows from $P_1(x)P_2(x)=x\cdot x^2=x^3$ being an odd function that $P_1(x)=x$ and $P_2(x)=x^2$ are orthogonal, thus

$$g(P_1(x),P_2(x))=\int_{-1}^{1}x^3\,dx=0.$$

1. Since

$$\|P_1\|^2=\int_{-1}^{1}x^2\,dx=\frac{2}{3}\quad\text{and}\quad \|P_2\|^2=\int_{-1}^{1}x^4\,dx=\frac{2}{5},$$

   we get e.g. an orthonormal basis of U by choosing

$$\mathbf{e}_1(x)=\sqrt{\frac{3}{2}}\cdot x\quad\text{and}\quad \mathbf{e}_2(x)=\sqrt{\frac{5}{2}}\cdot x^2.$$

2. Since

$$P(x)=x-x^2=\sqrt{\frac{2}{3}}\cdot\mathbf{e}_1(x)-\sqrt{\frac{2}{5}}\cdot\mathbf{e}_2(x),$$

   the coordinates with respect to $(\mathbf{e}_1,\mathbf{e}_2)$ are given by $\left(\sqrt{\frac{2}{3}},-\sqrt{\frac{2}{5}}\right)$.

Example 3.19 *Denote by S a set of vectors in an Euclidean vector space V (of finite dimension), and let $S^\perp$ denote the set of all vectors in V, which are orthogonal to any vector in S.*

1. *Prove that $S^\perp$ is a subspace of V. Let $U(S)$ denote the span of S,- Prove that*

$$U(S) \subseteq S^{\perp\perp} = \left(S^\perp\right)^\perp.$$

2. *If S also is a subspace, we call $S^{\perp\perp}$ the orthogonal complement of S. Prove in that case that*

$$V = S \oplus S^\perp \qquad \textit{and} \qquad S = S^{\perp\perp}.$$

1. Let $\mathbf{u}$, $\mathbf{v} \in S^\perp$ and $\lambda \in \mathbb{R}$.
   Let $g(\cdot,\cdot)$denote the inner product of V. Then we get for every $\mathbf{w} \in S$ that

$$g(\mathbf{w}, \mathbf{u} + \lambda\mathbf{v}) = g(\mathbf{w}, \mathbf{u}) + \lambda g(\mathbf{w}, \mathbf{v}) = 0 + \lambda \cdot 0 = 0,$$

   which shows that $\mathbf{u} + \lambda\mathbf{v} \in S^\perp$, thus $S^\perp$ is a subspace of V.

   Any element of the span $U(S)$ can be written

$$\mathbf{u} = \sum_{k=1}^n \lambda_k \mathbf{u}_k \in U(S), \quad \text{where } \mathbf{u}_k \in S \text{ and } \lambda_k \in \mathbb{R}, \;\; k = 1, \dots, n.$$

   If $\mathbf{v} \in S^\perp$, then

$$g(\mathbf{u}, \mathbf{v}) = g\left(\sum_{k=1}^n \lambda_k \mathbf{u}_k, \mathbf{v}\right) = \sum_{k=1}^n \lambda_k g(\mathbf{u}_k, \mathbf{v}) = 0,$$

   proving that $\mathbf{u} \in S^{\perp\perp}$, hence $U(S) \subseteq S^{\perp\perp}$.

2. Then assume that S is a subspace. Then $U(S) = S$, hence $S \subseteq S^{\perp\perp}$ according to 1).

   It is obvious that if $\mathbf{u} \in S \cap S^\perp$, then in particular,

$$\|\mathbf{u}\|^2 = g(\mathbf{u}, \mathbf{u}) = 0,$$

   hence $\mathbf{u} = \mathbf{0}$, and whence $S \cap S^\perp = \{\mathbf{0}\}$.

   The last claim follows from that since V is an Euclidean space, we can choose an orthonormal basis of S and supply it to an orthonormal basis of all of V. In this way we get a basis of $S^\perp$, hence $V = S \oplus S^\perp$.

   The same argument also gives

$$V = S^\perp \oplus S^{\perp\perp} = S^{\perp\perp} \oplus S^\perp.$$

   Now, $S = U(S) \subseteq S^{\perp\perp}$, and S and $^{\perp\perp}$ have the same dimension (just count the elements of the basis), hence $S = S^{\perp\perp}$.

Example 3.20 *Consider in $\mathbb{R}^3$ the usual scalar product. Find an orthonormal basis of the space of solutions of the homogeneous linear equation*

$$x_1 + x_2 + x_3 = 0.$$

The space of solutions is represented by a plane through $\mathbf{0}$ with the normal vector $(1,1,1)$, thus $(1,1,1)$ is perpendicular to all elements $\mathbf{x}$ in the space of solutions:

$$(1,1,1) \cdot (x_1, x_2, x_3) = x_1 + x_2 + x_3 = 0.$$

Two linearly independent vectors, satisfying this requirement, can e.g. be chosen as

$$\mathbf{e}_1 = \frac{1}{\sqrt{2}}(1,-1,0) \quad \text{and} \quad \mathbf{u}_2 = (1,0,-1),$$

where we have normed $\mathbf{e}_1$. It follows from $\mathbf{e}_1 \cdot u_2 = \dfrac{1}{\sqrt{2}}$ that

$$\mathbf{v}_2 = \mathbf{u}_2 - (\mathbf{e}_1 \cdot \mathbf{u}_2)\mathbf{e}_1 = (1,0,-1) - \frac{1}{\sqrt{2}}(1,-1,0) = \left(1 - \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -1\right)$$

is orthogonal on both $(1,1,1)$ and $\mathbf{e}_1$. Then from

$$\|\mathbf{v}_2\|^2 = \left(1-\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + 1 = 1-\sqrt{2}+\frac{1}{2}+\frac{1}{2}+1 = 3-\sqrt{2},$$

we get

$$\begin{aligned}\mathbf{e}_2 &= \frac{1}{3-\sqrt{2}}\left(1-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},-1\right) = \frac{3+\sqrt{2}}{7}\left(\frac{2-\sqrt{2}}{2},\frac{\sqrt{2}}{2},-1\right)\\ &= \frac{1}{14}(4-\sqrt{2},3\sqrt{2}+2,-6-2\sqrt{2}),\end{aligned}$$

and $\{\mathbf{e}_1,\mathbf{e}_2\}$ forms an orthonormal basis of the set of solutions.

Example 3.21 *Consider the usual scalar product in* $\mathbb{R}^4$.
Find an orthonormal basis of the subspace of $\mathbb{R}^4$*, which is spanned by* $(1,0,2,2)$ *and* $(2,-1,6,2)$.

It follows from

$$\|(1,0,2,2)\|^2 = 1^2+0^2+2^2+2^2 = 1+4+4 = 9 = 3^2,$$

that

$$\mathbf{e}_1 = \frac{1}{3}(1,0,2,2)$$

is a normed vector.

Now,

$$\mathbf{e}_1\cdot(2,-1,6,2) = \frac{1}{3}\{2+0+12+4\} = 6,$$

so an orthogonal vector is

$$\begin{aligned}\mathbf{u}_2 &= (2,-1,6,2)-\{\mathbf{e}_1\cdot(2,-1,6,2)\}\mathbf{e}_1 = (2,-1,6,2)-6\mathbf{e}_1\\ &= (2,-1,6,2)-(2,0,4,4) = (0,-1,2,-2).\end{aligned}$$

We get from $\|\mathbf{u}_2\|^2 = 1+4+4 = 9$ that

$$\mathbf{e}_2 = \frac{1}{3}\,\mathbf{u}_2 = \frac{1}{3}(0,-1,2,-2)$$

is an orthonormal vector of $\mathbf{e}_1$, thus

$$\{\mathbf{e}_1,\mathbf{e}_2\} = \left\{\frac{1}{3}(1,0,2,2),\frac{1}{3}(0,-1,2,-2)\right\}$$

is an orthonormal basis of the subspace.

Example 3.22 *Given in* $\mathbb{R}^3$ *the vectors* $\mathbf{x}_1 = (1,2,2)$, $\mathbf{x}_2 = (1,0,-2)$ *and* $\mathbf{x}_3 = (0,-1,-2)$ *and* $\mathbf{y}_1 = (1,1,0)$ *and* $\mathbf{y}_2 = (2,3,2)$. *Let* $U = \operatorname{span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$.

1. *Find the dimension of* U *and a basis of* U.
2. *Find an orthonormal basis of* U *with respect to the usual scalar product in* $\mathbb{R}^3$.
3. *Show that* $\operatorname{span}\{\mathbf{y}_1, \mathbf{y}_2\} = U$.

1. Clearly, $2 \leq \dim U \leq 3$.

   Now, $\mathbf{x}_1 - \mathbf{x}_2 + 2\mathbf{x}_3 = \mathbf{0}$, so $\dim U = 2$, and a basis is e.g. $\{\mathbf{x}, \mathbf{x}_2\}$, because these vectors are linearly independent.

2. Then $\|\mathbf{x}_1\|^2 = 1^2 + 2^2 + 2^2 = 9 = 3^2$, so

   $$\mathbf{e}_1 = \frac{1}{3}\,\mathbf{x}_1 = \frac{1}{3}(1,2,2)$$

   is a normed vector. Furthermore,

   $$\mathbf{e}_1 \cdot \mathbf{x}_2 = \frac{1}{3}\,(1,2,2)\cdot(1,0,-2) = \frac{1}{3}\{1-4\} = -1,$$

   so

   $$\begin{aligned} \mathbf{x}_2 - (\mathbf{e}_1 \cdot \mathbf{x}_2)\mathbf{e}_1 &= (1,0,-2) - (-\mathbf{e}_1) = (1,0,-2) + \frac{1}{3}\,(1,2,2) \\ &= \frac{1}{3}\,(4,2,-4) = \frac{2}{3}\,(2,1,-2) \end{aligned}$$

   lies in U and is orthogonal to $\mathbf{e}_1$. Then it followes by norming that

   $$\{\mathbf{e}_1, \mathbf{e}_2\} = \left\{\mathbf{e}_1, \frac{1}{2}(\mathbf{x}_2 - (\mathbf{e}_1 \cdot \mathbf{x}_2)\mathbf{e}_1)\right\} = \left\{\frac{1}{3}\,(1,2,2), \frac{1}{3}(2,1,-2)\right\}$$

   is an orthonormal basis of U.

3. It follows from

   $$\mathbf{y}_1 \cdot \mathbf{e}_1 = (1,1,0)\cdot\frac{1}{3}(1,2,2) = \frac{1}{3}\{1+2+0\} = 1$$

   and

   $$\mathbf{y}_1 \cdot \mathbf{e}_2 = (1,1,0)\cdot\frac{1}{3}(2,1,-2) = \frac{1}{3}\{2+1+0\} = 1,$$

   and

   $$\begin{aligned} \mathbf{y}_1 - (\mathbf{y}_1 \cdot \mathbf{e}_1)\mathbf{e}_1 - (\mathbf{y}_1 \cdot \mathbf{e}_2)\mathbf{e}_2 &= \mathbf{y}_1 - \{\mathbf{e}_1 + \mathbf{e}_2\} \\ &= (1,1,0) - \frac{1}{3}(1+2, 2+1, 2-2) \\ &= (1,1,0) - \frac{1}{3}(3,3,0) = \mathbf{0}, \end{aligned}$$

that $\mathbf{y}_1$ lies in U.

It follows from

$$\mathbf{y}_2 \cdot \mathbf{e}_1 = (2,3,2) \cdot \frac{1}{3}(1,2,2) = \frac{1}{3}\{2+6+4\} = 4,$$

$$\mathbf{y}_2 \cdot \mathbf{e}_2 = (2,3,2) \cdot \frac{1}{3}(2,1,-2) = \frac{1}{3}\{4+3-4\} = 1,$$

and

$$\begin{aligned}\mathbf{y}_2 - (\mathbf{y}_2 \cdot \mathbf{e}_1)\mathbf{e}_1 - (\mathbf{y}_2 \cdot \mathbf{e}_2)\mathbf{e}_2 &= \mathbf{y}_2 - \{4\mathbf{e}_1 + \mathbf{e}_2\} \\ &= (2,3,2) - \frac{1}{3}(4+2, 8+1, 8-2) = (2,3,2) - \frac{1}{3}(6,9,6) = \mathbf{0},\end{aligned}$$

that $\mathbf{y}_2$ also lies in U. Since $\mathbf{y}_1$ and $\mathbf{y}_2$ clearly are linearly independent, we have $\operatorname{span}\{\mathbf{y}_1, \mathbf{y}_2\} = U$.

Example 3.23 *Consider the usual scalar product in $\mathbb{R}^4$, where are given the vectors*

$$\mathbf{u}_1(1,1,-1,-1), \qquad \mathbf{u}_2 = (1,-1,1,-1),$$

$$\mathbf{v}_1(2,-2,-2,2), \qquad \mathbf{v}_2 = (1,0,0,1).$$

Let $U = \operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2\}$ and $V = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

1. *Prove that every vector of U is orthogonal on every vector in V.*
2. *Find an orthonormal basis $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4)$ of $\mathbb{R}^4$, such that $\mathbf{a}_1$, $\mathbf{a}_2 \in U$ and $\mathbf{a}_3$, $\mathbf{a}_4 \in V$.*

1. By some straightforward computations,

$$\mathbf{u}_1 \cdot \mathbf{v}_1 = 2-2+2-2 = 0, \qquad \mathbf{u}_1 \cdot \mathbf{v}_2 = 1-1 = 0,$$

$$\mathbf{u}_2 \cdot \mathbf{v}_1 = 2+2-2-2 = 0, \qquad \mathbf{u}_2 \cdot \mathbf{v}_2 = 1-1 = 0.$$

This shows that $\mathbf{u}_i \perp \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\} = V$, and that $\mathbf{v}_i \perp \operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2\} = U$, $i = 1, 2$, thus $U \perp V$.

2. Since $\|\mathbf{u}_1\|^2 = 1+1+1+1 = 4 = 2^2$, the normed vector is then $\mathbf{a}_1 = \frac{1}{2}(1,1,-1,-1)$.

It follows from

$$\mathbf{u}_2 \cdot \mathbf{a}_1 = \frac{1}{2}(1-1-1+1) = 0 \quad \text{and} \quad \|\mathbf{u}_2\|^2 = 2,$$

that $\mathbf{a}_2 = \frac{1}{2}(1,-1,1,-1)$. Hence $\mathbf{a}_1$ and $\mathbf{a}_2$ are orthonormal, and they are also perpendicular to V.

Now, $\|\mathbf{v}_1\|^2 = 16 = 4^2$, so the normed vector is $\mathbf{a}_3 = \frac{1}{2}(1,-1,-1,1)$.

Now,

$$\mathbf{v}_2 \cdot \mathbf{a}_3 = \frac{1}{2}\{1+1\} = 1$$

and

$$\begin{aligned} \mathbf{u}_2 &= \mathbf{v}_2 - (\mathbf{v}_2 \cdot \mathbf{a}_3)\mathbf{a}_3 = \mathbf{v}_2 - \mathbf{a}_3 \\ &= (1,0,0,1) - \frac{1}{2}(1,-1,-1,1) = \frac{1}{2}(1,1,1,1) \end{aligned}$$

where $\|\mathbf{u}_2\| = 1$, so $\mathbf{a}_4 = \mathbf{u}_2$, and an orthonormal basis of the wanted type is $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$ given by

$$\left\{\frac{1}{2}(1,1,-1,-1), \frac{1}{2}(1,-1,1,-1), \frac{1}{2}(1,-1,-1,1), \frac{1}{2}(1,1,1,1)\right\}.$$

Example 3.24 *Given the matrix*

$$\mathbf{A} = \begin{pmatrix} 0 & -a & 0 & a \\ a & 0 & a & 0 \\ 0 & -a & 0 & -a \\ -a & 0 & a & 0 \end{pmatrix}, \qquad \textit{where } a \in \mathbb{R}.$$

Find a, such that $\mathbf{A}$ becomes orthogonal .

By the definition, $\mathbf{A}$ is orthogonal, if and only if its columns, considered as vectors in $\mathbb{R}$ form an orthonormal basis.
A necessary condition is that $a^2 + a^2 = 2a^2 = 1$, thus $a = \pm\dfrac{1}{\sqrt{2}}$.

On the other hand, if $a = \pm\dfrac{1}{\sqrt{2}}$, then the columns are obviously mutually orthogonal and they are all of the length 1, hence $\mathbf{A}$ is orthogonal for $a = \pm\dfrac{1}{\sqrt{2}}$.

Example 3.25 *Given the matrix*

$$\mathbf{A} = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}, \qquad \textit{where } a \in \mathbb{R}.$$

1. *Prove that $\mathbf{I}+\mathbf{A}$ is regular for every a, and find $(\mathbf{I}+\mathbf{A})^{-1}$.*
2. *Compute $\mathbf{B} = (\mathbf{I}-\mathbf{A})(\mathbf{I}+\mathbf{A})^{-1}$, and check if $\mathbf{B}$ is orthogonal .*

1. It follows from

$$\mathbf{I}+\mathbf{A} = \begin{pmatrix} 1 & a \\ -a & 1 \end{pmatrix} \qquad \text{where} \qquad \det(\mathbf{I}+\mathbf{A}) = 1 + a^2 \neq 0,$$

that $\mathbf{I}+\mathbf{A}$ is regular. Its inverse is given by

$$(\mathbf{I}+\mathbf{A})^{-1} = \frac{1}{1+a^2}\begin{pmatrix} 1 & -a \\ a & 1 \end{pmatrix} = \frac{1}{1+a^2}(\mathbf{I}-\mathbf{A}).$$

2. By a small computation,

$$\begin{aligned} \mathbf{B} &= (\mathbf{I}-\mathbf{A})(\mathbf{I}+\mathbf{A})^{-1} = \frac{1}{1+a^2}(\mathbf{I}-\mathbf{A})^2 \\ &= \frac{1}{1+a^2}\begin{pmatrix} 1-a^2 & -2a \\ 2a & 1-a^2 \end{pmatrix}. \end{aligned}$$

The two columns are clearly orthogonal, and since

$$(1-a^2)^2 + (2a)^2 = 1 - 2a^2 + a^4 + 4a^2 = 1 + 2a^2 + a^4 = (1+a^2)^2$$

is precisely the square of the denominator, $\mathbf{B}$ is orthogonal .

Example 3.26 *Given the usual scalar product in $\mathbb{R}^4$. A linear map $f : \mathbb{R}^4 \to \mathbb{R}^4$ is defined by the matrix*

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 & -1 \\ 1 & 0 & -1 & 1 \\ 1 & -1 & 0 & 1 \\ -1 & 1 & 1 & 0 \end{pmatrix}$$

with respect to the usual basis of $\mathbb{R}^4$. Prove that one can choose an orthonormal basis of $\mathbb{R}^4$, such that f in this orthonormal basis is described by a diagonal matrix. Find such a basis and the matrix of the map with respect to this basis.

The matrix $\mathbf{A}$ is seen to be symmetric. We compute the characteristic polynomial

$$\begin{aligned}
\det(\mathbf{A} - \lambda\mathbf{I}) &= \begin{vmatrix} -\lambda & 1 & 1 & -1 \\ 1 & -\lambda & -1 & 1 \\ 1 & -1 & -\lambda & 1 \\ -1 & 1 & 1 & -\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & 1-\lambda & 1-\lambda & 1-\lambda \\ 0 & 1-\lambda & 0 & 1-\lambda \\ 0 & 0 & 1-\lambda & 1-\lambda \\ -1 & 1 & 1 & -\lambda \end{vmatrix} \\
&= (1-\lambda)^3 \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ -1 & 1 & 1 & -\lambda \end{vmatrix} \\
&= (1-\lambda)^3 \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & -\lambda \end{vmatrix} + (1-\lambda)^3 \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} \\
&= (1-\lambda)^3\{-\lambda - 1 - 1 + 1 - 1 - 1\} = (\lambda - 1)^3(\lambda + 3),
\end{aligned}$$

hence the eigenvalues are $\lambda = -3$ and $\lambda = 1$ (of multiplicity 3).

Remark 3.3 We note that the sum of all four eigenvalues is $-3 + 1 + 1 + 1 = 0$, which is also equal to the trace of $\mathbf{A}$. This could have been exploited at an earlier stage, once we have noticed that $\lambda = 1$ is a triple root, so we could avoid some of the reductions above. $\Diamond$

If $\lambda = -3$, then we reduce

$$\begin{aligned}
\begin{pmatrix} 3 & 1 & 1 & -1 \\ 1 & 3 & -1 & 1 \\ 1 & -1 & 3 & 1 \\ -1 & 1 & 1 & 3 \end{pmatrix} &\sim \begin{pmatrix} 4 & 4 & 4 & 4 \\ 0 & 4 & 0 & 4 \\ 0 & 0 & 4 & 4 \\ -1 & 1 & 1 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ -1 & 1 & 1 & 3 \end{pmatrix} \\
&\sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 2 & 2 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}$$

of rank 3. Choosing $x_4 = 1$, we get the eigenvector $(1, -1, -1, 1)$ corresponding to the eigenvalue $\lambda = -3$ of length $\sqrt{4} = 2$.

If $\lambda = 1$, then we get the reduction

$$\mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix} -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} -1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

of rank 1. Two orthogonal eigenvectors are e.g. $(1,1,0,0)$ and $(0,0,1,1)$, both of length $\sqrt{2}$.

A linearly independent eigenvector is e.g. $(0,1,-1,0)$. The vector

$$\begin{aligned} &(0,1,-1,0) - \left(\frac{1}{\sqrt{2}}\right)^2 \{(0,1,-1,0)\cdot(1,1,0,0)\}(1,1,0,0) \\ &\qquad\qquad - \left(\frac{1}{\sqrt{2}}\right)^2 \{(0,1,-1,0)\cdot(0,0,1,1)\}(0,0,1,1) \\ &\quad = (0,1,-1,0) - \frac{1}{2}(1,1,0,0) + \frac{1}{2}(0,0,1,1) \\ &\quad = \frac{1}{2}(0-1+0, 2-1+0, -2-0+1, 0-0+1) \\ &\quad = \frac{1}{2}(-1,1,-1,1) \end{aligned}$$

is then perpendicular to both of them, and it is obvious that the length is 1.

An orthonormal basis is $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$, where e.g.

$$\mathbf{e}_1 = \frac{1}{\sqrt{2}}(1,1,0,0), \qquad \mathbf{e}_2 = \frac{1}{\sqrt{2}}(0,0,1,1),$$

$$\mathbf{e}_3 = \frac{1}{2}(-1,1,-1,1), \qquad \mathbf{e}_4 = \frac{1}{2}(1,-1,-1,1).$$

The corresponding matrix is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}.$$

Example 3.27 *In $\mathbb{R}^5$ we are given the vectors*

$$\mathbf{a}_1 = (0,1,2,2,0), \mathbf{a}_2 = (1,1,4,0,0), \mathbf{a}_3 = (1,2,6,2,1) \text{ and } \mathbf{a}_4 = (-1,2,2,6,-1).$$

1. *Prove that $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$ span a 3-dimensional subspace U in $\mathbb{R}^5$, and that $\mathbf{a}_4 \in U$, and write $\mathbf{a}_4$ as a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$.*
2. *Using the usual scalar product in $\mathbb{R}^5$ (and hence in particular in U), we shall prove that there exists an orthonormal basis $(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3)$ of U, such that $\mathbf{q}_1$ is proportional to $\mathbf{a}_1$, and $\mathbf{q}_2$ is proportional to $\mathbf{a}_2 - \mathbf{a}_1$. Find such a, expressed by $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$.*
3. *Let $f : U \to U$ be the linear map for which $\mathbf{a}_1$ and $\mathbf{a}_2 - \mathbf{a}_1$ are eigenvectors corresponding to the eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$, respectively, and such that $f(\mathbf{a}_3) = \mathbf{a}_4$.*

   Find the representing matrix of f with respect to the basis $(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3)$. Show that f is symmetric and isometric.

1. If $\lambda_1\mathbf{a}_1 + \lambda_2\mathbf{a}_2 + \lambda_3\mathbf{a}_3 = \mathbf{0}$, then $\lambda_3 = 0$ because of the last coordinate. This implies that $\lambda_2 = 0$ (second last coordinate) and then finally, $\lambda_1 = 0$. This proves that $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ are linearly independent, hence they span a 3-dimensional subspace U.

   Then

   $$\begin{aligned} \mathbf{a}_4 &= (-1,2,2,6,-1) = -(1,2,6,2,1) + (0,4,8,8,0) \\ &= -\mathbf{a}_3 + 4\mathbf{a}_1 = 4\mathbf{a}_1 - \mathbf{a}_3 \in U. \end{aligned}$$

2. Since

   $$\|\mathbf{a}_1\|^2 = 1+4+4 = 9, \qquad \text{dvs.} \qquad \|\mathbf{a}_1\| = 3,$$

   we may choose $\mathbf{q}_1 = \frac{1}{3}\mathbf{a}_1 = \frac{1}{3}(0,1,2,2,0)$.

   Then

   $$\mathbf{a}_2 - \mathbf{a}_1 = (1,1,4,0,0) - (0,1,2,2,0) = (1,0,2,-2,0)$$

has the length, and it is obvious that

$$\mathbf{a}_1 \cdot (\mathbf{a}_2 - \mathbf{a}) = 0 + 0 + 4 - 4 + 0 = 0,$$

so $\mathbf{a}_1$ and $\mathbf{a}_2 - \mathbf{a}_1$ are perpendicular to each other. We may therefore choos

$$\mathbf{q}_2 = \frac{1}{3}(\mathbf{a}_2 - \mathbf{a}_1) = \frac{1}{3}(1, 0, 2, -2).$$

We adjust $\mathbf{a}_3$,

$$\begin{aligned}
&\mathbf{a}_3 - \{\mathbf{a}_3 \cdot \mathbf{q}_1\}\mathbf{q}_1 - \{\mathbf{a}_3 \cdot \mathbf{q}_2\}\mathbf{q}_2 \\
&\quad = (1, 2, 6, 2, 1) - \frac{1}{9}\{(1, 2, 6, 2, 1) \cdot (0, 1, 2, 2, 0)\}(0, 1, 2, 2, 0) \\
&\qquad\qquad -\frac{1}{9}\{(1, 2, 6, 2, 1) \cdot (1, 0, 2, -2)\}(1, 0, 2, -2, 0) \\
&\quad = (1, 2, 6, 2, 1) - \frac{1}{9}\{0 + 2 + 12 + 4 + 0\}(0, 1, 2, 2, 0) \\
&\qquad\qquad -\frac{1}{9}\{1 + 0 + 12 - 4 + 0\}(1, 0, 2, -2, 0) \\
&\quad = (1, 2, 6, 2, 1) - 2(0, 1, 2, 2, 0) - (1, 0, 2, -2, 0) \\
&\quad = (1 - 0 - 1, 2 - 2 - 0, 6 - 4 - 2, 2 - 4 + 2, 1 - 0 - 0) \\
&\quad = (0, 0, 0, 0, 1).
\end{aligned}$$

This is clearly a unit vector , $\mathbf{q}_3((0, 0, 0, 0, 1)$.

If we instead use $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, then

$$\begin{aligned}
\mathbf{q}_3 &= (0, 0, 0, 0, 1) = \mathbf{a}_3 - \{\mathbf{a}_3 \cdot \mathbf{q}_1\}\mathbf{q}_1 - \{\mathbf{a}_3 \cdot \mathbf{q}_2\}\mathbf{q}_2 \\
&= \mathbf{a}_3 - \frac{1}{9}\{\mathbf{a}_3 \cdot \mathbf{a}_1\}\mathbf{a}_1 - \frac{1}{9}\{\mathbf{a}_3 \cdot (\mathbf{a}_2 - \mathbf{a}_1)\}(\mathbf{a}_2 - \mathbf{a}_1) \\
&= \mathbf{a}_3 - 2\mathbf{a}_1 - (\mathbf{a}_2 - \mathbf{a}_1) = -\mathbf{a}_1 - \mathbf{a}_2 + \mathbf{a}_3.
\end{aligned}$$

Summing up, we get

$$\mathbf{q}_1 = \frac{1}{3}\mathbf{a}_1, \quad \mathbf{q}_2 = \frac{1}{3}(\mathbf{a}_2 - \mathbf{a}_1), \quad \mathbf{q}_3 = -\mathbf{a}_1 - \mathbf{a}_2 + \mathbf{a}_3.$$

3. Now,

$$f(\mathbf{a}_1) = \mathbf{a}_1, \qquad f(\mathbf{a}_2 - \mathbf{a}_1) = -(\mathbf{a}_2 - \mathbf{a}_1),$$

and by 1),

$$f(\mathbf{a}_3) = \mathbf{a}_4 = 4\mathbf{a}_1 - \mathbf{a}_3.$$

Expressed by $\mathbf{q}_1$, $\mathbf{q}_2$, $\mathbf{q}_3$ we first get

$$f(\mathbf{q}_1) = \mathbf{q}_1 \qquad \text{and} \qquad f(\mathbf{q}_2) = -\mathbf{q}_2.$$

From

$$\begin{aligned}\mathbf{a}_3 &= \mathbf{q}_3+\mathbf{a}_1+\mathbf{a}_2=\mathbf{q}_3+(\mathbf{a}_2-\mathbf{a}_1)+2\mathbf{a}_1\\ &= 6\mathbf{q}_1+3\mathbf{q}_2+\mathbf{q}_3,\end{aligned}$$

and

$$\mathbf{a}_4=4\mathbf{a}_1-\mathbf{a}_3=12\mathbf{q}_1-\{6\mathbf{q}_1+3\mathbf{q}_2+\mathbf{q}_3\}=6\mathbf{q}_1-3\mathbf{q}_2-\mathbf{q}_3,$$

we get by the linearity that

$$\begin{aligned}f(\mathbf{a}_3) &= 6f(\mathbf{q}_1)+3f(\mathbf{q}_2)+f(\mathbf{q}_3)=6\mathbf{q}_1-3\mathbf{q}_2+f(\mathbf{q}_3)\\ &= \mathbf{a}_4=6\mathbf{q}_1-3\mathbf{q}_2-\mathbf{q}_3,\end{aligned}$$

hence by reduction, $f(\mathbf{q}_3)=-\mathbf{q}_3$.

The matrix representing the map is now

$$\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix}.$$

This is clearly symmetric (a diagonal matrix), and since its eigenvalues are ± 1, it is also isometric.

Example 3.28 *A linear map* $f : \mathbb{R}^4 \to \mathbb{R}^4$ *is given by the matrix* $\mathbf{y} = \mathbf{A}\mathbf{x}$, *where*

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix}$$

is the matrix of f *with respect to the usual basis in* $\mathbb{R}^4$.

1. *Show that the vectors*

$$\mathbf{v}_1 = (1,1,1,1), \mathbf{v}_2 = (1,1,-1,-1), \mathbf{v}_3 = (1,-1,1,-1), \mathbf{v}_4 = (1,0,-1,0),$$

   are eigenvectors of f, *and find the corresponding eigenvalues.*

2. *Find all eigenvectors of* f.

3. *Using the usual scalar product in* $\mathbb{R}^4$, *find an orthonormal basis consisting of eigenvectors of* f, *and then find the matrix of* f *in this basis.*

1. By a small computation,

$$\mathbf{A}\mathbf{v}_1 = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 0 \cdot \mathbf{v}_1,$$

   hence $\mathbf{v}_1$ is an eigenvector corresponding to the eigenvalue $\lambda_1 = 0$.

   Furthermore,

$$\mathbf{A}\mathbf{v}_2 = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -2 \\ -2 \end{pmatrix} = 2\mathbf{v}_2,$$

   hence $\mathbf{v}_2$ is an eigenvector corresponding to the eigenvalue $\lambda_2 = 2$.

   Then we get

$$\mathbf{A}\mathbf{v}_3 = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ -4 \\ 4 \\ -4 \end{pmatrix} = 4\mathbf{v}_3,$$

   so $\mathbf{v}_3$ is an eigenvector corresponding to $\lambda_3 = 4$.

   Finally, r

$$\mathbf{A}\mathbf{v}_4 = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -2 \\ 0 \end{pmatrix} = 2\mathbf{v}_4,$$

   so $\mathbf{v}_4$ is an eigenvector corresponding to $\lambda_4 = 2$.

2. If $\lambda_1 = 0$, then all eigenvectors are $k \cdot \mathbf{v}_1$, $k \in \mathbb{R}$.

   If $\lambda_2 = \lambda_4 = 2$ then all eigenvectors are $k_2\mathbf{v}_2 + k_4\mathbf{v}_4$, because clearly $\mathbf{v}_2$ and $\mathbf{v}_4$ are linearly independent. An orthogonal basis of the eigenspace is e.g.

   $$\mathbf{v}_2 - \mathbf{v}_4 = (0,1,0,-1) \quad \text{and} \quad \mathbf{v}_4 = (1,0,-1,0).$$

   If $\lambda_3 = 4$, then all eigenvectors are $k \cdot \mathbf{v}_3$, $k \in \mathbb{R}$.

3. Since eigenvectors corresponding to different eigenvalues are orthogonal, an orthogonal basis consisting of eigenvectors of f is given by

   $$\mathbf{v}_1 = (1,1,1,1), \qquad \mathbf{v}_2 - \mathbf{v}_4 = (0,1,0,-1),$$
   $$\mathbf{v}_4 = (1,0,-1,0), \qquad \mathbf{v}_3 = (1,-1,1,-1).$$

   Then by norming,

   $$\mathbf{q}_1 = \frac{1}{2}(1,1,1,1), \quad \mathbf{q}_2 = \frac{1}{\sqrt{2}}(\mathbf{v}_2 - \mathbf{v}_4) = \frac{1}{\sqrt{2}}(0,1,0,-1),$$
   $$\mathbf{q}_3 = \frac{1}{\sqrt{2}}\mathbf{v}_4 = \frac{1}{\sqrt{2}}(1,0,-1,0), \quad \mathbf{q}_4 = \frac{1}{2}\mathbf{v}_3 = \frac{1}{2}(1,-1,1,-1).$$

   The corresponding matrix is

   $$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}.$$

Example 3.29 *Let $\mathbf{v}_1$, $\mathbf{v}_2$, …, $\mathbf{v}_n$ be an orthonormal basis of $\mathbb{R}^n$ in the usual scalar product, and let $\mathbf{v}_i$, $i = 1, 2, \dots, n$, denote the column matrix, the elements of which are the coordinates of $\mathbf{v}_i$ with respect to the usual basis of $\mathbb{R}^n$. Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be n given real numbers, and consider the matrix*

$$\mathbf{A} = \lambda_1\mathbf{v}_1\mathbf{v}_1^T + \lambda_2\mathbf{v}_2\mathbf{v}_2^T + \cdots + \lambda_n\mathbf{v}_n\mathbf{v}_n^T.$$

1. *Prove that $\mathbf{A}$ is symmetric and that $\mathbf{v}_1$, $\mathbf{v}_2$, …, $\mathbf{v}_n$ are eigenvectors of the linear map given by the matrix $\mathbf{A}$ with the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$, respectively.*

2. *Using the result of 1), one shall construct a symmetric (3×3)-matrix $\mathbf{A}$ of the eigenvalues 1, 2 and 3 corresponding to the eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ with the coordinates*

   $$\mathbf{v}_1 = \frac{1}{3}\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}, \quad \mathbf{v}_2 = \frac{1}{3}\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \quad \textit{and} \quad \mathbf{v}_3 = \frac{1}{3}\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}.$$

1. Clearly, $\mathbf{A} = \mathbf{A}^T$, so $\mathbf{A}$ is symmetric.

   Let $\mathbf{Q}$ denote then $(n \times n)$-matrix, where the i-th column is given by $\mathbf{v}_i$. Since $\mathbf{v}_1$, $\mathbf{v}_2$, …, $\mathbf{v}_n$ form an orthonormal basis, we infer that $\mathbf{Q}$ is an orthogonal matrix, and it follows from the structure that

   $$\mathbf{A} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^{-1}, \quad \text{dvs. } \mathbf{Q}^{-1}\mathbf{A}\mathbf{Q} = \Lambda = \text{ diag}\{\lambda_1, \dots, \lambda_n\}.$$

   Then $\lambda_1, \dots, \lambda_n$ are the eigenvalues of $\mathbf{A}$, and the column $\mathbf{v}_i$ in $\mathbf{Q}$ is an eigenvector corresponding to λ_i.

2. We shall first check that $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ indeed form an orthonormal basis. It is, however, straightforward to see that

$$\|\mathbf{v}_1\| = \|\mathbf{v}_1\| = \|\mathbf{v}_3\| = \frac{1}{3}\sqrt{9} = 1,$$

and that $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ for $i \neq j$.

Then we compute

$$\mathbf{v}_1\mathbf{v}_1^T = \frac{1}{9}\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}(-1 \;\; 2 \;\; 2) = \frac{1}{9}\begin{pmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{pmatrix},$$

$$\mathbf{v}_2\mathbf{v}_2^T = \frac{1}{9}\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}(2 \;\; -1 \;\; 2) = \frac{1}{9}\begin{pmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{pmatrix},$$

$$\mathbf{v}_3\mathbf{v}_3^T = \frac{1}{9}\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}(2 \;\; 2 \;\; -1) = \frac{1}{9}\begin{pmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{pmatrix},$$

hence

$$\begin{aligned}
\mathbf{A} &= \frac{1}{9}\begin{pmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{pmatrix} + \frac{2}{9}\begin{pmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{pmatrix} + \frac{3}{9}\begin{pmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{pmatrix} \\
&= \frac{1}{9}\begin{pmatrix} 1+8+12 & -2-4+12 & -2+8-6 \\ -2-4+12 & 4+2+12 & 4-4-6 \\ -2+8-6 & 4-4-6 & 4+8+3 \end{pmatrix} \\
&= \frac{1}{9}\begin{pmatrix} 21 & 6 & 0 \\ 6 & 18 & -6 \\ 0 & -6 & 15 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 7 & 2 & 0 \\ 2 & 6 & -2 \\ 0 & -2 & 5 \end{pmatrix}.
\end{aligned}$$

Example 3.30 *Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ denote the linear map, given in the usual basis of $\mathbb{R}^3$ by the symmetric matrix*

$$\mathbf{A} = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ -1 & -1 & 5 \end{pmatrix}.$$

1. *Find the eigenvalues of f and an orthonormal basis $(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3)$ of $\mathbb{R}^3$ (with respect to the usual scalar product in $\mathbb{R}^3$) consisting of eigenvectors for f.*

2. *Let a bilinear function $g : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ be given in the usual basis of $\mathbb{R}^3$ by the equation*

$$g(\mathbf{x}, \mathbf{y}) = (x_1 \;\; x_2 \;\; x_3)\mathbf{A}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}.$$

Prove that g is a scalar product, and show that the vectors $\mathbf{q}_1$, $\mathbf{q}_2$, $\mathbf{q}_3$ are also mutually orthogonal with respect to the scalar product g. Then find a basis of $\mathbb{R}^3$, which is orthonormal with respect to the scalar product g.

1. The eigenvalues are the roots of the characteristic polynomial

$$\begin{aligned}
\det(\mathbf{A} - \lambda\mathbf{I}) &= \begin{vmatrix} 3-\lambda & 1 & -1 \\ 1 & 3-\lambda & -1 \\ -1 & -1 & 5-\lambda \end{vmatrix} = \begin{vmatrix} 3-\lambda & 1 & -1 \\ 1 & 3-\lambda & -1 \\ 3-\lambda & 3-\lambda & 3-\lambda \end{vmatrix} \\
&= (3-\lambda)\begin{vmatrix} 3-\lambda & 1 & -1 \\ 1 & 3-\lambda & -1 \\ 1 & 1 & 1 \end{vmatrix} = (3-\lambda)\begin{vmatrix} 4-\lambda & 2 & -1 \\ 2 & 4-\lambda & -1 \\ 0 & 0 & 1 \end{vmatrix} \\
&= (3-\lambda)\{(\lambda-4)^2 - 2^2\} = (3-\lambda)(\lambda-2)(\lambda-6).
\end{aligned}$$

(The difference between two squares).

The eigenvalues are $\lambda_1 = 2$, $\lambda_2 = 3$ and $\lambda_3 = 6$.

If $\lambda_1 = 2$, then we get the reduction

$$\begin{pmatrix} 3-\lambda & 1 & -1 \\ 1 & 3-\lambda & -1 \\ -1 & -1 & 5-\lambda \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

An eigenvector is e.g. $\mathbf{v}_1 = (1, -1, 0)$ with the normed eigenvector

$$\mathbf{q}_1 = \frac{1}{\sqrt{2}}(1, -1, 0).$$

If $\lambda_2 = 3$, then we get analogously

$$\begin{pmatrix} 3-\lambda & 1 & -1 \\ 1 & 3-\lambda & -1 \\ -1 & -1 & 5-\lambda \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ -1 & -1 & 2 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

An eigenvector is e.g. $\mathbf{v}_2 = (1, 1, 1)$ with the normed eigenvector

$$\mathbf{q}_2 = \frac{1}{\sqrt{3}}(1, 1, 1).$$

If $\lambda_3 = 6$, then

$$\begin{pmatrix} 3-\lambda & 1 & -1 \\ 1 & 3-\lambda & -1 \\ -1 & -1 & 5-\lambda \end{pmatrix} = \begin{pmatrix} -3 & 1 & -1 \\ 1 & -3 & -1 \\ -1 & -1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & -4 & -2 \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

An eigenvector is e.g. $\mathbf{v}_4 = (1, 1, -2)$ with the corresponding normed eigenvector

$$\mathbf{q}_3 = \frac{1}{\sqrt{6}}(1, 1, -2).$$

Now, $\mathbf{q}_1$, $\mathbf{q}_2$, $\mathbf{q}_3$ are normed eigenvectors corresponding to each its own eigenvalue, so

$$\mathbf{q}_1 = \frac{1}{\sqrt{2}}(1, -1, 0), \quad \mathbf{q}_2 = \frac{1}{\sqrt{3}}(1, 1, 1), \quad \mathbf{q}_3 = \frac{1}{\sqrt{6}}$$

form an orthonormal basis consisting of eigenvectors for f.

2. Since $\mathbf{A}$ is symmetric, and $\lambda_1 = 2$, $\lambda_2 = 3$, $\lambda_3 = 6$ are all positive, g is a scalar product . An ortho*gonal* basis is still $\mathbf{q}_1$, $\mathbf{q}_2$, $\mathbf{q}_3$, while the norms have been changed:

$$g(\mathbf{q}_1, \mathbf{q}_1) = 2\|\mathbf{q}_1\|^2 = 2, \quad g(\mathbf{q}_2, \mathbf{q}_2) = 3, \quad g(\mathbf{q}_3, \mathbf{q}_3) = 6.$$

When we norm, we obtain the orthonormal basis with respect to g,

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}}\mathbf{q}_1 = \frac{1}{2}(1,-1,0), \quad \mathbf{u}_2 = \frac{1}{\sqrt{3}}\mathbf{q}_2 = \frac{1}{3}(1,1,1),$$

$$\mathbf{u}_3 = \frac{1}{\sqrt{6}}\mathbf{q}_3 = \frac{1}{6}(1,1,-2).$$

Example 3.31 *Denote by $f : \mathbb{R}^3 \to \mathbb{R}^3$ the linear map which in the usual basis of $\mathbb{R}^3$ is given by the matrix*

$$\mathbf{A} = \begin{pmatrix} 5 & 2 & -2 \\ 2 & 8 & -4 \\ -2 & -4 & 8 \end{pmatrix}.$$

1. *Prove, using the usual scalar product in $\mathbb{R}^3$, that $\mathbf{v}_1 = (2,1,2)$, $\mathbf{v}_2 = (1,2,-1)$, $\mathbf{v}_3 = (-2,2,1)$ form a set of orthogonal eigenvectors for f, and then find all eigenvalues of f.*
2. *A function $g : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ is in the usual basis of $\mathbb{R}^3$ given by the equation*

$$g(\mathbf{x}, \mathbf{y}) = (x_1 \;\; x_2 \;\; x_3)\mathbf{A}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}.$$

Prove that g is a scalar product, and show that using this scalar product in $\mathbb{R}^3$, the angle between $\mathbf{e}_2 = (0,1,0)$ and $\mathbf{e}_3 = (0,0,1)$ is $\dfrac{2\pi}{3}$.

1. We compute

$$\mathbf{A}\mathbf{v}_1 = \begin{pmatrix} 5 & 2 & -2 \\ 2 & 8 & -4 \\ -2 & -4 & 8 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ 8 \end{pmatrix} = 4\mathbf{v}_1, \qquad \lambda_1 = 4,$$

$$\mathbf{A}\mathbf{v}_2 = \begin{pmatrix} 5 & 2 & -2 \\ 2 & 8 & -4 \\ -2 & -4 & 8 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 13 \\ 26 \\ -26 \end{pmatrix} = 13\mathbf{v}_2, \qquad \lambda_2 = 13,$$

$$\mathbf{A}\mathbf{v}_3 = \begin{pmatrix} 5 & 2 & -2 \\ 2 & 8 & -4 \\ -2 & -4 & 8 \end{pmatrix}\begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -8 \\ 8 \\ 4 \end{pmatrix} = 4\mathbf{v}_3, \qquad \lambda_3 = 4.$$

Since $\lambda_1 = \lambda_3 = 4$, we shall only prove that $\mathbf{v}_1$ and $\mathbf{v}_3$ are orthogonal. This follows from

$$\mathbf{v}_1 \cdot \mathbf{v}_3 = (2,1,2) \cdot (-2,2,1) = -4 + 2 + 2 = 0,$$

and 1) is proved.

2. Now, $\mathbf{A}$ is symmetric, and the eigenvalues $\lambda_1 = \lambda_3 = 4$ and $\lambda_2 = 13$ are all positive, hence g is an inner product.

   We have

$$g(\mathbf{e}_2, \mathbf{e}_2) = (0 \;\; 1 \;\; 0) \begin{pmatrix} 5 & 2 & -2 \\ 2 & 8 & -4 \\ -2 & -4 & 8 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 8,$$

$$g(\mathbf{e}_3, \mathbf{e}_3) = (0 \;\; 0 \;\; 1) \begin{pmatrix} 5 & 2 & -2 \\ 2 & 8 & -4 \\ -2 & -4 & 8 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 8,$$

   and

$$\begin{aligned} g(\mathbf{e}_2, \mathbf{e}_3) &= (0 \;\; 1 \;\; 0) \begin{pmatrix} 5 & 2 & -2 \\ 2 & 8 & -4 \\ -2 & -4 & 8 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = -4 \\ &= \sqrt{g(\mathbf{e}_2, \mathbf{e}_2)} \cdot \sqrt{g(\mathbf{e}_3, \mathbf{e}_3)} \cdot \cos(\angle(\mathbf{e}_2, \mathbf{e}_3)) \\ &= 8 \cos(\angle(\mathbf{e}_2, \mathbf{e}_3)), \end{aligned}$$

   hence

$$\cos(\angle(\mathbf{e}_2, \mathbf{e}_3)) = -\frac{4}{8} = -\frac{1}{2}, \quad \text{dvs.} \quad \angle(\mathbf{e}_2, \mathbf{e}_3) = \frac{2\pi}{3}.$$

Example 3.32 1. *Prove that the determinant of order* n*,*

$$A_n = \begin{vmatrix} b & a & 0 & 0 & \cdots & 0 & 0 & 0 \\ a & b & a & 0 & \cdots & 0 & 0 & 0 \\ 0 & a & b & a & \cdots & 0 & 0 & 0 \\ 0 & 0 & a & b & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & b & a & 0 \\ 0 & 0 & 0 & 0 & \cdots & a & b & a \\ 0 & 0 & 0 & 0 & \cdots & 0 & a & b \end{vmatrix}$$

satisfies the recursion formula

$$A_n = bA_{n-1} - a^2 A_{n-2}, \qquad n \geq 3.$$

2. *Find the eigenvalues of the matrix*

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}.$$

3. *Let* $f : \mathbb{R}^5 \to \mathbb{R}^5$ *denote the linear map, which in the usual basis of* $\mathbb{R}^5$ *is given by the matrix* $\mathbf{A}$*.*

   Prove (with respect to the usual scalar product in $\mathbb{R}^5$*) that there exists an orthonormal ' basis* $(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4, \mathbf{q}_5)$ *of* $\mathbb{R}^5$*, such that the matrix of* f *with respect to this basis is given by*

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 2+\sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & 2-\sqrt{3} \end{pmatrix},$$

   and find $\mathbf{q}_1$*.*

4. *A bilinear function* $g : \mathbb{R}^5 \times \mathbb{R}^5 \to \mathbb{R}$ *is in the usual basis of* $\mathbb{R}^5$ *given by*

$$g(\mathbf{x}, \mathbf{y}) = (x_1 \ \ \ldots \ \ x_5)\mathbf{A} \begin{pmatrix} y_1 \\ \vdots \\ y_5 \end{pmatrix}.$$

5. *Prove that* g *is a scalar product, and find the set of vectors which in this scalar product are perpendicular to the vector* $(1, 1, 1, 1, 1)$*.*

1. When we start by expanding the first column, and then expand the first row, we get

$$\begin{aligned} A_n &= \begin{vmatrix} b & a & 0 & \cdots & 0 & 0 & 0 \\ a & b & a & \cdots & 0 & 0 & 0 \\ 0 & a & b & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & b & a & 0 \\ 0 & 0 & 0 & \cdots & a & b & a \\ 0 & 0 & 0 & \cdots & 0 & a & b \end{vmatrix} = bA_{n-1} - a \begin{vmatrix} a & 0 & 0 & \cdots & 0 & 0 & 0 \\ a & b & a & \cdots & 0 & 0 & 0 \\ 0 & a & b & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & b & a & 0 \\ 0 & 0 & 0 & \cdots & a & b & a \\ 0 & 0 & 0 & \cdots & 0 & a & b \end{vmatrix} \\ &= bA_{n-1} - a^2 A_{n-2}, \end{aligned}$$

where we of course must assume that $n \geq 3$.

Then notice that

$$A_2 = \begin{vmatrix} b & a \\ a & b \end{vmatrix} = b^2 - a^2 \qquad \text{and} \qquad A_1 = b.$$

2. The eigenvalues are the roots of the determinant

$$\det(\mathbf{A} - \lambda \mathbf{I}) = A_5 = \begin{vmatrix} 2-\lambda & -1 & 0 & 0 & 0 \\ -1 & 2-\lambda & -1 & 0 & 0 \\ 0 & -1 & 2-\lambda & -1 & 0 \\ 0 & 0 & -1 & 2-\lambda & -1 \\ 0 & 0 & 0 & -1 & 2-\lambda \end{vmatrix}$$

of the type of 1) where $b = 2 - \lambda$ and $a = -1$. Then according to the recursion formula,

$$\begin{aligned} A_5 &= bA_4 - a^2 A_3 = b\{bA_3 - a^2 A_2\} - a^3 A_3 \\ &= (b^2 - a^2)A_3 - a^2 b A_2 = (b^2 - a^2)\{bA_2 - a^2 A_1\} - a^2 b A_2 \\ &= (b^2 - a^2)\{b(b^2 - a^2) - a^2 b\} - a^2 b(b^2 - a^2) \\ &= (b^2 - a^2) b (b^2 - 3a^2) \\ &= (b-a)(b+a)b(b - \sqrt{3}a)(b + \sqrt{3}a) \\ &= (2 - \lambda + 1)(2 - \lambda - 1)(2 - \lambda)(2 - \lambda + \sqrt{3})(2 - \lambda - \sqrt{3}) \\ &= -(\lambda - 1)(\lambda - 2)(\lambda - 3)(\lambda - \{2 + \sqrt{3}\})(\lambda - \{2 - \sqrt{3}\}). \end{aligned}$$

The eigenvalues are

$$\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3, \lambda_4 = 2 + \sqrt{3}, \lambda_5 = 2 - \sqrt{3}.$$

3. The matrix $\mathbf{A}$ has five simple eigenvalues (the diagonal elements of the new matrix). Therefore, if we choose normed eigenvectors as basis $\mathbf{q}_1, \dots, \mathbf{q}_5$, we obtain the matrix.

If $\lambda_1 = 1$, then

$$\begin{pmatrix} 2-\lambda & -1 & 0 & 0 & 0 \\ -1 & 2-\lambda & -1 & 0 & 0 \\ 0 & -1 & 2-\lambda & -1 & 0 \\ 0 & 0 & -1 & 2-\lambda & -1 \\ 0 & 0 & 0 & -1 & 2-\lambda \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

An eigenvector is e.g. $(1,1,0,-1,-1)$ of length $\sqrt{4} = 2$, thus

$$\mathbf{q}_1 = \frac{1}{2}(1,1,0,-1,-1).$$

4. Since $\mathbf{A}$ is symmetric with only positive eigenvalues, g is a scalar product. If $\mathbf{x}$ is perpendicular to $(1,1,1,1,1)$, then

$$\begin{aligned} 0 &= (1,1,1,1,1)\begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \\ &= (1,0,0,0,1)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = x_1 + x_5. \end{aligned}$$

The solutions are

$$\alpha(1,0,0,0,-1) + \beta(0,1,0,0,0) + \gamma(0,0,1,0,0) + \delta(0,0,0,1,0).$$

Example 3.33 *Given two linear maps $f : \mathbb{R}^3 \to \mathbb{R}^3$ and $g : \mathbb{R}^6 \to \mathbb{R}^6$ with the matrices, resp.,*

$$\mathbf{A} = \begin{pmatrix} 11 & -9 & 7 \\ -9 & 27 & -9 \\ 7 & -9 & 11 \end{pmatrix} \quad \textit{and} \quad \mathbf{B} = \begin{pmatrix} 11 & -9 & 7 & 0 & 0 & 0 \\ -9 & 27 & -9 & 0 & 0 & 0 \\ 7 & -9 & 11 & 0 & 0 & 0 \\ 0 & 0 & 0 & 11 & -9 & 7 \\ 0 & 0 & 0 & -9 & 27 & -9 \\ 0 & 0 & 0 & 7 & -9 & 11 \end{pmatrix}$$

(with respect to the usual basis of $\mathbb{R}^3$ and $\mathbb{R}^6$, resp.).

1. *Prove that $\mathbf{v}_1 = (1,1,1)$ is an eigenvector of f, and find the corresponding eigenvalue λ_1.*
2. *Prove that $\mathbf{w}_1 = (1,-2,1,0,0,0)$ and $\mathbf{w}_2 = (0,0,0,1,1,1)$ are both eigenvectors of g, and find the corresponding eigenvalues μ_1 and μ_2.*
3. *Find (e.g. by using the results of 1 and 2) every eigenvalue of f.*
4. *Find all eigenvectors of f.*
5. *Find all eigenvectors of for g.*
6. *Explain that the matrix $\mathbf{B}$ can be diagonalized, and find an orthonormal basis of $\mathbb{R}^6$, where the matrix of g is a diagonal matrix $\mathbf{\Lambda}$, and find $\mathbf{\Lambda}$.*

Notice the structure

$$\mathbf{B} = \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{A} \end{pmatrix}.$$

We infer from this that if $\mathbf{x}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue λ, then both $(\mathbf{x},\mathbf{0})$ and $(\mathbf{0},\mathbf{x})$ are eigenvectors of $\mathbf{B}$ corresponding to the same eigenvalue. Here we have put $\mathbf{0} = (0,0,0)$.

1. It follows by insertion that

$$\mathbf{A}\mathbf{v}_1 = \begin{pmatrix} 11-9+7 \\ -9+27-9 \\ 7-0+11 \end{pmatrix} = \begin{pmatrix} 9 \\ 9 \\ 9 \end{pmatrix} = 9\mathbf{v}_1,$$

so $\mathbf{v}_1 = (1,1,1)$ is an eigenvector with the corresponding eigenvalue $\lambda_1 = 9$.

2. It follows from the remark in the beginning of this example that $\mathbf{w}_2 = (\mathbf{0}, \mathbf{v}_1)$ is an eigenvector of $\mathbf{B}$ corresponding to the eigenvalue $\mu_2 = \lambda_1 = 9$.

If we put $\mathbf{v}_2 = (1,-2,1)$, then

$$\mathbf{A}\mathbf{v}_2 = \begin{pmatrix} 11 & -9 & 7 \\ -9 & 27 & -9 \\ 7 & -9 & 11 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 11+18+7 \\ -9-54-9 \\ 7+18+11 \end{pmatrix} = \begin{pmatrix} 36 \\ -72 \\ 36 \end{pmatrix} = 36\mathbf{v}_2,$$

so $\mathbf{v}_2$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 36, thus $\mathbf{w}_1 = (\mathbf{v}_2, \mathbf{0})$ is by the introductory remark an eigenvector of $\mathbf{B}$ with the eigenvalue $\mu_2 = 36$.

3. Since $\lambda_1 + \lambda_2 + \lambda_3 = \text{ trace } \mathbf{A}$, we get

$$\lambda_3 = \text{ trace } \mathbf{A} - \lambda_1 - \lambda_2 = 11 + 27 + 11 - 9 - 36 = 49 - 45 = 4.$$

4. It follows by inspection from

$$\mathbf{A} - \lambda_3\mathbf{I} = \begin{pmatrix} 7 & -9 & 7 \\ -9 & 23 & -9 \\ 7 & -9 & 7 \end{pmatrix} \sim \begin{pmatrix} 7 & -9 & 7 \\ -2 & 14 & -2 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -7 & 1 \\ 7 & -9 & 7 \\ 0 & 0 & 0 \end{pmatrix},$$

that $\mathbf{v}_3 = (1,0,-1)$ is an eigenvector corresponding to $\lambda_3 = 4$.

If f, then the eigenvalues and there corresponding eigenvectors are

$$\begin{array}{lll} \lambda_1 = 9, & \text{and} & \mathbf{v}_1 = (1,1,1), \\ \lambda_2 = 36, & \text{and} & \mathbf{v}_2 = (1,-2,1), \\ \lambda_3 = 4, & \text{and} & \mathbf{v}_3 = (1,0,-1). \end{array}$$

5. According to the introductory remark we get analogously for g that

$$\begin{array}{lllll} \mu_1 = \lambda_2 = 36 & \text{med} & \mathbf{w}_1 = (1,-2,1,0,0,0) & \text{and} & \mathbf{w}_3 = (0,0,0,1,-2,1), \\ \mu_2 = \lambda_1 = 9 & \text{med} & \mathbf{w}_4 = (1,1,1,0,0,0) & \text{and} & \mathbf{w}_2 = (0,0,0,1,1,1), \\ \mu_3 = \lambda_3 = 4 & \text{med} & \mathbf{w}_5 = (1,0,-1,0,0,0) & \text{and} & \mathbf{w}_6 = (0,0,0,1,0,-1). \end{array}$$

Notice that all the chosen eigenvectors are orthogonal.

The eigenvectors corresponding to $\mu_1 = 36$ are the linear combinations of $\mathbf{w}_1$ and $\mathbf{w}_3$.

Analogously for the other eigenvalues. It will in each case be sufficient to indicate two linearly independent eigenvectors.

6. Since all eigenvectors in 5) are orthogonal, we shall only norm these eigenvectors:

$$\begin{array}{llll}\mu_1 = 36 & \text{med} \quad \mathbf{q}_1 = \frac{1}{\sqrt{6}}(1,-2,1,0,0,0) & \text{and} & \mathbf{q}_2 = \frac{1}{\sqrt{6}}(0,0,0,,1,-2,1), \\ \mu_2 = 9 & \text{med} \quad \mathbf{q}_3 = \frac{1}{\sqrt{3}}(1,1,1,0,0,0) & \text{and} & \mathbf{q}_4 = \frac{1}{\sqrt{3}}(0,0,0,1,1,1), \\ \mu_3 = 4 & \text{med} \quad \mathbf{q}_5 = \frac{1}{\sqrt{2}}(1,0,-1,0,0,0) & \text{and} & \mathbf{q}_6 = \frac{1}{\sqrt{2}}(0,0,0,1,0,-1).\end{array}$$

We get in this basis,

$$\mathbf{\Lambda} = \ \text{diag}\{36,36,9,9,4,4\} = \begin{pmatrix} 36 & 0 & 0 & 0 & 0 & 0 \\ 0 & 36 & 0 & 0 & 0 & 0 \\ 0 & 0 & 9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 9 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{pmatrix},$$

Example 3.34 *A linear map $f : \mathbb{R}^4 \to \mathbb{R}^4$ is in the usual basis described by the matrix*

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & 1 & -1 \\ -1 & 2 & 1 & -1 \\ 1 & 1 & 2 & 1 \\ -1 & -1 & 1 & 2 \end{pmatrix}.$$

Given also the vectors $\mathbf{v}_1 = (1,1,-1,1)$ and $\mathbf{v}_2 = (1,0,1,0)$.

1. *Prove that $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors for f and find their corresponding eigenvalues λ_1 and λ_2. Then find all eigenvectors corresponding to λ_1 and λ_2.*

2. *Find an orthogonal matrix $\mathbf{Q}$, such that $\mathbf{\Lambda} = \mathbf{Q}^T\mathbf{A}\mathbf{Q}$ is a diagonal matrix, and find $\mathbf{\Lambda}$.*

1. It follows by insertion that

$$\mathbf{A}\mathbf{v}_1 = \begin{pmatrix} 2 & -1 & 1 & -1 \\ -1 & 2 & 1 & -1 \\ 1 & 1 & 2 & -1 \\ -1 & -1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2-1-1-1 \\ -1+2-1-1 \\ 1+1-1+1 \\ -1-1-1+2 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \\ 1 \\ -1 \end{pmatrix} = -\mathbf{v}_1,$$

and

$$\mathbf{A}\mathbf{v}_2 = \begin{pmatrix} 2 & -1 & 1 & -1 \\ -1 & 2 & 1 & -1 \\ 1 & 1 & 2 & 1 \\ -1 & -1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 3 \\ 0 \end{pmatrix} = 3\mathbf{v}_2,$$

hence $\lambda_1 = -1$ and $\lambda_2 = 3$.

Now

$$\mathbf{A}-\lambda_1\mathbf{I}=\begin{pmatrix} 3 & -1 & 1 & -1 \\ -1 & 3 & 1 & -1 \\ 1 & 1 & 3 & 1 \\ -1 & -1 & 1 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 3 & 1 \\ 0 & -4 & -8 & -4 \\ 0 & 4 & 4 & 0 \\ 0 & 0 & 4 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

is of rank 3, so $k\mathbf{v}_1$ are the only eigenvectors corresponding to the eigenvalue λ_1.

Since

$$\mathbf{A}-\lambda_2\mathbf{I}=\begin{pmatrix} -1 & -1 & 1 & -1 \\ -1 & -1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ -1 & -1 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

is of rank 1, the eigenspace has dimension $4-1=3$. We have already been given the eigenvector $\mathbf{v}_2=(1,0,1,0)$, and we see by inspection that $\mathbf{v}_3=(0,1,1,0)$ and $\mathbf{v}_4=(0,0,1,1)$ are two linearly independent eigenvectors.

The complete set of eigenvectors corresponding to $\lambda_2=3$ is $\operatorname{span}\{\mathbf{v}_2,\mathbf{v}_3,\mathbf{v}_4\}$.

2. From $\|\mathbf{v}_1\|=\sqrt{4}=2$ follows by taking the norm,

$$\mathbf{q}_1=\frac{1}{2}(1,1,-1,1).$$

Then

$$\mathbf{q}_2=\frac{1}{\|\mathbf{v}_2}\,\mathbf{v}_2=\frac{1}{\sqrt{2}}\,(1,0,1,0).$$

We compute

$$\begin{aligned}\mathbf{v}_3-(\mathbf{v}_3\cdot\mathbf{q}_2)\,\mathbf{q}_2 &= (0,1,1,0)-\left(\frac{1}{\sqrt{2}}\right)^2\{(0,1,1,0)\cdot(1,0,1,0)\}\,(1,0,1,0)\\ &= (0,1,1,0)-\frac{1}{2}\,(1,0,1,0)=\frac{1}{2}\,(-1,2,1,0).\end{aligned}$$

Now, $\|(-1,2,1,0)\|^2=1+4+1=6$, so

$$\mathbf{q}_3=\frac{1}{\sqrt{6}}\,(-1,2,1,0).$$

Finally,

$$\begin{aligned}&\mathbf{v}_4-(\mathbf{v}_4\cdot\mathbf{q}_2)\,\mathbf{q}_2-(\mathbf{v}_4\cdot\mathbf{q}_3)\,\mathbf{q}_3\\ &\quad=(0,0,1,1)-\frac{1}{2}\{(0,0,1,1)\cdot(1,0,1,0)\}\,(1,0,1,0)\\ &\qquad\qquad-\frac{1}{6}\{(0,0,1,1)\cdot(-1,2,1,0)\}\,(-1,2,1,0)\\ &\quad=\frac{1}{6}\{(0,0,6,6)-(3,0,3,0)-(-1,2,1,0)\}=\frac{1}{6}\,(-2,-2,2,6)\\ &\quad=-\frac{1}{3}\,(1,1,-1,-3).\end{aligned}$$

As $\|(1,1,-1,-3)\|=\sqrt{1+1+1+9}=\sqrt{12}=2\sqrt{3}$, we choose

$$\mathbf{q}_4=\frac{1}{2\sqrt{3}}\,(1,1,-1,-3).$$

Since $\mathbf{q}_1$, $\mathbf{q}_2$, $\mathbf{q}_3$, $\mathbf{q}_4$ is an orthonormal basis consisting of eigenvectors, we get

$$\mathbf{Q}=\{\mathbf{q}_1\ \ \mathbf{q}_2\ \ \mathbf{q}_3\ \ \mathbf{q}_4\}=\begin{pmatrix}\frac{1}{2} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{2\sqrt{3}}\\ \frac{1}{2} & 0 & \frac{2}{\sqrt{6}} & \frac{1}{2\sqrt{3}}\\ -\frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{2\sqrt{3}}\\ \frac{1}{2} & 0 & 0 & -\frac{\sqrt{3}}{2}\end{pmatrix}$$

and

$$\mathbf{\Lambda}=\begin{pmatrix}-1 & 0 & 0 & 0\\ 0 & 3 & 0 & 0\\ 0 & 0 & 3 & 0\\ 0 & 0 & 0 & 3\end{pmatrix}.$$

Example 3.35 *Consider the vector space $\mathbb{R}^3$ with the usual scalar product. Given of a linear symmetric map $f : \mathbb{R}^3 \to \mathbb{R}^3$ that*

- $f(4,-1,-8) = (-4,1,8)$,
- $(1,-1,4)$ *is an eigenvector for f, corresponding to the eigenvalue -1,*
- *and 2 is an eigenvalue of f.*

1. *Show that $(4,-1,-8) \times (1,-1,4)$ is an eigenvector of f corresponding to the eigenvalue 2.*
2. *Find an orthonormal basis of $\mathbb{R}^3$ consisting of eigenvectors of f, and find the matrix of f with respect to this basis.*

1. We infer from the given that $(4,-1,-8)$ and $(1,-1,4)$ are linearly independent eigenvector corresponding to the eigenvalue $\lambda_1 = -1$, so this eigenspace has at least dimension 2. Now, the eigenspace corresponding to $\lambda_2 = 2$, is of at least dimension 1, and since the sum of these lower bounds of the dimensions is 3, we conclude that

   - the eigenspace corresponding to the eigenvalue $\lambda_1 = -1$ is precisely of dimension 2,
   - the eigenspace corresponding to the eigenvalue $\lambda_2 = 2$ is precisely of dimension 1.

   Since $(4,-1,-8)$ and $(1,-1,4)$ are linearly independent,

   $$(4,-1,-8) \times (1,-1,4) = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 4 & -1 & -8 \\ 1 & -1 & 4 \end{vmatrix} = (-12,24,-3) = -3(4,8,1)$$

   is a proper vector, which is perpendicular to the eigenspace corresponding to $\lambda_1 = -1$. Since there is "no space" for other possibilities, this vector product must belong to the eigenspace corresponding to $\lambda_2 = 2$, because it is also orthogonal to the eigenspace corresponding to λ_1. Hence, $(4,8,1)$ is a (generating) eigenvector of the eigenspace corresponding to $\lambda_2 = 2$.

2. Now, $\|(4,-1,-8)\| = \sqrt{16-1+64} = \sqrt{81} = 9$, so we choose

   $$\mathbf{q}_1 = \frac{1}{9}\,(4,-1,8).$$

   Since

   $$\begin{aligned}&(1,-1,4) - \frac{1}{9^2}\{(1,-1,4)\cdot(4,-1,-8)\}(4,-1,-8)\\ &\qquad = (1,-1,4) - \frac{1}{81}\{4+1-32\}(4,-1,8) = (1,-1,4) + \frac{1}{3}\,(4,-1,-8)\\ &\qquad = \frac{1}{3}\{(3,-3,12)+(4,-1,-8)\} = \frac{1}{3}\,(7,-4,4),\end{aligned}$$

   where $\|(7,-4,4)\|^2 = 49+16+16 = 81 = 9^2$, we choose

   $$\mathbf{q}_2 = \frac{1}{9}\,(7,-4,4)$$

   as an eigenvector of λ_1, which is perpendicular to $\mathbf{q}_1$.

Finally, $\|(4,8,1)\| = \sqrt{16+64+1} = 9$, and since $(4,8,1)$ is orthogonal to $\mathbf{q}_1$ and $\mathbf{q}_2$, we choose

$$\mathbf{q}_3 = \frac{1}{9}\,(4,8,1).$$

Summing up we get the orthonormal basis consisting of

$$\mathbf{q}_1 = \frac{1}{9}\,(4,-1,-8), \quad \mathbf{q}_2 = \frac{1}{9}\,(7,-4,4), \quad \mathbf{q}_3 = \frac{1}{9}\,(4,8,1),$$

and the matrix of f in this basis is

$$\mathbf{\Lambda} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Example 3.36 *Given the symmetric matrices*

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 1 & 2 \\ 2 & 2 & -2 \end{pmatrix} \quad \textit{and} \quad \mathbf{B} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

1. *Prove that the characteristic polynomial of* $\mathbf{A}$ *has a simple root and a double root.*
   Find a proper eigenvector $\mathbf{v}_1$ *corresponding to the simple root.*
2. *Prove that the characteristic polynomial of* $\mathbf{B}$ *also has a simple root and a double root.*
   Find a proper eigenvector $\mathbf{v}_2$ *corresponding to the simple root.*
3. *Prove that* $\mathbf{v}_1^T\mathbf{v}_2 = 0$ *(i.e.* $\mathbf{v}_1$ *and* $\mathbf{v}_2$ *are orthogonal).*
4. *Find by e.g. applying the results of 1), 2) and 3) above an orthogonal matrix* $\mathbf{Q}$*, which reduces both* $\mathbf{A}$ *and* $\mathbf{B}$ *to the diagonal form. Indicate the results of both* $\mathbf{Q}^{-1}\mathbf{A}\mathbf{Q}$ *and* $\mathbf{Q}^{-1}\mathbf{B}\mathbf{Q}$.

1. We compute the characteristic polynomial of $\mathbf{A}$,

$$\begin{aligned}
\det(\mathbf{A}-\lambda\mathbf{I}) &= \begin{vmatrix} 1-\lambda & -1 & 2 \\ -1 & 1-\lambda & 2 \\ 2 & 2 & -2-\lambda \end{vmatrix} = \begin{vmatrix} 2-\lambda & 2-\lambda & 2-\lambda \\ -1 & 1-\lambda & 2 \\ 2 & 2 & -2-\lambda \end{vmatrix} \\
&= (2-\lambda)\begin{vmatrix} 1 & 1 & 1 \\ -1 & 1-\lambda & 2 \\ 2 & 2 & -2-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 1 & 0 & 0 \\ -1 & 2-\lambda & 3 \\ 2 & 0 & -4-\lambda \end{vmatrix} \\
&= -(\lambda-2)^2(\lambda+4).
\end{aligned}$$

It follows immediately that $\lambda_1 = -4$ is a simple eigenvalue, and that $\lambda_2 = 2$ is an eigenvalue of multiplicity 2.

We infer from

$$\begin{aligned}\mathbf{A}-\lambda_1\mathbf{I} &= \begin{pmatrix} 5 & -1 & 2 \\ -1 & 5 & 2 \\ 2 & 2 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & -5 & -2 \\ -1 & 5 & 2 \\ 1 & 1 & 1 \end{pmatrix} \\ &\sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix},\end{aligned}$$

that we can choose an eigenvector corresponding to $\lambda_1 = -4$ as e.g. $\mathbf{v}_1 = (1, 1, -2)$.

2. We compute the characteristic e polynomial of $\mathbf{B}$,

$$\begin{aligned}\det(\mathbf{B}-\mu\mathbf{I}) &= \begin{vmatrix} 2-\mu & -1 & -1 \\ -1 & 2-\mu & -1 \\ -1 & -1 & 2-\mu \end{vmatrix} = \begin{vmatrix} -\mu & -\mu & -\mu \\ -1 & 2-\mu & -1 \\ -1 & -1 & 2-\mu \end{vmatrix} \\ &= -\mu \begin{vmatrix} 1 & 1 & 1 \\ -1 & 2-\mu & -1 \\ -1 & -1 & 2-\mu \end{vmatrix} = -\mu \begin{vmatrix} 1 & 0 & 0 \\ -1 & 3-\mu & 0 \\ -1 & 0 & 3-\mu \end{vmatrix} \\ &= -\mu(\mu-3)^2.\end{aligned}$$

We see that we get a simple root $\mu_1 = 0$ and a double root $\mu_2 = 3$.

It follows from

$$\mathbf{B} - \mu_1\mathbf{I} = \mathbf{B} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \sim \begin{pmatrix} 2 & -1 & -1 \\ 3 & -3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 2 & -1 & -1 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

that an eigenvector can be chosen as $\mathbf{v}_2 = (1,1,1)$.

3. Clearly,

$$\mathbf{v}_1^T\mathbf{v}_2 = (1,1,-2)\cdot(1,1,1) = 1+1-2 = 0,$$

så $\mathbf{v}_1$ and $\mathbf{v}_2$ er orthogonal e.

4. Then notice that the matrices are symmetric, so we can in both cases write $\mathbb{R}^3$ as a direct sum of orthogonal eigenspaces.

If $\mathbf{A}$, then $\mathbf{v}_1$ the eigenspace corresponding to $\lambda_1 = -4$. The vector $\mathbf{v}_2$ is perpendicular to this eigenspace, so $\mathbf{v}_2$ must be an eigenvector corresponding to the eigenvalue $\lambda_2 = 2$. Since $\mathbf{v}_1 \times \mathbf{v}_2$ is perpendicular to both $\mathbf{v}_1$ and $\mathbf{v}_2$, and $\lambda_2 = 2$ is a double root, it follows that $\mathbf{v}_1 \times \mathbf{v}_2$ is an eigenvector corresponding to $\lambda_2 = 2$. We compute

$$\mathbf{v}_1 \times \mathbf{v}_2 = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 1 & 1 & -2 \\ 1 & 1 & 1 \end{vmatrix} = (3,-3,0) = 3(1,-1,0).$$

For $\mathbf{A}$ har vi, at eigenspace mene udspændes af

$$\mathbf{q}_1 = \frac{1}{\sqrt{6}}(1,1,-2) \qquad \text{for } \lambda_1 = -4,$$

and

$$\mathbf{q}_2 = \frac{1}{\sqrt{3}}(1,1,1) \text{ and } \mathbf{q}_3 = \frac{1}{\sqrt{2}}(1,-1,0) \qquad \text{for } \lambda_2 = 2,$$

where $\mathbf{q}_1$, $\mathbf{q}_2$, $\mathbf{q}_3$ er en orthonormal basis.

Concerning $\mathbf{B}$, the eigenspaces are spanned by

$$\mathbf{q}_2 = \frac{1}{\sqrt{3}}(1,1,1) \qquad \text{for } \mu_1 = 0,$$

and

$$\mathbf{q}_1 = \frac{1}{\sqrt{6}}(1,1,-2) \text{ and } \mathbf{q}_3 = \frac{1}{\sqrt{2}}(1,-1,0) \qquad \text{for } \mu_2 = 3,$$

where we as before conclude that both $\mathbf{q}_1$ and $\mathbf{q}_3$ must lie in the eigenspace corresponding to $\mu_2 = 3$.

We may now choose the orthogonal matrix as

$$\mathbf{Q} = (\mathbf{q}_1 \;\; \mathbf{q}_2 \;\; \mathbf{q}_3) = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \end{pmatrix}.$$

Using this matrix we obtain the transformed matrices

$$\mathbf{Q}^{-1}\mathbf{AQ} = \begin{pmatrix} -4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \text{ and } \mathbf{Q}^{-1}\mathbf{BQ} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Example 3.37 *Let* $\mathbf{A}$, $\mathbf{P}_{r,s} \in \mathbb{R}^{n\times n}$, *where* $\mathbf{P}_{r,s}$ *is an elementary permutation matrix.*

1. *Prove that* $\mathbf{B} = \mathbf{P}_{r,s}\mathbf{A}\mathbf{P}_{r,s}$ *is similar to* $\mathbf{A}$.
2. *Describe in words how* $\mathbf{B}$ *is obtained from* $\mathbf{A}$.
3. *Exploit the above together with Sylvester's theorem to find the number of positive and negative eigenvalues of the symmetric e matrix*

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 3 \\ 1 & 1 & 2 \\ 3 & 2 & 4 \end{pmatrix}.$$

1. This is obvious, because $\mathbf{P}_{r,s} = \mathbf{P}_{r,s}^{\perp} = \mathbf{P}_{r,s}^{-1}$.
2. We see that $\mathbf{B}$ is obtained from $\mathbf{A}$ by first interchange the r-th and the s-th row, and then interchange the r-th and the s-th column in the result [or vice versa].
3. In order to be able to perform an elementary Gauß elimination it will be convenient to have the number 1 on the place number $(1,1)$. Since we have the number 1 on place number $(2,2)$, we first apply $\mathbf{P}_{1,2}$ as described above,

$$\begin{aligned} \mathbf{A} &= \begin{pmatrix} 0 & 1 & 3 \\ 1 & 1 & 2 \\ 3 & 2 & 4 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \end{pmatrix} \xrightarrow{S_1 \leftrightarrow S_2} \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 3 \\ 2 & 3 & 4 \end{pmatrix} \\ &\sim \begin{pmatrix} 1 & 1 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \end{aligned}$$

and then it follows from Sylvester's theorem that we have two positive and one negative eigenvalue of $\mathbf{A}$.

ALTERNATIVELY,

$$\begin{aligned} \det(\mathbf{A} - \lambda\mathbf{I}) &= \begin{vmatrix} -\lambda & 1 & 3 \\ 1 & 1-\lambda & 2 \\ 3 & 2 & 4-\lambda \end{vmatrix} \\ &= -\lambda(\lambda-1)(\lambda-4) + 6 + 6 + 9(\lambda-1) + (\lambda-4) + 4\lambda \\ &= -\lambda^3 + 5\lambda^2 - 4\lambda + 12 + 9\lambda - 9 + \lambda - 4 + 4\lambda \\ &= -\lambda^3 + 5\lambda^2 + 10\lambda - 1 = -\left\{\lambda^3 - 5\lambda^2 - 10\lambda + 1\right\}. \end{aligned}$$

Since $\lambda_1 \cdot \lambda_2 \cdot \lambda_3 = -1$, and all the roots are real, we have 1 or 3 negative roots.

Since $\lambda_1 + \lambda_2 + \lambda_3 = 5 > 0$, we have at least one, and hence two positive roots.

Example 3.38 *Given for every $a \in \mathbb{R}$ the symmetric matrix*

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & a \\ 2 & 3 & 0 \\ a & 0 & -3 \end{pmatrix}.$$

1. *Find the LU-factorization of* $\mathbf{A}$.

2. *Use this LU-factorization to find for every $a \in \mathbb{R}$ the number (counted by multiplicity) of positive eigenvalues of $\mathbf{A}$, of negative eigenvalues of $\mathbf{A}$, and – if necessary – the multiplicity of the eigenvalue 0 for $\mathbf{A}$.*

1. By an elementary Gauß elimination,

$$\begin{aligned} \mathbf{A} &= \begin{pmatrix} 1 & 2 & a \\ 2 & 3 & 0 \\ a & 0 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & a \\ 0 & -1 & -2a \\ 0 & -2a & -3-a^2 \end{pmatrix} \\ &\sim \begin{pmatrix} 1 & 2 & a \\ 0 & -1 & -2a \\ 0 & 0 & 3a^2-3 \end{pmatrix} = \mathbf{U}, \end{aligned}$$

where

$$\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ a & 2a & 1 \end{pmatrix}$$

was implicitly derived above.

CHECK:

$$\mathbf{LU} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ a & 2a & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & a \\ 0 & -1 & -2a \\ 0 & 0 & 3a^2-3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & a \\ 2 & 3 & 0 \\ a & 0 & -3 \end{pmatrix}. \qquad \Diamond$$

2. According to Sylvester's theorem we have at least 1 positive and at least 1 negative eigenvalue. The correct numbers are determined by the signs of $3a^2 - 3 = 3(a^2 - 1)$.

   (a) If $|a| < 1$, then we have 1 positive and 2 negative eigenvalues-

   (b) If $a = \pm 1$, then we have 1 positive and 1 negative eigenvalue, and 0.

   (c) If $|a| > 1$, then we have 2 positive and 1 negative eigenvalue.

Example 3.39 *Given the symmetric matrix*

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & -a \\ 2 & 8 & 0 \\ -a & 0 & 6a^2 \end{pmatrix}, \qquad a \in \mathbb{R}.$$

1. *Find a lower triangular unit matrix* $\mathbf{L}$ *and an upper triangular matrix* $\mathbf{U}$*, such that1* $\mathbf{A} = \mathbf{LU}$.
2. *Find a lower triangular unit matrix* $\mathbf{L}$ *and a diagonal matrix* $\mathbf{D}$*, such that* $\mathbf{A} = \mathbf{LDL}^T$.

1. We get by an elementary Gauß elimination

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & -a \\ 2 & 8 & 0 \\ -a & 0 & 6a^2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -a \\ 0 & 4 & 2a \\ 0 & 2a & 5a^2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -a \\ 0 & 4 & 2a \\ 0 & 0 & 4a^2 \end{pmatrix} = \mathbf{U},$$

where

$$\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -a & \frac{a}{2} & 1 \end{pmatrix}$$

is implicitly found in the computations.

Check:

$$\mathbf{LU} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 1 & 0 \\ -a & \frac{a}{2} & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & -a \\ 0 & 4 & 2a \\ 0 & 0 & 4a^2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & -a \\ 2 & 8 & 0 \\ -a & 0 & 6a^2 \end{pmatrix}. \qquad \Diamond$$

2. What is the problem? According to a theorem,

$$\mathbf{A} = \mathbf{LU} = \mathbf{LDL}^T = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -a & \frac{a}{2} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4a^2 \end{pmatrix} \begin{pmatrix} 1 & 2 & -a \\ 0 & 1 & \frac{a}{2} \\ 0 & 0 & 1 \end{pmatrix}.$$

Check:

$$\mathbf{DL}^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4a^2 \end{pmatrix} \begin{pmatrix} 1 & 2 & -a \\ 0 & 1 & \frac{a}{2} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & -a \\ 0 & 4 & 2a \\ 0 & 0 & 4a^2 \end{pmatrix} = \mathbf{U}.$$

Example 3.40 *Given in an ordinary rectangular coordinate system of positive orientation XYZ in space three planes α, β and γ by the equations*

$$\begin{array}{ll} \alpha: & x+y-2z=0, \\ \beta: & 2x-y+(3a-4)z=3, \\ \gamma: & ay-z=1, \end{array}$$

where $a \in \mathbb{R}$.

1. *Find all a, for which the planes α, β and γ have*
   (a) *precisely one point in common,*
   (b) *a straight line in common,*
   (c) *no point in common,*

   and find in case 1a) the coordinates of the common point.

   In the remaining part of the example we put $a = 1$.

2. *Prove that α, β and γ pairwise intersect each other, and find a parametric descriptions of the intersection lines*

   $$\ell_{\alpha\beta} = \alpha \cap \beta, \quad \ell_{\alpha\gamma} = \alpha \cap \gamma \quad \text{and} \quad \ell_{\beta\gamma} = \beta \cap \gamma.$$

   Prove that the intersection lines are parallel, and find the distance between $\ell_{\alpha\beta}$ and $\ell_{\beta\gamma}$.

3. *The planes α, β and γ form a prismatic tube of the edges $\ell_{\alpha\beta}$, $\ell_{\alpha\gamma}$ and $\ell_{\beta\gamma}$.*
   Find the three angles between the planes inside the tube.

1. The corresponding inhomogeneous system of equation is in matrix form

   $$\begin{pmatrix} 1 & 1 & -2 \\ 2 & -1 & 3a-4 \\ 0 & a & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}.$$

   By reduction,

   $$\begin{aligned} (\mathbf{A} \mid \mathbf{b}) &= \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 2 & -1 & 3a-4 & 3 \\ 0 & a & -1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & -3 & 3a & 3 \\ 0 & 1 & -1 & 1 \end{array}\right) \\ &\sim \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & -a & -1 \\ 0 & a & -1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & a-2 & 1 \\ 0 & 1 & -a & -1 \\ 0 & 0 & a^2-1 & a+1 \end{array}\right). \end{aligned}$$

   (a) If $a \neq \pm 1$, then the rank of the matrix of coefficients is 3, so we have precisely one common point of the planes. By a further reduction it follows that the common point is given by

   $$\left(\frac{1}{a-1}, \frac{1}{a-1}, \frac{1}{a-1}\right) = \frac{1}{a-1}(1,1,1) \qquad \text{for } a \neq \pm 1.$$

(b) If $a=-1$, then both the matrix of coefficients and the total matrix have rank 2, and the planes have a line in common.

(c) If $a=1$, then the matrix of coefficients has rank 2 and the total matrix has rank 3, and we have no solution, so the three planes have no point in common.

If we put $a=1$, then

$$\begin{array}{rrcrcrcl} \alpha: & x & + & y & - & 2z & = & 0, \\ \beta: & 2x & - & y & - & z & = & 3, \\ \gamma: & & & y & - & z & = & 1, \end{array} \qquad \text{med} \qquad \begin{pmatrix} 1 & 1 & -2 \\ 2 & -1 & -1 \\ 0 & 1 & -1 \end{pmatrix}.$$

2. For $\alpha\cap\beta$ we reduce

$$\left(\begin{array}{rrr|r} 1 & 1 & -2 & 0 \\ 2 & -1 & -1 & 3 \end{array}\right) \sim \left(\begin{array}{rrr|r} 1 & 1 & -2 & 0 \\ 0 & -3 & 3 & 3 \end{array}\right) \sim \left(\begin{array}{rrr|r} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & -1 \end{array}\right)$$
$$\sim \left(\begin{array}{rrr|r} 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & -1 \end{array}\right),$$

hence, by using z as parameter,

$$x=z+1 \qquad \text{and} \qquad y=z-1.$$

The line is described by

$$\ell_{\alpha\beta}: \quad (s+1,s-1,s)=(1,-1,0)+s(1,1,1), \qquad s\in\mathbb{R}.$$

For $\alpha\cap\gamma$ we reduce

$$\left(\begin{array}{rrr|r} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right) \sim \left(\begin{array}{rrr|r} 1 & 0 & -1 & -1 \\ 0 & 1 & -1 & 1 \end{array}\right).$$

Using z as parameter we get

$$x=z-1 \qquad \text{and} \qquad y=z+1.$$

The line is described by

$$\ell_{\alpha\gamma}: \quad (s-1,s+1,s)=(-1,1,0)+s(1,1,1), \qquad s\in\mathbb{R}.$$

For $\beta\cap\gamma$ we reduce

$$\left(\begin{array}{rrr|r} 2 & -1 & -1 & 3 \\ 0 & 1 & -1 & 1 \end{array}\right) \sim \left(\begin{array}{rrr|r} 2 & 0 & -2 & 4 \\ 0 & 1 & -1 & 1 \end{array}\right) \sim \left(\begin{array}{rrr|r} 1 & 0 & -1 & 2 \\ 0 & 1 & -1 & 1 \end{array}\right).$$

Using z as parameter we get

$$x=z+2 \qquad \text{and} \qquad y=z+1.$$

The line is described by

$$\ell_{\beta\gamma}: \quad (s+2, s+1, s) = (2,1,0) + s(1,1,1), \qquad s \in \mathbb{R}.$$

Since the direction of all three lines is given by the vector $(1,1,1)$, the three lines are parallel.

Now,

$$\ell_{\alpha\beta}: \quad (1,-1,0) + s(1,1,1), \qquad s \in \mathbb{R},$$

$$\ell_{\beta\gamma}: \quad (2,1,0) + t(1,1,1), \qquad t \in \mathbb{R}.$$

We first find $s \in \mathbb{R}$, such that

$$(1,-1,0) + s(1,1,1) - (2,1,0)$$

is perpendicular to $(1,1,1)$, thus

$$0 = (1,1,1) \cdot \{(1,-1,0) + s(1,1,1) - (2,1,0)\} = 3s - 3.$$

If $s = 1$, then $(2,0,1)$ is a point on $\ell_{\alpha\beta}$, hence

$$(2,0,1) - (2,1,0) = (0,-1,1)$$

is perpendicular to the direction $(1,1,1)$ of the two lines.
The length $\sqrt{2}$ of this vector is the distance between $\ell_{\alpha\beta}$ and $\ell_{\beta\gamma}$.

3. Now,

$$\beta: \quad 2x - y - z = 3, \qquad \text{thus } (2,-1,-1) \cdot (x,y,z) = 3,$$

$$\ell_{\alpha\gamma}: \quad (-1,1,0+) + s(1,1,1), \qquad s \in \mathbb{R}.$$

A normed normal vector og β is $\mathbf{q}_\beta = \dfrac{1}{\sqrt{6}}(2,-1,-1)$.

The line through the point $(-1,1,0)$ on $\ell_{\alpha\gamma}$ in the direction $\mathbf{q}_\beta$ has the parametric description

$$(x,y,z) = (-1,1,0) + \frac{s}{\sqrt{6}}(2,-1,-1), \qquad s \in \mathbb{R}.$$

We shall find s, such that this point lies in β, because $|s|$ is then the searched length (because $\mathbf{q}_\beta$ is normed and perpendicular to β).

The condition is

$$3 = 2x - y - z = -2 - 1 + \frac{s}{\sqrt{6}}(4+1+1) = -3 + s\sqrt{6},$$

thus $s = \sqrt{6}$.

4. We first compute the distance between $\ell_{\alpha\beta}$ and $\ell_{\alpha\gamma}$, where

$$\ell_{\alpha\beta}: \quad (1,-1,0)+s(1,1,1), \quad s\in\mathbb{R},$$

$$\ell_{\alpha\gamma}: \quad (-1,1,0)+t(1,1,1), \quad t\in\mathbb{R}.$$

It is immediately seen that $(1,-1,0)-(-1,1,0)=2(1,-1,0)$ is perpendicular to the direction $(1,1,1)$ of the lines, hence the distance between them is $2\sqrt{2}=\sqrt{8}$.

It is left to the reader to sketch the corresponding triangle which is perpendicular to the direction $(1,1,1)$. If the edges are symbolized by α, β and γ, corresponding to each of the planes, then the height from $\ell_{\alpha\gamma}$ to β is of length $\sqrt{6}$. The edge from β is of length $\sqrt{2}$, and the edge from α is of length $\sqrt{8}$. Since

$$(\sqrt{2})^2+(\sqrt{6})^2=8=(\sqrt{8})^2,$$

the triangle must necessarily be right-angled, so the edge from γ is equal to the height onto β, i.e. of length $\sqrt{6}$.

The remaining part of the example is concerned with finding the angles in a right-angled triangle. We conclude from

$$\frac{\sqrt{2}}{2\sqrt{2}} = \frac{1}{2} = \cos\frac{\pi}{3} \quad \text{and} \quad \frac{\sqrt{6}}{2\sqrt{2}} = \frac{\sqrt{3}}{2} = \cos\frac{\pi}{6},$$

that

- the angle between the planes α and β is $\frac{\pi}{3}$,
- the angle between the planes α and γ is $\frac{\pi}{6}$,
- the angle between β and γ is $\frac{\pi}{2}$.

Example 3.41 *Given in an ordinary rectangular coordinate system XYZ in space of positive orientation a sphere of the equation*

$$x^2 + y^2 + z^2 - 4x - 6y - 2z + 13 = 0$$

and the line ℓ of the parametric description

$$(x, y, z) = (3, 4, 0) + t(1, 0, -1), \qquad t \in \mathbb{R}.$$

1. *Find the coordinates of the centrum C of the sphere, and find the radius r of the sphere.*
2. *Find the distance between ℓ and the centrum C of the sphere.*
3. *The sphere is now illuminated by a parallel bundle of light rays, of of the vector of direction $(1, 0, -1)$.*

   Find an equation of the contour of the shadow, which the sphere produces on the XY-plane, and give a name of this contour.

1. It follows by some manipulations that

$$\begin{aligned} 0 &= x^2 + y^2 + z^2 - 4x - 6y - 2z + 13 \\ &= \{x^2 - 4x + 4 - 4\} + \{y^2 - 6y + 9 - 9\} + \{z^2 - 2z + 1 - 1\} + 13 \\ &= (x-2)^2 + (y-3)^2 + (z-1)^2 - 4 - 9 - 1 + 13 \\ &= (x-2)^2 + (y-3)^2 + (z-1)^2 - 1, \end{aligned}$$

   from which we obtain the equation of the sphere

$$(x-2)^2 + (y-3)^2 + (z-1)^2 = 1^2$$

   of centrum $C : (2, 3, 1)$ and radius 1.

2. We shall first find the vector from C to the point on ℓ, which is perpendicular to ℓ. The condition is

$$\begin{aligned} 0 &= \{(3, 4, 0) + t(1, 0, -1) - (2, 3, 1)\} \cdot (1, 0, -1) \\ &= 3 - 2 + 1 + 2t = 2 + 2t, \end{aligned}$$

which is fulfilled for $t = -1$. Then the vector is

$$(1,1,1) - (1,0,-1) = (0,1,0).$$

The distance between C and ℓ is equal to the length 1 of this vector.

3. We note that ℓ is a line from the parallel bundle, which at the same time is tangent to the sphere at the point $(2,4,1)$.

   By a reflection in C we obtain the point $(2,2,1)$, which is projected into $(3,2,0)$, hence the centre of the projection is $(3,3,0)$, and the half axis in the direction of the Y-axis has length 1.

   In order to find the second half axis we notice that $\frac{1}{\sqrt{2}}(1,0,1)$ is a unit vector, which is perpendicular to both $(1,0,-1)$ and $(0,1,0)$, where the latter vector is not changed by the projection. Then the second half axis (which is in parallel to the X-axis) is derived from the projection of

$$(2,3,1) - \frac{1}{\sqrt{2}}(1,0,1) \quad \text{and} \quad (2,3,1) + \frac{1}{\sqrt{2}}(1,0,1).$$

   This means that we shall find s and t, such that the Z-coordinates become 0 (the images by the projection). Hence

$$s = 1 - \frac{1}{\sqrt{2}} \qquad \text{and} \qquad t = 1 + \frac{1}{\sqrt{2}},$$

   and the two points are

$$(2,3,1) - \frac{1}{\sqrt{2}}(1,0,1) + (1,0,-1) - \frac{1}{\sqrt{2}}(1,0,-1) = (3-\sqrt{2},3,0)$$

   and

$$(2,3,1) + \frac{1}{\sqrt{2}}(1,0,1) + (1,0,-1) + \frac{1}{\sqrt{2}}(1,0,-1) = (3+\sqrt{2},3,0).$$

   The centrum of the ellipse is $(3,3)$, one of the half axis has length $\sqrt{2}$, the other 1, so the limit curve of the shadow has the equation

$$\frac{1}{2}(x-3)^2 + (y-3)^2 = 1.$$

   The shadow itself is the set

$$\left\{(x,y) \in \mathbb{R}^2 \mid (x-3)^2 + 2(y-3)^2 \leq 2\right\}.$$

4 Quadratic forms

Example 4.1 *Find a reducing orthogonal substitution of the quadratic form*

1. $2x_1x_2 + 2x_3 4_4$;
2. $x_1^2 + x_2^2 + x_3^2 + x_4^2 - 2x_1x_2 + 6x_1x_3 - 4x_1x_4 - 4x_2x_3 + 6x_2x_4 - 2x_3x_4$.

1. It follows that

$$K(x_1, x_2, x_3, x_4) = \mathbf{x}^T \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \mathbf{x},$$

hence

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} \mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{pmatrix},$$

where

$$\mathbf{B} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

If $\mathbf{v}$ is an eigenvector of $\mathbf{B}$, then $(\mathbf{v}, \mathbf{0})$ and $(\mathbf{0}, \mathbf{v})$ are orthogonal eigenvectors of $\mathbf{A}$ corresponding to the same eigenvalue.

The characteristic polynomial of $\mathbf{B}$ is

$$\det(\mathbf{B} - \lambda\mathbf{I}) = \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 = (\lambda - 1)(\lambda + 1),$$

hence the eigenvalues are λ_1 with e.g. the eigenvector $(1, 1)$, and $\lambda_2 = -1$ with e.g. the eigenvector $(1, -1)$, both of length $\sqrt{2}$.

A reducing orthogonal substitution is given by

$$\mathbf{Q} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{pmatrix} \quad \text{med} \quad \mathbf{\Lambda} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

2. Here,

$$K(x_1, x_2, x_3, x_4) = \mathbf{x}^T \begin{pmatrix} 1 & -1 & 3 & -2 \\ -1 & 1 & -2 & 3 \\ 3 & -2 & 1 & -1 \\ -2 & 3 & -1 & 1 \end{pmatrix} \mathbf{x},$$

hence

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 3 & -2 \\ -1 & 1 & -2 & 3 \\ 3 & -2 & 1 & -1 \\ -2 & 3 & -1 & 1 \end{pmatrix}.$$

The characteristic polynomial is

$$\begin{aligned}
\det(\mathbf{A}-\lambda\mathbf{I}) &= \begin{vmatrix} 1-\lambda & -1 & 3 & -2 \\ -1 & 1-\lambda & -2 & 3 \\ 3 & -2 & 1-\lambda & -1 \\ -2 & 3 & -1 & 1-\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & 1-\lambda & 1-\lambda & 1-\lambda \\ 0 & 2-\lambda & -2-\lambda & 3-\lambda \\ 1 & 1 & -\lambda & -\lambda \\ -2 & 3 & -1 & 1-\lambda \end{vmatrix} \\
&= (1-\lambda)\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 2-\lambda & -2-\lambda & 3-\lambda \\ 1 & 1 & -\lambda & -\lambda \\ -2 & 3 & -1 & 1-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 2-\lambda & -2-\lambda & 3-\lambda \\ 0 & 0 & -1-\lambda & -1-\lambda \\ 0 & 5 & 1 & 3-\lambda \end{vmatrix} \\
&= (\lambda-1)(\lambda+1)\begin{vmatrix} 2-\lambda & -2-\lambda & 3-\lambda \\ 0 & 1 & 1 \\ 5 & 1 & 3-\lambda \end{vmatrix} = (\lambda^2-1)\begin{vmatrix} 2-\lambda & 0 & 5 \\ 0 & 1 & 1 \\ 5 & 1 & 3-\lambda \end{vmatrix} \\
&= (\lambda^2-1)\begin{vmatrix} 2-\lambda & 0 & 5 \\ 0 & 1 & 1 \\ 5 & 0 & 2-\lambda \end{vmatrix} = (\lambda^2-1)\begin{vmatrix} 2-\lambda & 5 \\ 5 & 2-\lambda \end{vmatrix} \\
&= (\lambda^2-1)\{(\lambda-2)^2-5^2\} = (\lambda-1)(\lambda+1)(\lambda+3)(\lambda-7),
\end{aligned}$$

thus the eigenvalues are $\lambda_1=-3$, $\lambda_2=-1$, $\lambda_3=1$ and $\lambda_4=7$.

If $\lambda_1=-3$, then

$$\begin{aligned}
\mathbf{A}-\lambda_1\mathbf{I} &= \begin{pmatrix} 4 & -1 & 3 & -2 \\ -1 & 4 & -2 & 3 \\ 3 & -2 & 4 & -1 \\ -2 & 3 & -1 & 4 \end{pmatrix} \sim \begin{pmatrix} 4 & 4 & 4 & 4 \\ 0 & 5 & 1 & 6 \\ 1 & 1 & 3 & 3 \\ -2 & 3 & -1 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 5 & 1 & 6 \\ 0 & 0 & 2 & 2 \\ 0 & 5 & 1 & 6 \end{pmatrix} \\
&\sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 5 & 1 & 6 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\end{aligned}$$

An eigenvector is e.g. $\mathbf{v}_1=(1,-1,-1,1)$ of length $\sqrt{1+1+1+1}=2$. Hence a normed eigenvector is

$$\mathbf{q}_1 = \frac{1}{2}\,(1,-1,-1,1).$$

If $\lambda_2 = -1$, then

$$\begin{aligned}
\mathbf{A} - \lambda_2 \mathbf{I} &= \begin{pmatrix} 2 & -1 & 3 & -2 \\ -1 & 2 & -2 & 3 \\ 3 & -2 & 2 & -1 \\ -2 & 3 & -1 & 2 \end{pmatrix} \sim \begin{pmatrix} 2 & 2 & 2 & 2 \\ 0 & 3 & -1 & 4 \\ 1 & 1 & 1 & 1 \\ -2 & 3 & -1 & 2 \end{pmatrix} \\
&\sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 3 & -1 & 4 \\ 0 & 5 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 3 & -1 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
&\sim \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\end{aligned}$$

An eigenvector is e.g. $\mathbf{v}_2 = (1, 1, -1, -1)$, which is of length

$$\|\mathbf{v}_2\| = \sqrt{1+1+1+1+} = 2.$$

Thus a normed eigenvector is

$$\mathbf{q}_2 = \frac{1}{2}\,(1, 1, -1, -1).$$

If $\lambda_3 = 1$, then

$$\begin{aligned}
\mathbf{A} - \lambda_3 \mathbf{I} &= \begin{pmatrix} 0 & -1 & 3 & -2 \\ -1 & 0 & -2 & 3 \\ 3 & -2 & 0 & -1 \\ -2 & 3 & -1 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -3 & 2 \\ 1 & 1 & -1 & -1 \\ -2 & 3 & -1 & 0 \end{pmatrix} \\
&\sim \begin{pmatrix} 1 & 1 & -1 & -1 \\ 0 & 5 & -3 & -2 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -1 & -1 \\ 0 & 1 & -3 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
&\sim \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\end{aligned}$$

We see that eigenvector $\mathbf{v}_3 = (1,1,1,1)$ is an eigenvector of length 2, so a normed eigenvector is

$$\mathbf{q}_2 = \frac{1}{2}\,(1,1,1,1).$$

If $\lambda_4 = 7$, then

$$\begin{aligned}
\mathbf{A} - \lambda_4 \mathbf{I} &= \begin{pmatrix} -6 & -1 & 3 & -2 \\ -1 & -6 & -2 & 3 \\ 3 & -2 & -6 & -1 \\ -2 & 3 & -1 & -6 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -5 & -9 & -4 \\ 1 & 1 & -7 & -7 \\ -2 & 3 & -1 & -6 \end{pmatrix} \\
&\sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 5 & 1 & -4 \\ 0 & -5 & -9 & -4 \\ 0 & 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 5 & 1 & -4 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \\
&\sim \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\end{aligned}$$

An eigenvector is e.g. $(1,-1,1,-1)$ of length 2, hence a normed eigenvector is

$$\mathbf{q}_4 = \frac{1}{2}\,(1,-1,1,-1).$$

A reducing orthogonal substitution is given by

$$\mathbf{Q} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}, \quad \text{and vi har} \quad \mathbf{\Lambda} = \begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 7 \end{pmatrix}.$$

Example 4.2 *Prove that the two quadratic forms*

(1) $6x^2 + 5y^2 + 7z^2 - 4\sqrt{2}\,yz$

and

(2) $7x^2 + 6y^2 + 5z^2 + 4yz + 4xy$

can be reduced to the same quadratic form

$$ax_1^2 + by_1^2 + cz_1^2 \qquad\qquad (a < b < c),$$

and find for each of these forms an orthogonal substitution, which carries out this reduction. Then find an orthogonal substitution, which carries the quadratic form (2) into (1).

The matrix $\mathbf{A}$ corresponding to (1) is

$$\mathbf{A} = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 5 & -2\sqrt{2} \\ 0 & -2\sqrt{2} & 7 \end{pmatrix}$$

with the characteristic polynomial

$$\begin{aligned}
\det(\mathbf{A} - \lambda\mathbf{I}) &= \begin{vmatrix} 6-\lambda & 0 & 0 \\ 0 & 5-\lambda & -2\sqrt{2} \\ 0 & -2\sqrt{2} & 7-\lambda \end{vmatrix} = (6-\lambda)\begin{vmatrix} 5-\lambda & -2\sqrt{2} \\ -2\sqrt{2} & 7-\lambda \end{vmatrix} \\
&= -(\lambda-6)\{(\lambda-5)(\lambda-7)-8\} = -(\lambda-6)(\lambda^2-12\lambda+27) \\
&= -(\lambda-6)\{(\lambda-6)^2-3^2\} = -(\lambda-3)(\lambda-6)(\lambda-9).
\end{aligned}$$

The matrix $\mathbf{B}$ corresponding to (2) is

$$\mathbf{B} = \begin{pmatrix} 7 & 2 & 0 \\ 2 & 6 & 2 \\ 0 & 2 & 5 \end{pmatrix}$$

and its characteristic polynomial is

$$\begin{aligned}
\det(\mathbf{B} - \lambda\mathbf{I}) &= \begin{vmatrix} 7-\lambda & 2 & 0 \\ 2 & 6-\lambda & 2 \\ 0 & 2 & 5-\lambda \end{vmatrix} \\
&= (7-\lambda)\begin{vmatrix} 6-\lambda & 2 \\ 2 & 5-\lambda \end{vmatrix} - 2\begin{vmatrix} 2 & 0 \\ 2 & 5-\lambda \end{vmatrix} \\
&= (7-\lambda)\{(\lambda-6)(\lambda-5)-4\} + 4(\lambda-5) \\
&= -(\lambda-6)\{\lambda^2-12\lambda+35-8\} = -(\lambda-3)(\lambda-6)(\lambda-9).
\end{aligned}$$

In both cases we get the simple roots 3, 6, 9, hence we can reduce to the quadratic form

$$3x_1^2 + 6y_1^2 + 9z_1^2.$$

If $\lambda_1 = 3$, then we get for $\mathbf{A}$ that

$$\mathbf{A} - \lambda_1 \mathbf{I} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & -2\sqrt{2} \\ 0 & -2\sqrt{2} & 4 \end{pmatrix} \sim \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & -\sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}.$$

An eigenvector is e.g. $\mathbf{v}_{1,A} = (0, \sqrt{2}, 1)$ of length $\sqrt{3}$. Then by norming,

$$\mathbf{q}_{1,A} = \left(0, \sqrt{\frac{2}{3}}, \sqrt{\frac{1}{3}}\right).$$

If $\lambda_1 = 3$, then we get for $\mathbf{B}$,

$$\mathbf{B} - \lambda_1 \mathbf{I} = \begin{pmatrix} 4 & 2 & 0 \\ 2 & 3 & 2 \\ 0 & 2 & 2 \end{pmatrix} \sim \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

An eigenvector is e.g. $\mathbf{v}_{1,B} = (1, -2, 2)$ of length $\sqrt{1+4+4} = 3$. Then by norming,

$$\mathbf{q}_{1,B} = \frac{1}{3}(1, -2, 2).$$

If $\lambda_2 = 6$, then we get for $\mathbf{A}$ that

$$\mathbf{A} - \lambda_2 \mathbf{I} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & -2\sqrt{2} \\ + & -2\sqrt{2} & 1 \end{pmatrix}.$$

A normed eigenvector is

$$\mathbf{q}_{2,A} = (1, 0, 0).$$

If $\lambda_2 = 6$, then we get for $\mathbf{B}$,

$$\mathbf{B} - \lambda_2 \mathbf{I} = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

An eigenvector is e.g. $\mathbf{v}_{2,B} = (2, -1, -2)$ of length $\sqrt{4+1+4} = 3$. Then by norming,

$$\mathbf{q}_{2,B} = \frac{1}{3}(2, -1, -2).$$

If $\lambda_3 = 9$, then we get for $\mathbf{A}$ that

$$\mathbf{A} - \lambda_3 \mathbf{I} = \begin{pmatrix} -3 & 0 & 0 \\ 0 & -4 & -2\sqrt{2} \\ 0 & -2\sqrt{2} & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2} & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

An eigenvector is e.g. $\mathbf{v}_{3,A} = (0, 1, -\sqrt{2})$ of length $\sqrt{3}$. Then by norming,

$$\mathbf{q}_{3,A} = \left(0, \sqrt{\frac{1}{3}}, -\sqrt{\frac{2}{3}}\right).$$

If $\lambda_3 = 9$, then we get for $\mathbf{B}$ that

$$\mathbf{B} - \lambda_3\mathbf{I} = \begin{pmatrix} -2 & 2 & 0 \\ 2 & -3 & 2 \\ 0 & 2 & -4 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

An eigenvector is e.g. $\mathbf{v}_{3,B} = (2, 2, 1)$ of length 3. Hence a normed eigenvector is given by

$$\mathbf{q}_{3,B} = \frac{1}{3}\,(2, 2, 1).$$

We have

$$\mathbf{\Lambda} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 9 \end{pmatrix} = \mathbf{Q}_A^T\mathbf{A}\mathbf{Q}_A = \mathbf{Q}_B^T\mathbf{B}\mathbf{Q}_B,$$

where

$$\mathbf{Q}_A = (\mathbf{q}_{1,A} \;\; \mathbf{q}_{2,A} \;\; \mathbf{q}_{3,A}) = \begin{pmatrix} 0 & 1 & 0 \\ \sqrt{\frac{2}{3}} & 0 & \sqrt{\frac{1}{3}} \\ \sqrt{\frac{1}{3}} & 0 & -\sqrt{\frac{2}{3}} \end{pmatrix}$$

and

$$\mathbf{Q}_B = (\mathbf{q}_{1,B} \;\; \mathbf{q}_{2,B} \;\; \mathbf{q}_{3,B}) = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{pmatrix}.$$

Since $\mathbf{Q}_A^{-1} = \mathbf{Q}_A^T$, it follows that

$$\mathbf{A} = \mathbf{Q}_A \mathbf{Q}_B^T \mathbf{B} \mathbf{Q}_B \mathbf{Q}_A^T = \left(\mathbf{Q}_A \mathbf{Q}_B^T\right) \mathbf{B} \left(\mathbf{Q}_A \mathbf{Q}_B^T\right)^T.$$

The transformation is uniquely determined by

$$\begin{aligned}
\mathbf{Q}_A\mathbf{Q}_B^T &= \begin{pmatrix} 0 & 1 & 0 \\ \sqrt{\frac{2}{3}} & 0 & \sqrt{\frac{1}{3}} \\ \sqrt{\frac{1}{3}} & 0 & -\sqrt{\frac{2}{3}} \end{pmatrix} \frac{1}{3}\begin{pmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{pmatrix} \\
&= \frac{1}{3\sqrt{3}}\begin{pmatrix} 0 & 1 & 0 \\ \sqrt{2} & 0 & 1 \\ 1 & 0 & -\sqrt{2} \end{pmatrix}\begin{pmatrix} 1 & -2 & 2 \\ 2 & -1 & -1 \\ 2 & 2 & 1 \end{pmatrix} \\
&= \frac{1}{3\sqrt{3}}\begin{pmatrix} 2 & -1 & -1 \\ 2+\sqrt{2} & 2-2\sqrt{2} & 1+2\sqrt{2} \\ 1-2\sqrt{2} & -2-2\sqrt{2} & 2-\sqrt{2} \end{pmatrix}.
\end{aligned}$$

Example 4.3 *Given the symmetric matrix*

$$\mathbf{A} = \begin{pmatrix} 2 & -2 & 1 \\ -2 & 5 & -2 \\ 1 & -2 & 2 \end{pmatrix}.$$

1. *Reduce the quadratic form*

$$(x \;\; y \;\; z)\mathbf{A}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

to a form $\lambda_1 x_1^2 + \lambda_2 y_1^2 + \lambda_3 z_1^2$*, where* $\lambda_1 \geq \lambda_2 \geq \lambda_3$*, and find a proper orthogonal substitution, which carries out this reduction.*

2. *Find all pairs* (p, q)*, for which*

$$(\mathbf{A} - p\mathbf{I})(\mathbf{A} - q\mathbf{I}) = \mathbf{0}.$$

1. We compute the characteristic polynomial,

$$\begin{aligned}
\det(\mathbf{A} - \lambda\mathbf{I}) &= \begin{vmatrix} 2-\lambda & -2 & 1 \\ -2 & 5-\lambda & -2 \\ 1 & -2 & 2-\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & 1-\lambda & 1-\lambda \\ -2 & 5-\lambda & -2 \\ 1 & -2 & 2-\lambda \end{vmatrix} \\
&= (1-\lambda)\begin{vmatrix} 1 & 1 & 1 \\ -2 & 5-\lambda & -2 \\ 1 & -2 & 2-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 1 & 1 & 0 \\ -2 & 5-\lambda & -2 \\ 1 & -2 & 1-\lambda \end{vmatrix} \\
&= (1-\lambda)^2\begin{vmatrix} 1 & 1 \\ -2 & 5-\lambda \end{vmatrix} = (1-\lambda)^2(5-\lambda+2) \\
&= -(\lambda-7)(\lambda-1)^2.
\end{aligned}$$

The roots are $\lambda_1 = 7 > \lambda_2 = 1 = \lambda_3 = 1$.

If $\lambda_1 = 7$, then

$$\begin{pmatrix} -5 & -2 & 1 \\ -2 & -2 & -2 \\ 1 & -2 & -5 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 1 & -2 & -5 \\ -4 & -4 & -4 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

An eigenvector is e.g. $\mathbf{v}_1 = (1,-2,1)$ of length $\|\mathbf{v}_1\| = \sqrt{1+4+1} = \sqrt{6}$. Then by norming

$$\mathbf{q}_1 = \frac{1}{\sqrt{6}}\,(1,-2,1).$$

If $\lambda_2 = \lambda_3 = 1$, then

$$\begin{pmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Two linearly independent, though not orthogonal eigenvectors are e.g. $\mathbf{v}_2 = (1,1,1)$ of length $\|\mathbf{v}_2\| = \sqrt{3}$, and $\mathbf{v}_3 = (2,1,0)$, where we using the Gram-Schmidt method reduce to

$$\mathbf{v}_3 - (\mathbf{v}_3 \cdot \mathbf{v}_2)\,\frac{1}{\|\mathbf{v}_2\|^2}\,\mathbf{v}_2 = (2,1,0) - \frac{3}{3}\,(1,1,1) = (1,0,-1),$$

which is orthogonal to $\mathbf{v}_2$. We therefore choose

$$\mathbf{q}_2 = \frac{1}{\sqrt{3}}\,(1,1,1) \qquad \text{and} \qquad \mathbf{q}_3 = \frac{1}{\sqrt{2}}\,(1,0,-1).$$

The orthogonal substitution, which is defined by the matrix

$$\mathbf{Q} = (\mathbf{q}_1\ \ \mathbf{q}_2\ \ \mathbf{q}_3 =) = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix},$$

reduces the quadratic form to

$$7x_1^2 + y_1^2 + z_1^2.$$

2. Let

$$\mathbf{B} = \begin{pmatrix} 7 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then

$$(\mathbf{B} - p\mathbf{I})(\mathbf{B} - q\mathbf{I}) = \mathbf{0},$$

if and only if $(p,q) \in \{(7,1),(1,7)\}$.

Since $\mathbf{A}$ can be derived from $\mathbf{B}$ by an orthogonal substitution, we have

$$(\mathbf{A} - p\mathbf{I})(\mathbf{A} - q\mathbf{I}) = \mathbf{0} \qquad \text{if and only if } (p,q) \in \{(7,1),(1,7)\}.$$

Example 4.4 *Given the symmetric matrix*

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & -3a \\ 2 & 5 & -4a \\ -3a & -4a & 13a \end{pmatrix}, \qquad \text{where } a \in \mathbb{R}.$$

1. *Find a lower triangular unit matrix* $\mathbf{L}$ *and an upper triangular matrix* $\mathbf{U}$*, such that* $\mathbf{A} = \mathbf{LU}$.
2. *Find all* a*, for which* $\mathbf{A}$ *is positive definite.*

1. By a simple Gauß reduction,

$$\begin{aligned} \mathbf{A} &= \begin{pmatrix} 1 & 2 & -3a \\ 2 & 5 & -4a \\ -3a & -4a & 13a \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -3a \\ 0 & 1 & 2a \\ 0 & 2a & -9a^2+13a \end{pmatrix} \\ &\sim \begin{pmatrix} 1 & 2 & -3a \\ 0 & 1 & 2a \\ 0 & 0 & -13a^2+13a \end{pmatrix} \end{aligned}$$

hence

$$\mathbf{U} = \begin{pmatrix} 1 & 2 & -3a \\ 0 & 1 & 2a \\ 0 & 0 & -13a(a-1) \end{pmatrix} \qquad \text{and} \qquad \mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3a & 2a & 1 \end{pmatrix}.$$

2. Now, $\mathbf{A}$ is positive definite, if all diagonal elements of $\mathbf{U}$ are positive, so the condition becomes $0 < a < 1$.

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